CSCI 361 Lecture 4:
Regular Expressions

Shikha Singh

Announcements & Logistics

- **Assignment 2** due Sept 17 (tomorrow) at 10 pm
  - Please check LaTeX formatting
  - Please assign pages to each question on Gradescope
- Hand in Exercise #3, pick up Exercise #4
- Assignment 1 graded feedback will be released soon
- Reminder: Midterm 1 (Oct 7) and Midterm 2 (Nov 6)
- **Questions?**

Last Time

- Defined non-deterministic finite automata
  - Relaxed transition rules compared to DFAs
  - Equivalent in power
- Practice with NFAs
- Showed regular languages are closed under concatenation using NFAs

Today

- More closure: Kleene star operation
- Regular expressions and equivalence with DFAs/NFAs

Kleene Star

- Let A be a language on Σ
- Definition. Kleene star of A, denoted A^* is defined as:

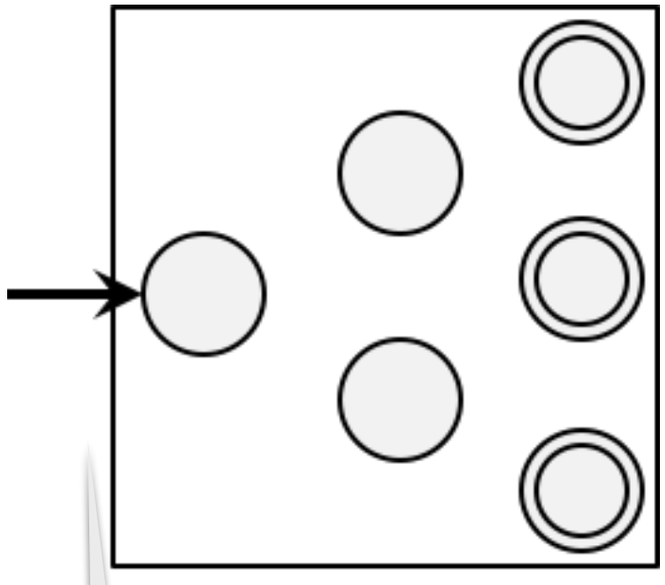
$$A^* = \{w_1 w_2 \cdots w_k \mid k \geq 0 \text{ and each } w_i \in A\}$$

- **Example**. Suppose $L_1 = \{01, 11\}$, what is L^*?

- **Question**. Are regular languages closed under Kleene star?

Kleene Star

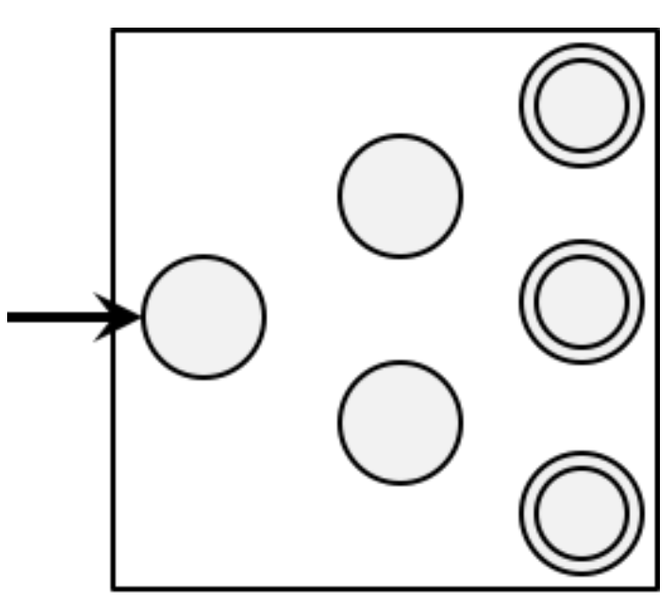
- **Theorem.** The class of regular languages is closed under Kleene star.

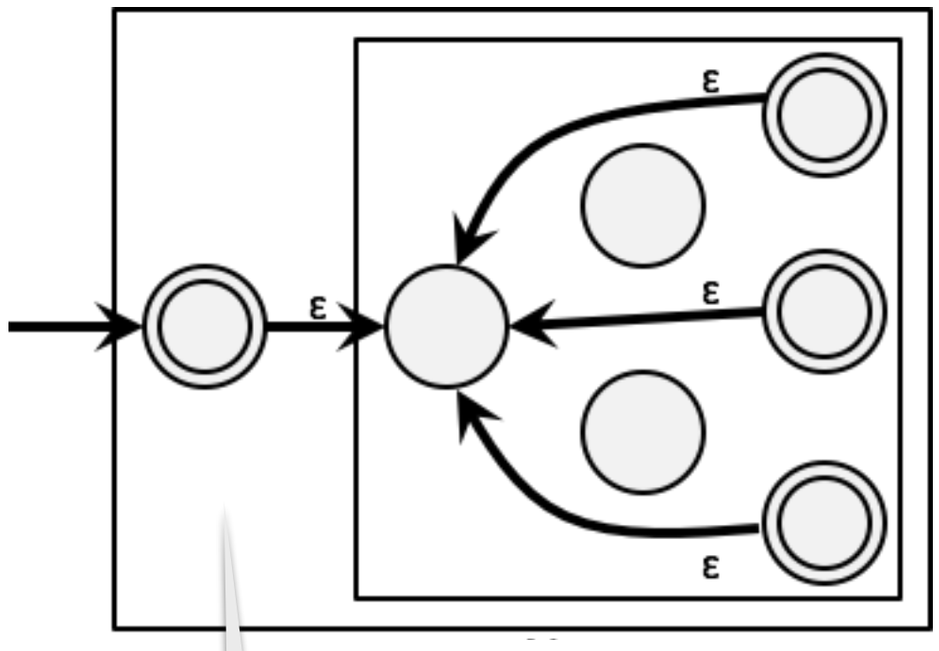


Suppose this is the NFA for L. How to draw an NFA for L^*?

Kleene Star

- **Theorem.** The class of regular languages is closed under Kleene star.





Do we need this new state? Why?

