Lecture 12: Locality-Sensitive Hashing and MinHash

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Williams College

Introduction: Finding Similar Items

Back to Normal Inputs

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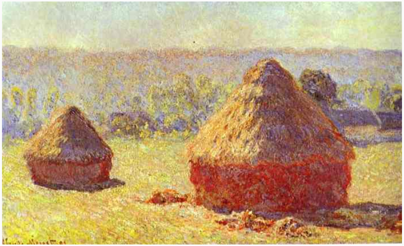
- But data is still big!
  - In particular: high-dimensional
  - Table with many columns
  - For each netflix user, what movies have they seen
- Goal: solve a difficult, but important, problem

Finding Similar Pair



- Given a set of objects

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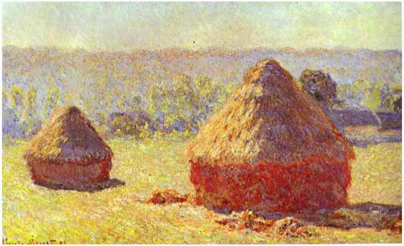

- Given a set of objects
- Find the most similar pair of objects in the set

Why Find Similar Objects?

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- Similar music: Spotify suggests music by finding similar users, and selecting what they listen to
- Machine learning in general (training, evaluation, actual algorithms, etc.)
- Data deduplication, etc.
- “Give me a similar pair in this dataset” is a common query!

Strategies for Similarity Search

First attempt: 1-dimensional data

92
44
7
65
60
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80
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- Given a list of numbers

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- Given a list of numbers
- “Similarity” is the difference between them
- How can we find the closest numbers (i.e. ones with smallest difference)?

First attempt: 1-dimensional data

- How efficiently can we do this?

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- How efficiently can we do this?
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- Step 2: Scan through list, find most similar adjacent elements.

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- How efficiently can we do this?
- Step 1: Sort!
- Step 2: Scan through list, find most similar adjacent elements.
- $O(n \log n)$ time

First attempt: 1-dimensional data

Aside: can we do better? Yes, there's a clever $O(n)$ algorithm based on sampling.

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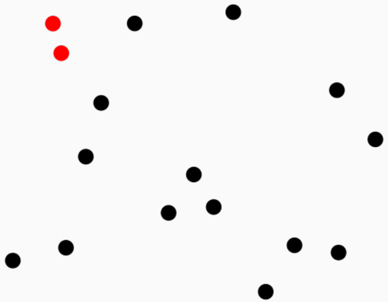
- Step 2: Scan through list, find most similar adjacent elements.
- $O(n \log n)$ time

Two-dimensional Data?



- You likely saw this in CS 256.

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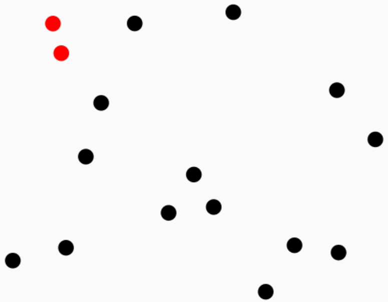

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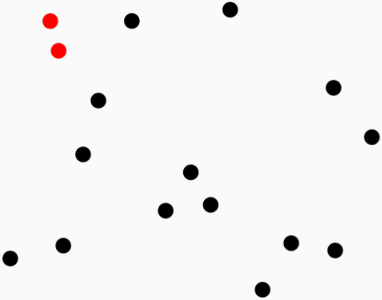

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- Divide and conquer, $O(n \log n)$ time.
- (Again, possible in $O(n)$)

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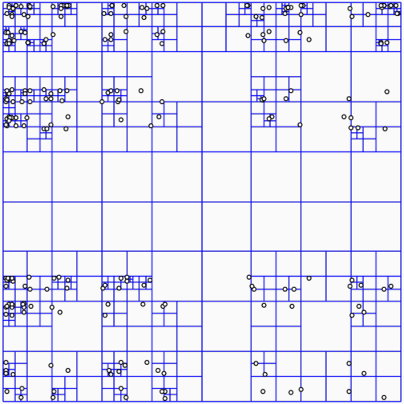
- Songs listened to, movies watched, image tags, etc.
- Words that appear in a book, k-grams that appear in a DNA sequence
- Classic options: quad trees, kd trees

How Efficient are High-dimensional Algorithms?