Closed Under Kleene Star

- **Theorem**. The class of languages are closed under Kleene star.
- Proof. Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be the NFA for L_1
- Construct NFA $N = (Q, \Sigma, \delta, q_0, F)$ to recognize L_1^*
  - $Q = Q_1 \cup \{q_0\}$ (add a new start state)
  - $F = F_1 \cup \{q_0\}$
  - $$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F_1 \\ \end{cases}$$

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Regular Expressions

Regular Expressions

- A "generative" way to characterize regular languages: **regular expressions**
- Have many applications in programming languages
  - `grep/ awk` in UNIX
- Define regular expressions and give examples
- Show that regular expressions are equivalent to DFA/NFA

Revisit Formal Definition

A regular expression R over the alphabet Σ is defined inductively as follows. R is regular expression if

- **(base cases)**. R is either a for some $a \in \Sigma$, or an empty string ε or an empty set $\emptyset$
- **(recursive cases using regular operators).** R is the union, concatenation or Kleene star of smaller regular expressions that is, $R = R_1 \cup R_2$ or $R = R_1 \circ R_2$ or $R = R_1^*$ where R_1, R_2 are regular expressions

- Let the language of a regular expression $L(R)$ be the set of strings that can be generated by the regular expression
- Examples: 0^*10^*, $(01)^* \cup (10)^*$, $\Sigma\Sigma$, etc

Working with Regular Expressions

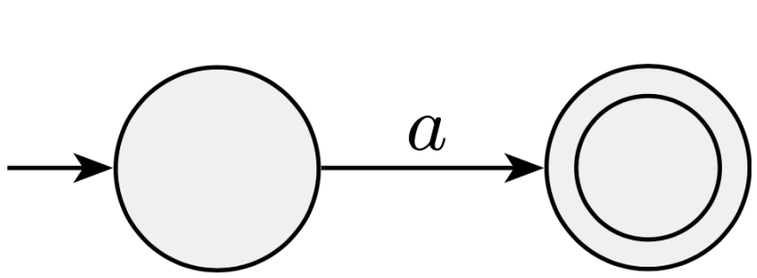
- If R is a regular expression then:
  - $R \cup \emptyset = ?$
  - $R \circ \varepsilon = ?$
- But the following may not hold:
  - $R \cup \varepsilon = ? R$
  - What is $R \circ \emptyset$?

Working with Regular Expressions

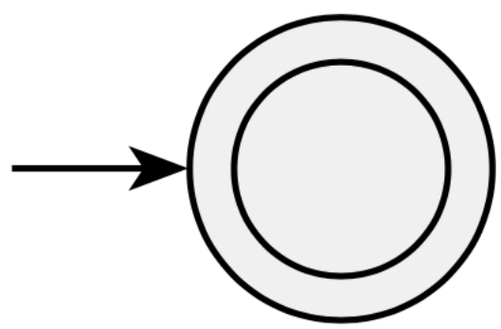
- If R is a regular expression then:
  - $R \cup \emptyset = R$
  - $R \circ \varepsilon = R$
- But the following may not hold:
  - $R \cup \varepsilon$ is not necessarily the same as R
  - $R \circ \emptyset = \emptyset$

Equivalence with Finite Automata

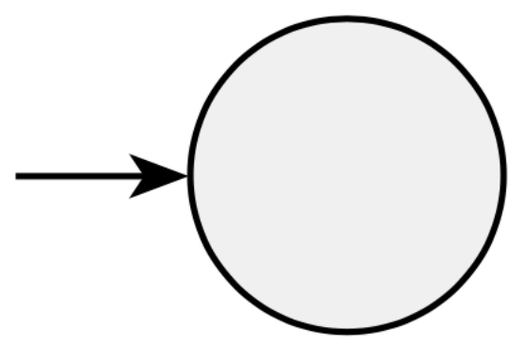
- **Lemma (1.55 in Sipser)**. If a language is described by a regular expression, then is regular
- **Proof.** Let R is the regular expression, sufficient to create an NFA that recognizes $L(R)$
    - (base cases). It is easy to create an NFA for each of the base cases for R



$R = a$



$R = \varepsilon$



$R = \emptyset$

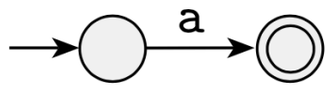
Equivalence with Finite Automata

- **Lemma (1.55 in Sipser)**. If a language is described by a regular expression, then is regular
- **Proof.** Let R is the regular expression, sufficient to create an NFA that recognizes $L(R)$
    - (recursive cases). Suppose by induction we have an NFA for any regular expression smaller than R, we can create an NFA for R using the union/concatenation/Kleene star of these NFA

Regular Expression to NFA: Example 1.56

- Convert regular expression (ab ∪ a*) to an NFA

a



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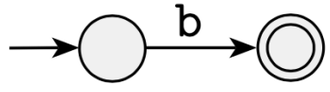

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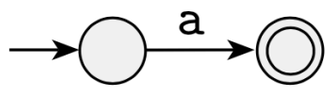

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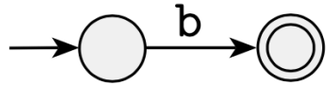
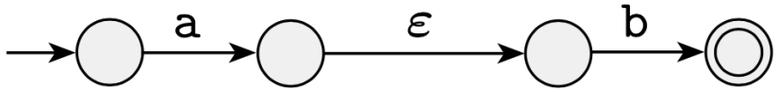


ab



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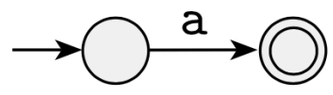

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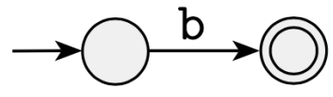
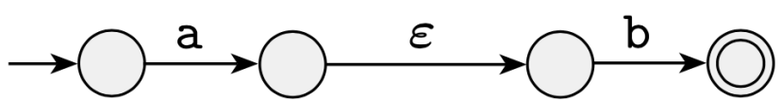
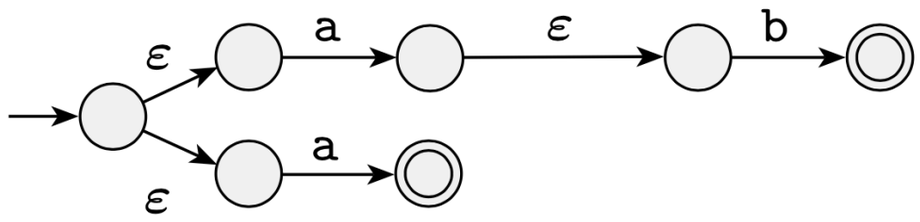


ab



ab ∪ a



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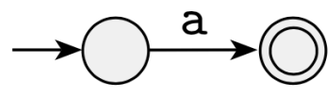

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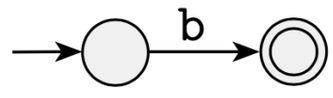
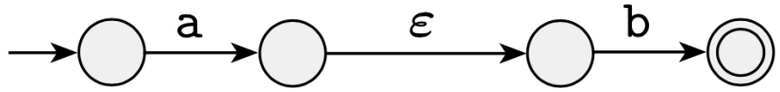
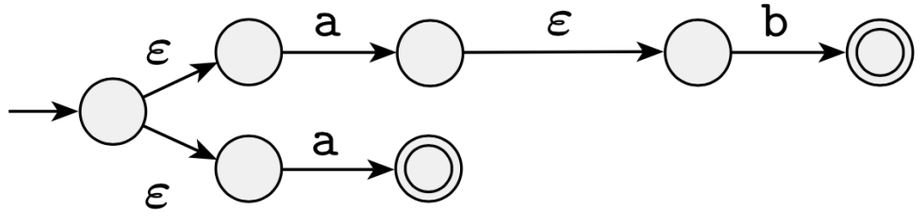


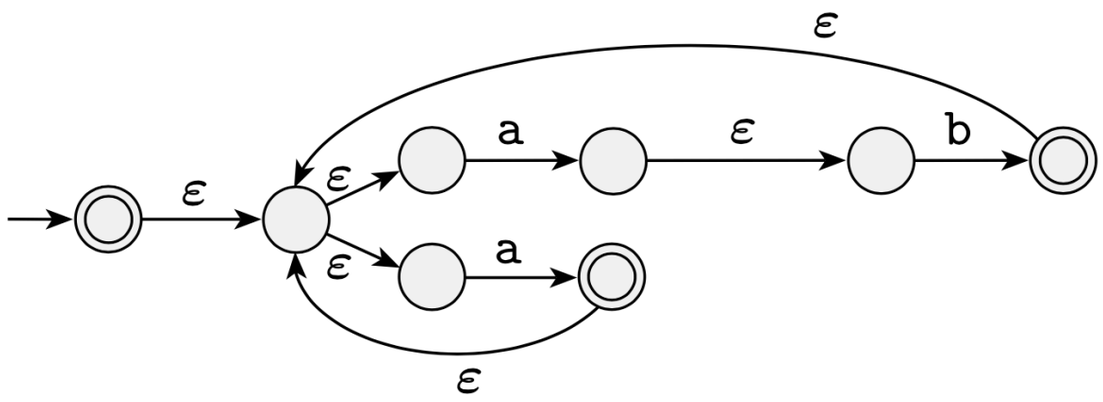
ab



ab ∪ a



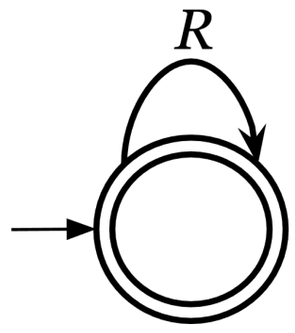
(ab ∪ a)*



NFAs to Regular Expression

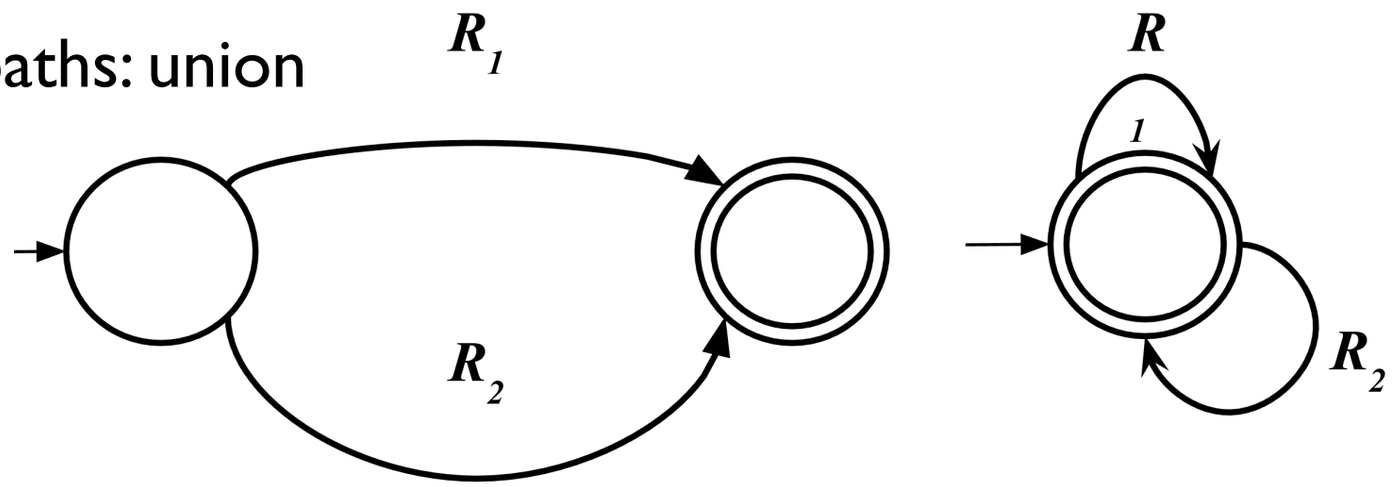
- Every NFA can be converted to an equivalent regular expression