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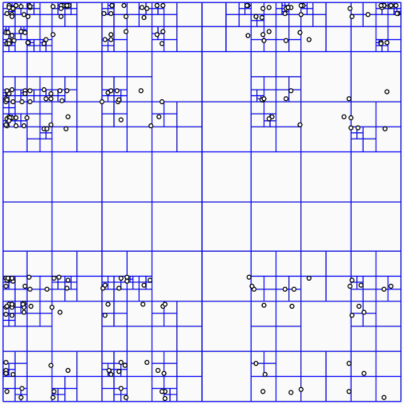

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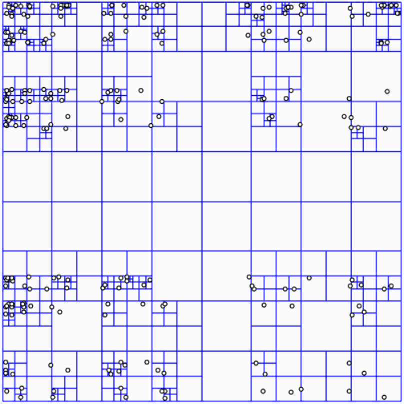

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- But: exponential in dimension!
- Worse than trying all pairs if $> \log n$ dimensions

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- Many problems have running time exponential in the dimension of the objects.
- Well-known phenomenon
- Applies to similarity search, machine learning, combinatorics

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  - Use hashing! ...A special kind of hashing

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- Locality-sensitive hashing tries to hash similar items together

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- High level: close items are likely to collide. Far items are unlikely to collide.
- Generally want p_2 to be about $1/n$; then we get a normal hash table for far (i.e. distance $\geq cr$) elements.

Why Locality-Sensitive Hashing Helps

	(101, 37, 65) (103,37,64)	(91,84,3)		(100,18,79)
0	1	2	3	4

Ideally, close items hash to the same bucket.

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- If we have $p_2 = 1/n$, then p_1 is usually very small.
- How can we increase this probability?
- Repetitions! Maintain many hash tables, each with a different locality-sensitive hash function, and try all of them.

LSH with Repetitions

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Similarity

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    - Words that appear in a document
- Need a way to measure set similarity

Set Similarity

User 1	User 2
Post Malone	Ariana Grande
Ariana Grande	Khalid
Khalid	Drake
Drake	Travis Scott
Travis Scott	

- When are two sets similar?

Set Similarity

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- When are two sets similar?
- Let's look at our two sets. Similar if they have a lot of *overlap*

Set Similarity

Very similar!

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- I.e. : lots of artists in common, compared to total artists in either list

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Set Similarity

Not very similar!

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- When are two sets similar?
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Set Similarity

Moderately similar

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Ariana Grande	Ed Sheerhan
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Jaccard Similarity

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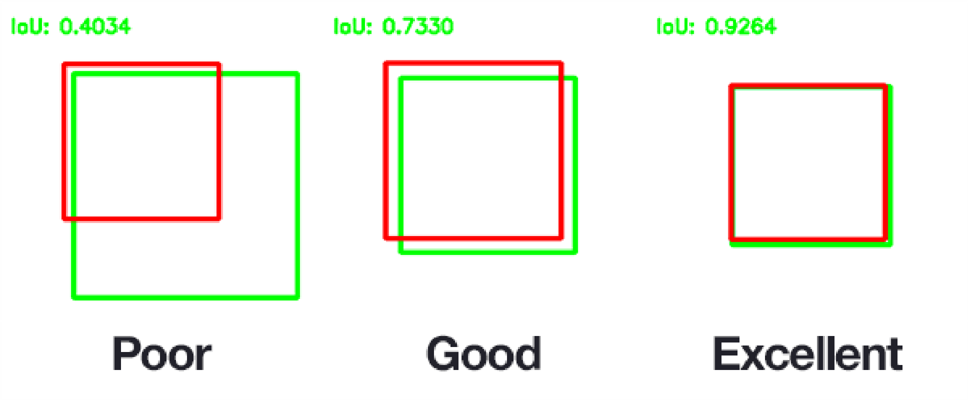
- Intuitively: what fraction of these sets overlaps?