Self loops: Kleene star

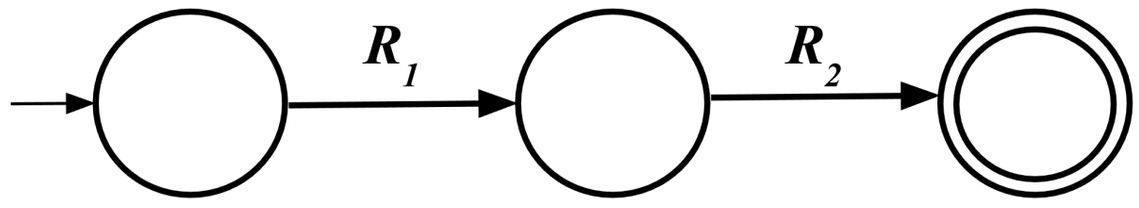


R^*

Alternate paths: union



$R_1 \cup R_2$



$R_1 \circ R_2$

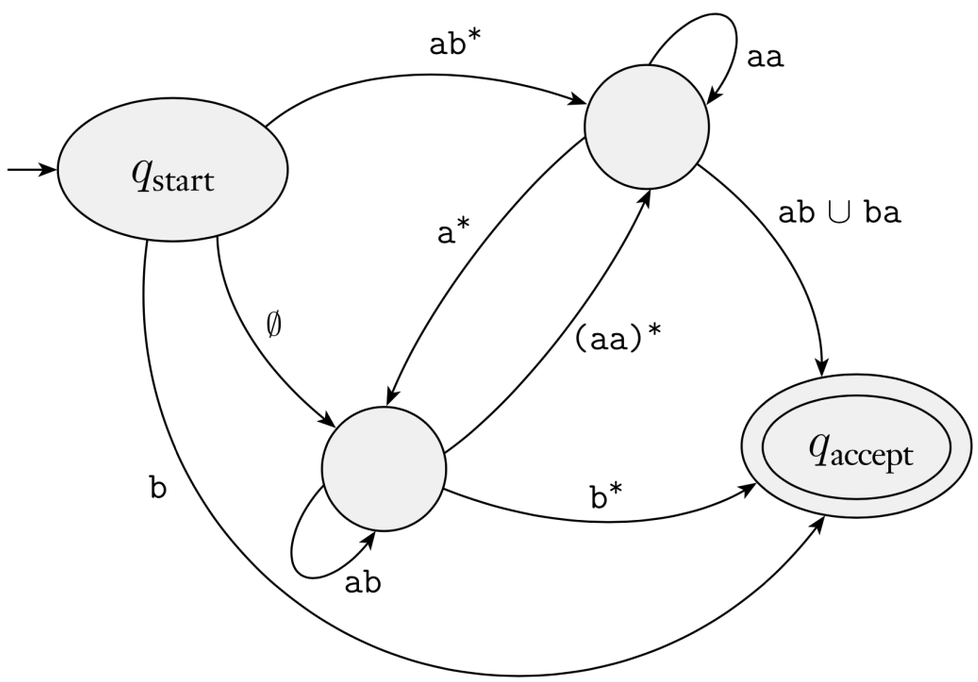
Adjacent paths: concatenation

Converting a DFA to Regular Expression

- **Lemma (1.60 in Sipser)**. If a language is regular (recognized by a DFA), then it can be described by some regular expression.
- **Proof outline.**
  - Convert the DFA into a GNFA (generalized NFA) with $k \geq 2$ states (NFA where transitions occur on regular expressions)
  - Eliminate states of the GNFA one by one until two states left
  - Output the regular expression from the start to accept state

Generalized NFA

- A GNFA is a generalized NFA with the following conditions:
  - Transitions are on regular expressions (not just symbols or ε)
  - Start state has an arrow to every other state and not arrows coming in from any state
  - Only one accept state that has arrows coming in from every other state and no arrows leaving it
  - Every other state has arrows to every other state including itself

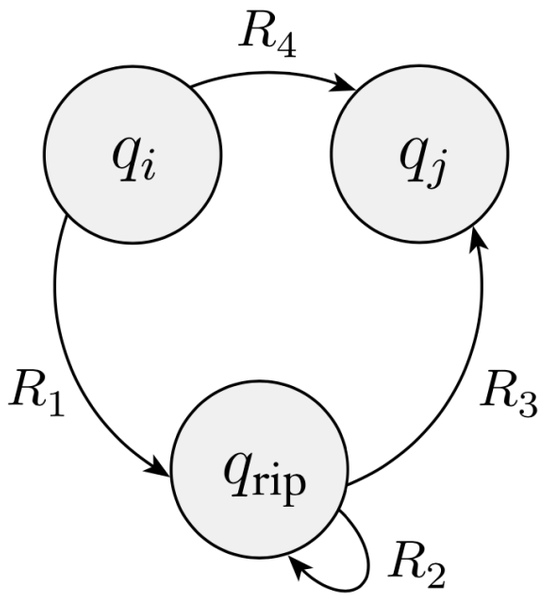


DFA $\Longrightarrow$ GNFA $\Longrightarrow$ RegularExp

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, we can convert it to a regular expression as follows
- GNFA G: add a new start state q_s and accept state q_f to M
    - If there is an arrow missing from q_s to a state $q \in Q$, add an arrow labelled with $\emptyset$
        - Add ε arrows from F to q_f
- For any pair of states (p, q) that are neither start or accept states of M, add additional $\emptyset$ arrows to create a valid GNFA
- Now perform the state-elimination algorithm described next to convert G with k states to G with $k - 1$ states

State Elimination Algorithm

- Consider a GNFA with $k > 2$ states and let q_{rip} be a state that is neither the start or accept state
- Reduce: create a GNFA with $k - 1$ states by removing q_{rip}; replace all paths that go through it with an equivalent regular expression



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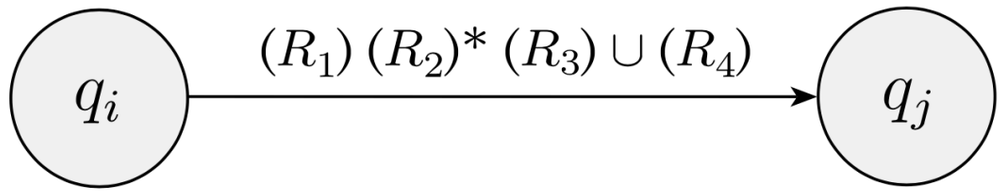

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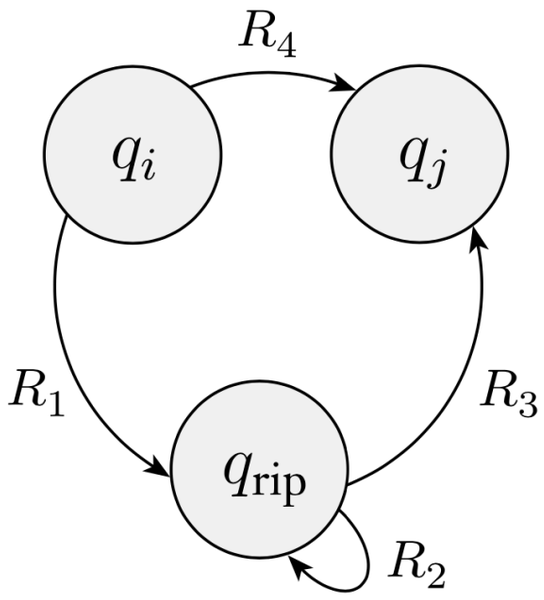
Final Regular Expression

- Perform the state elimination algorithm until there are $k = 2$ states (start and accept) states left
- Output the regular expression on the only remaining transition
- **Correctness:** by induction on the number of states of GNFA