Jaccard Similarity Intuition 1



Jaccard Similarity Intuition 2

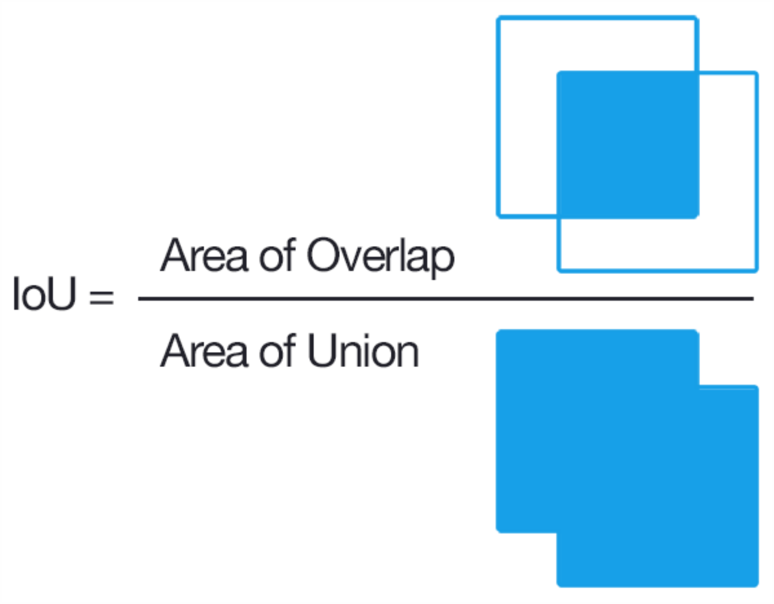
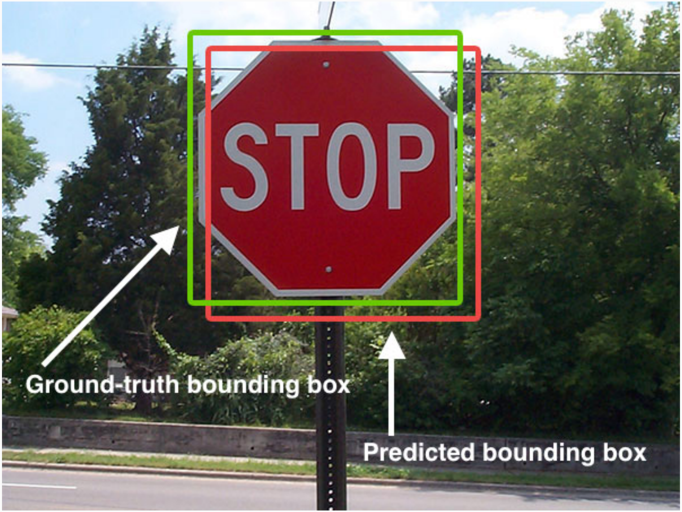


Image Search Example



Jaccard Example 1

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- $|A \cap B| = 4$

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- $|A \cap B| = 4$
- $|A \cup B| = 5$

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- $|A \cap B| = 4$
- $|A \cup B| = 5$
- Jaccard Similarity: $4/5 = .8$

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- $|A \cap B| = 1$
- $|A \cup B| = 5$
- Jaccard Similarity: $1/5 = .2$

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- $|A \cap B| = 3$

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- $|A \cap B| = 3$
- $|A \cup B| = 7$

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- $|A \cap B| = 3$
- $|A \cup B| = 7$
- Jaccard Similarity: $3/7 = 0.428$

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- 1 means identical, 0 means no items in common
- Jaccard similarity ignores items not in either set. So we learn nothing if neither of us like an artist. (Is this good?)
- Still works if one list is much longer than the other. Generally, they'll have small overlap

Locality-Sensitive Hash for Jaccard Similarity

- Want: items with high Jaccard Similarity are likely to hash together

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- (AltaVista was probably the most popular search engine before Google, they wanted to detect similar web pages to eliminate them from search results)
- Now used for similarity search, database joins, clustering—LOTS of things.

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- If someone likes red and blue, we can store that information as 101000.
- Effective if universe is smallish; use a list for larger universe

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- We want the size of these sets—need to count the number of 1s in A & B, or A | B.