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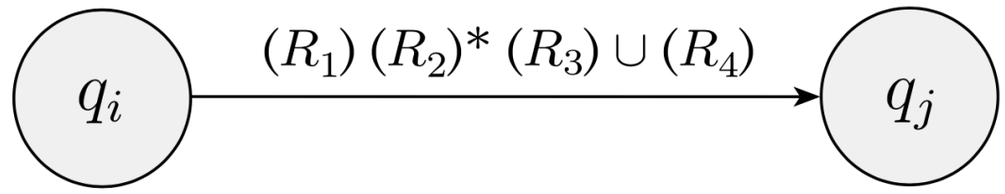

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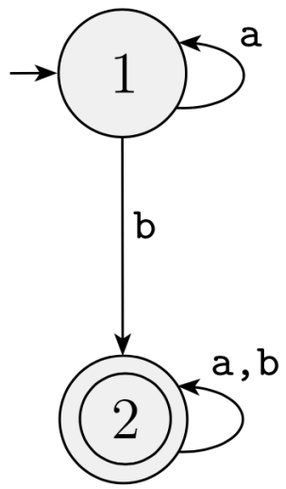
DFA to Regular Expression Example



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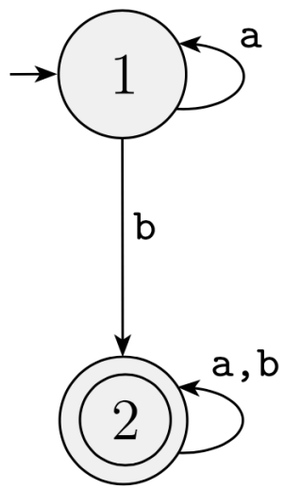
DFA to Regular Expression Example



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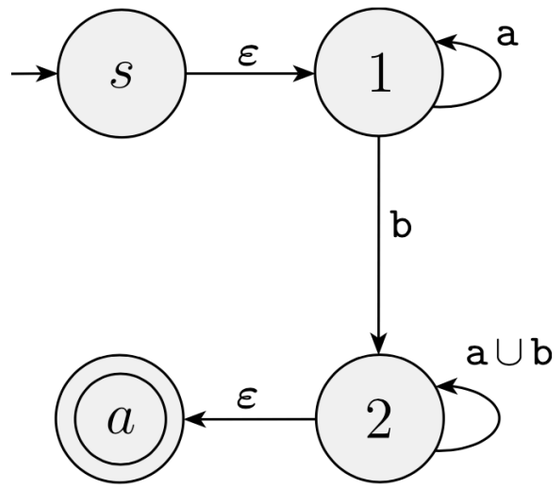

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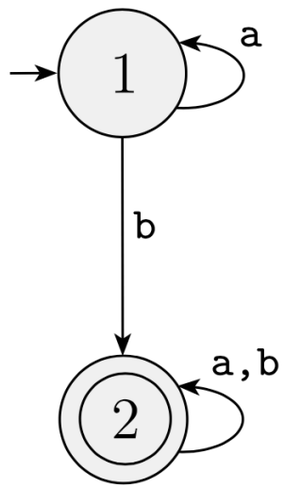
DFA to Regular Expression Example



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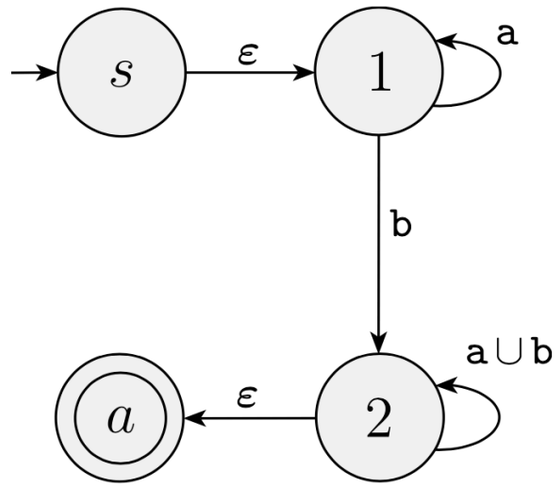

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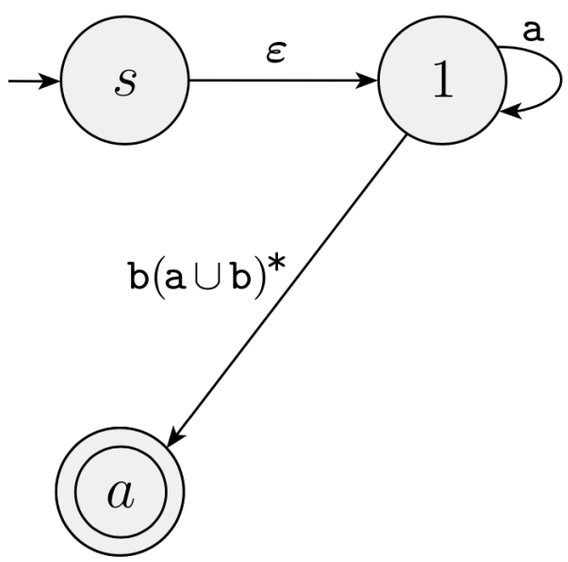

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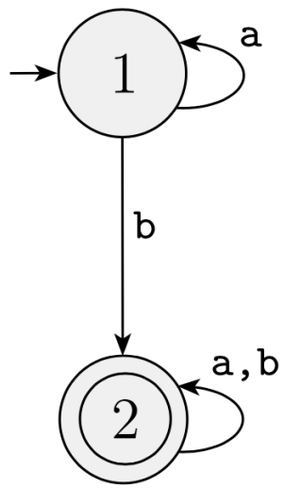
DFA to Regular Expression Example



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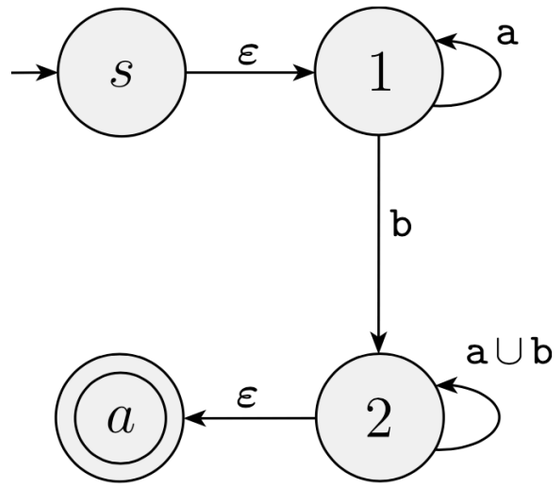

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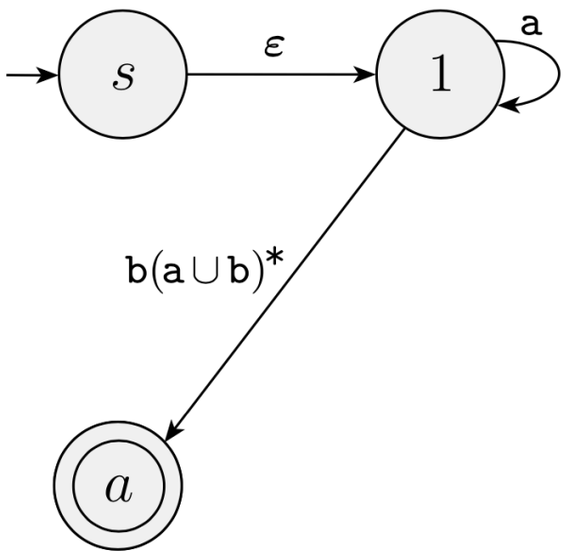

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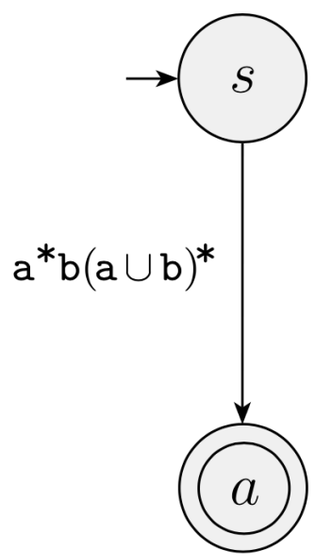


(d)

DFA $\Longrightarrow$ GNFA $\Longrightarrow$ RegularExp

- Why does this reduction work?
- **Claim.** Consider a GNFA G with k states, let G' be GNFA after the state-elimination algorithm is performed once, then both G and G' accept the same language.
    - By construction

- **Corollary.** The language of the regular expression constructed from this algorithm is the same as the starting DFA

Takeaways

- Regular expressions provide an alternate "generative" way to describe regular languages
- Three ways to characterize regular languages:
    - DFAs
    - NFAs
    - Regular languages

Not All Languages Are Regular

- Any language does not a DFA that recognizes it is not-regular
- How we do ***prove no such DFA*** exists?
    - First example of an impossibility results in this class
    - Many more to come

- Intuitively, any decision problem that requires ***finite*** memory "to solve" is regular
- **Question.** Are finite languages regular?
    - $L \subseteq \Sigma^*$ and $|L|$ is finite
    - All finite languages are regular

All Finite Languages are Regular

- **Theorem.** All finite languages are regular.
- $L = \{w_1, \ldots, w_n\}$ for some $n \in \mathbb{N}$
- Let $L_i = \{w_i\}$ for each $i \in \{1,\ldots,n\}$
- $L = \cup_{i=1}^{n} L_i$

- **Claim 1.** Each L_i is regular.

- **Claim 2.** A finite union of regular languages is regular.

- Using Claim 1 and 2, L is regular

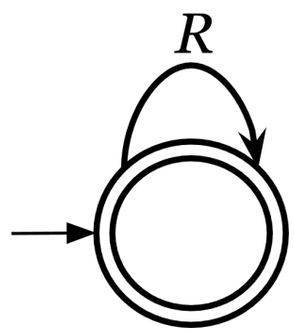
Infinite Regular Languages

- Have seen many infinite regular languages
- What do they have in common?

Structure of Infinite Regular Languages

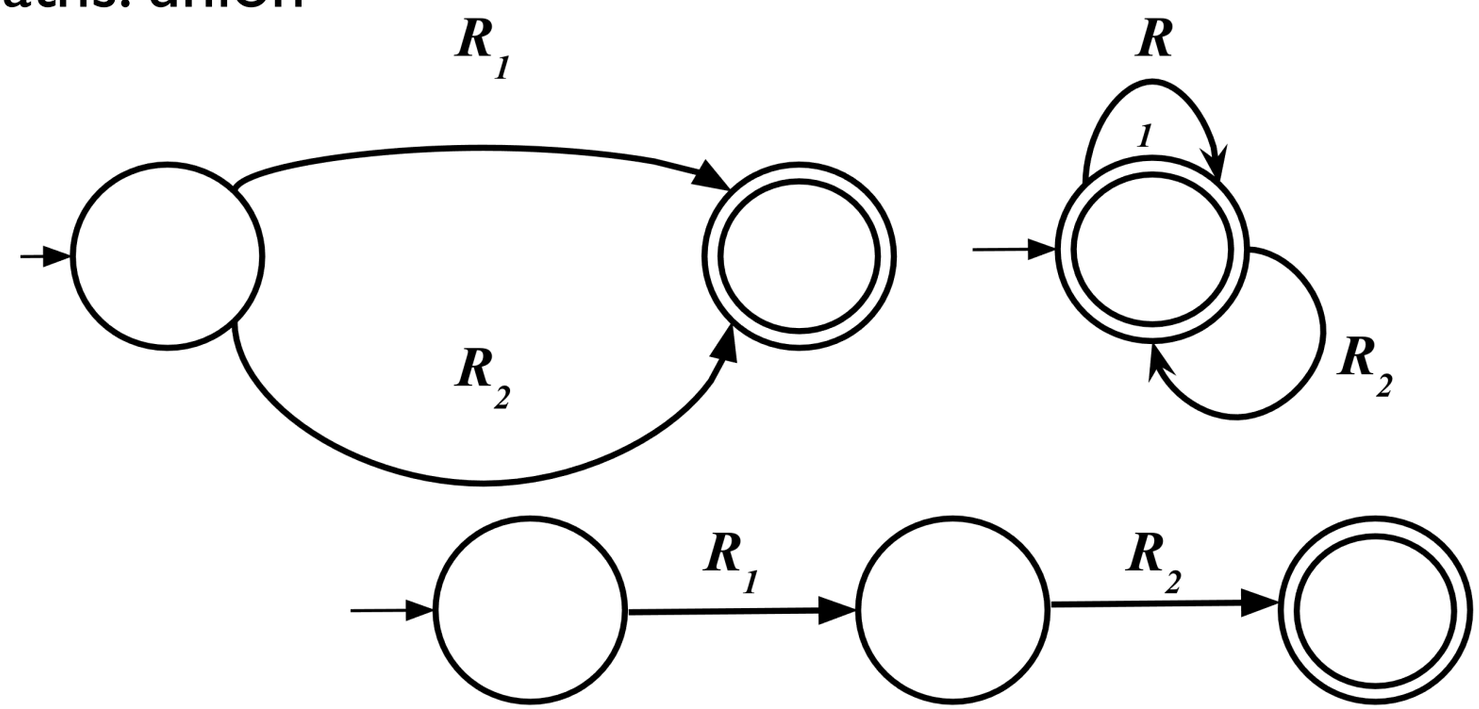
- Which of these are responsible for going from finite to infinite?

Self loops: Kleene star



R^*

Alternate paths: union



$R_1 \cup R_2$

$R_1 \circ R_2$

Adjacent paths: concatenation

Loops in DFA: Intuition

- Consider the DFA $M's$ transitions on an input string w
- It enters some states $q_0, \ldots, q_1, q_2, \ldots, q_n$
- **Question.** If there is a "loop" what does that mean about the states visited?

- Now suppose two different strings x, y bring M to the same state q
- Consider attaching the same suffix z to both
- **Question.** What can we say about the state M is in after reading input string xz versus after reading input string yz?

Indistinguishability (DFA)

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Let x, y be any string over Σ.

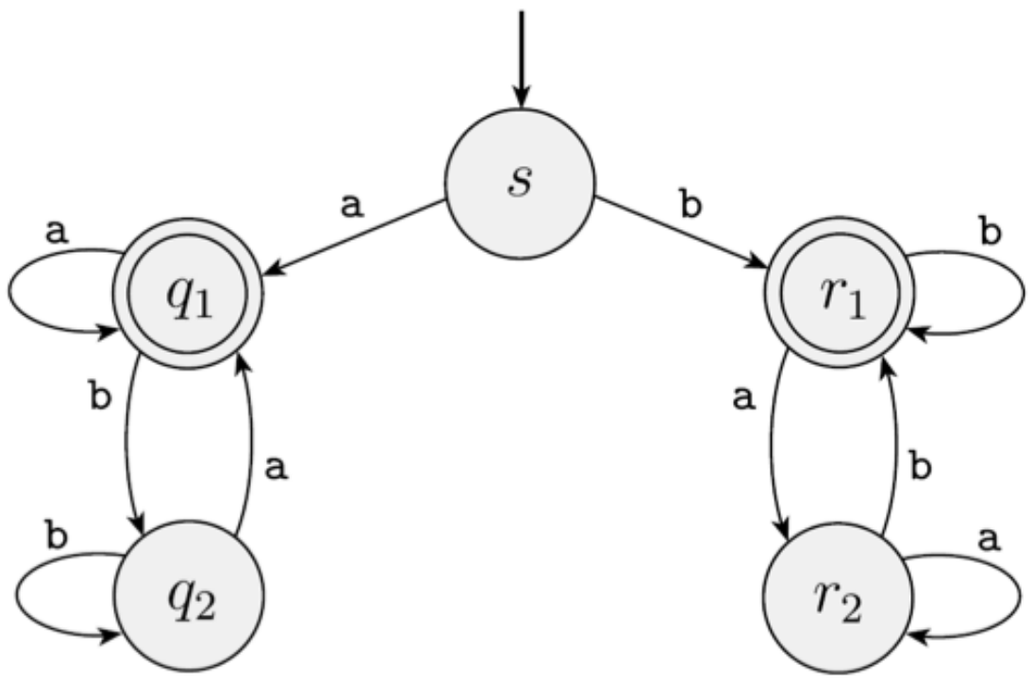
Definition. x **indistinguishable to** y with respect to a DFA M, denoted $x \sim_M y$ if and only if $\delta^*(q_0, x) = \delta^*(q_0, y)$ (i.e., the state reached by M on x is the same as the state reached by M on y)

Corollary. If $x \sim_M y$ then for all $z \in \Sigma^*$, then

$xz \in L(M) \iff yz \in L(M)$

Class Exercise

- **Example.** $L = \{w \in \{a, b\}^* \mid w$ starts and ends with the same symbol$\}$
- **Definition.** x **indistinguishable to** y with respect to a DFA M, denoted $x \sim_M y$ if and only if $\delta^*(q_0, x) = \delta^*(q_0, y)$ (i.e., the state reached by M on x is the same as the state reached by M on y)
- **Question:** for each state in the DFA for L, write a regular expression characterizing all strings that bring the DFA to that state.



Indistinguishability (Languages)

Let L be any language over an alphabet Σ.

Definition. x **indistinguishable to** y with respect to L, denoted $x \equiv_L y$ if and only if for all $z \in \Sigma^*$, we have that $xz \in L \iff yz \in L$

Observation: $\equiv_L$ is an equivalence relation over Σ^*

Thus, $\equiv_L$ ***partitions*** Σ^* into equivalence classes.

Distinguishing Suffixes

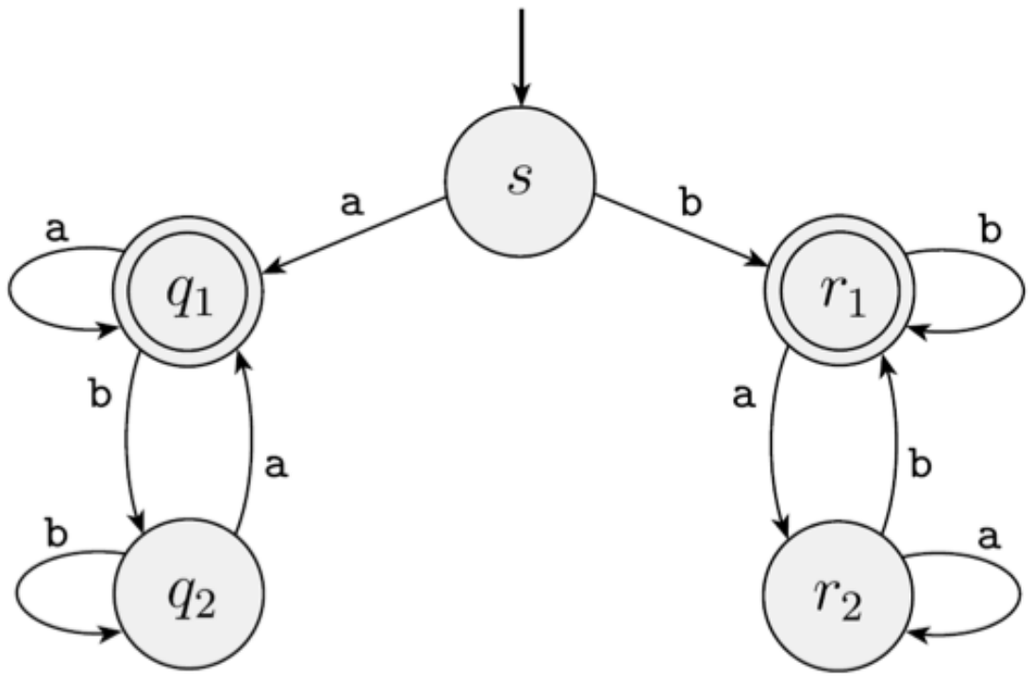
- Every string in the same equivalence class $[x]$ of $\equiv_L$ are indistinguishable with each other
- Two strings $x, y \in \Sigma^*$ are in different equivalence iff they are ***distinguishable***
  - Can find a suffix $z \in \Sigma^*$ that distinguishes them, that is, $xz \in L$ and $yz \notin L$ or $xz \notin L$ and $yz \in L$
- **Question.** Suppose $x \in L$ and $y \notin L$, are they distinguishable?

Indistinguishability (Languages)

- **Example.**

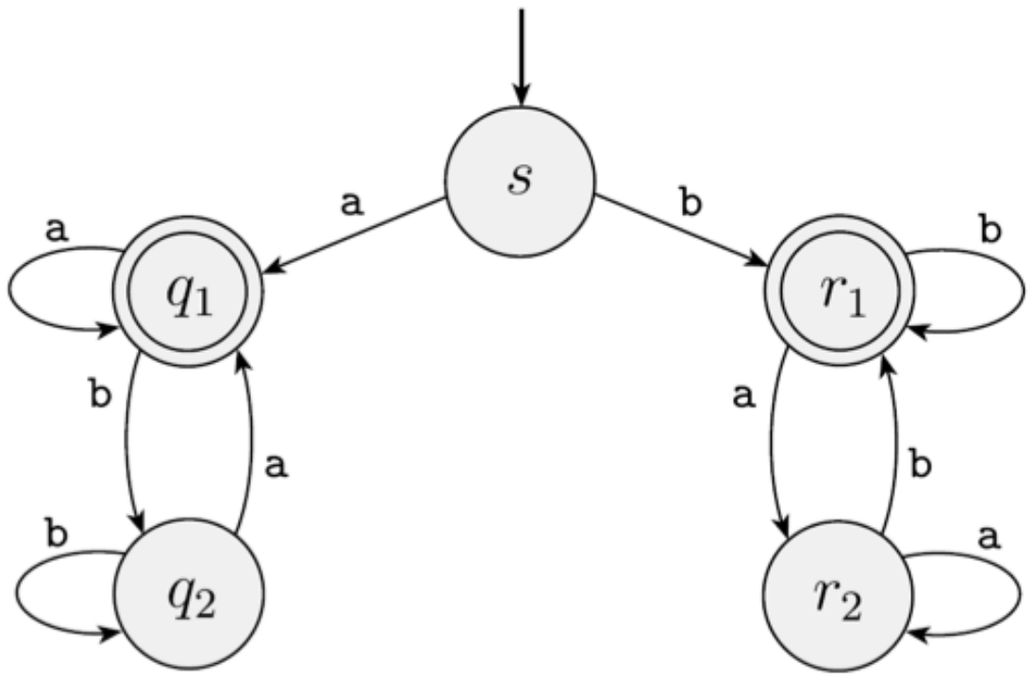
  $L = \{w \in \{a, b\}^* \mid w \text{ starts and ends with the same symbol}\}$

- **Problem.** Find the equivalence classes of the relation $\equiv_L$.



Indistinguishability DFA vs Languages

- **Claim.** If $x \sim_M y$, then $x \equiv_{L(M)} y$.



Minimal DFA

- **Claim.** If a language L over Σ has k equivalence classes defined by $\equiv_L$, then any DFA for L must have at least k states.
- **Corollary.** If a DFA M for L has number of states equal to the number of equivalence classes of $\equiv_L$ then such a DFA is minimal.

Myhill-Nerode Theorem

Let L be a language over Σ^*, then L is regular **if and only if** the relation $\equiv_L$ over Σ^* has a finite number of equivalence classes.

Myhill-Nerode Theorem

Let L be a language over Σ^*, then L is regular **if and only if** the relation $\equiv_L$ over Σ^* has a finite number of equivalence classes.

Necessary condition. For L to be regular, it must have finitely many equivalence classes. Equivalently, if $\equiv_L$ over Σ^* has an infinite number of equivalence classes, then L cannot be regular.

Sufficient condition. If $\equiv_L$ has finitely many equivalence classes, then L must be regular.

Proving Non Regularity

- Myhill-Nerode theorem says that any language that has infinitely many equivalence classes with respect to $\equiv_L$ is not regular
- Typically, we don't need to find all of equivalence classes
- Sufficient to find an infinite subset of strings that are mutually distinguishable

Fooling Sets

Definition. A set of strings $S \subseteq \Sigma^*$ is a **fooling set** with respect to a language $L \subseteq \Sigma^*$ if every pair of strings in S is distinguishable with respect to each other.

Example. $L = \{w \in \{a, b\}^* \mid w$ starts and ends with the same symbol$\}$

An example fooling set for L?

Question. Can the size of a fooling set be bigger than the number of equivalence classes?

- Max size of a fooling set for L = # of equivalence class of $\equiv_L$
- Size of any fooling set for $L \leq$ # of equivalence class of $\equiv_L$

Myhill-Nerode Theorem

Maximum fooling set size of L

$=$ # equivalence classes of $\equiv_L$

$=$ minimum states of DFA for L

Takeaway. If we could prove that there exists an infinite number of distinguishable sets for a language, it would mean that even the smallest DFA for the language would require an infinite number of states. Therefore, no such DFA exists and the language cannot be regular.

Proving Non-Regularity

Problem. Prove that the language $L = \{a^i b^i \mid i \in \mathbb{N}\}$ is not regular.

Hint. Identify and prove that L has an infinite fooling set.

Proving Non-Regularity

Problem. Prove that the langueg

$L = \{a^n \mid n \in \mathbb{N} \text{ and } n \text{ is a power of 2}\}$ is not regular.

Hint. Identify and prove that L has an infinite fooling set.