MinHash

- The hash consists of an *permutation* of all possible items in the universe

MinHash

128 in the assignment

- The hash consists of an *permutation* of all possible items in the universe

- To hash a set A: find the first item of A in the order given by the permutation. That item is the hash value!

MinHash example

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MinHash example

- Let's stick with favorite colors, out of $\{$red, yellow, blue, green, purple, orange$\}$
- To hash, we randomly permute them. Let's say our current hash is given by the permutation (blue, orange, green, purple, red, yellow)

MinHash example

- Let's stick with favorite colors, out of {red, yellow, blue, green, purple, orange}
- To hash, we randomly permute them. Let's say our current hash is given by the permutation (blue, orange, green, purple, red, yellow)
- First set is 101000 (same as {red, blue}). blue is in the set, so the hash value is blue.

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- First set is 101000 (same as {red, blue}). blue is in the set, so the hash value is blue.
- Second set is 110010 (we could also write {red, yellow, purple}). blue is not in the set; nor is orange; nor is green. purple is, so purple is the hash value

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For the sake of space let's do 8 bits

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- Let's say $x = 10011001$, and the permutation is $(1, 5, 2, 0, 7, 6, 4, 3)$.
- Then the hash of x is 5.

Analysis of Basic MinHash

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- Therefore: probability of hashing together is $|A \cap B|/|A \cup B|$.

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Analysis: Phrased as bit vectors

- What is the probability that $h(A) = h(B)$?
- Let's look at the permutation that defines h. We can ignore any index that is 0 in both A and B.
- Look at the first index in the permutation that is 1 in A or B
  - If this index is in *both* A and B, then $h(A) = h(B)$
  - If this index is in only one of A or B, then $h(A) \neq h(B)$
- Any index that is 1 in $A|B$ is equally likely to be first. If the index is in $A\&B$, they hash together; otherwise they do not
- Therefore: probability of hashing together is (number of 1s in $A\&B$)/(number of 1s in $A|B$).

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- Probability that red or green is first out of $\{\text{red, blue, green, orange, purple}\}$ is $2/5$.
- Therefore, A and B hash together with probability $2/5$.

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- Let's say our close pair has similarity .5. How many times do we need to repeat?
- Each repetition has the close pair in the same bucket with probability .5. So need 2 repetitions in expectation.

An Aside on Expectation

Lemma 1

If a random process succeeds with probability p, then in expectation it takes $1/p$ iterations of the process before success.

Examples:

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Proof:

$$\sum_{i=1}^{\infty} ip(1-p)^{i-1} = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

Concatenations and Repetitions

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- How can we decrease the probability that items hash together?
- Answer: concatenate multiple hashes together.

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- Rather than one hash h, concatenate k independent hashes $h_1, h_2, \ldots h_k$, each with its own permutation $P_1, P_2, \ldots P_k$.

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  - We only obtain the same concatenated hashes if *all* of the hashes are the same.
  - They are independent, so we can multiply to obtain probability $(|A \cap B|/|A \cup B|)^k$ of A and B colliding.

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- Concatenating, we have $h(A) = \text{redbluered}$

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- Let's hash B.
  - First hash: red is in B.
  - Second hash: orange is in B.
  - Third hash: red is in B.
- Concatenating, we have $h(B) = \text{redorangered}$

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- So: overall need kR permutations
- What kind of values work for k and R?

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- How should we set k? How many repetitions R is it likely to take?

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- We can then solve $(n-1)(1/3)^k = 1$ to get $k = \log_3 n - 1$.

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- So we expect about $R = n^{.26}$ repetitions. That's a lot!
- But it's essentially the best we know how to do.

Practical MinHash Considerations

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- So let's say we have $A = \{$black, red, green, blue, orange$\}$, and we're looking at a permutation $P = \{$purple, red, white, orange, yellow, blue, green, black$\}$.

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- So let's say we have $A = \{$black, red, green, blue, orange$\}$, and we're looking at a permutation $P = \{$purple, red, white, orange, yellow, blue, green, black$\}$.
- Then A hashes to redorangeblue

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- Does this affect the analysis?
    - Yes; the k we're concatenating for each hash table are no longer independent!
    - But this works fine in practice (and is used all the time)

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- So getting unlucky is super costly!
- What can we do if we happen to get a big bucket?

Handling Big Buckets

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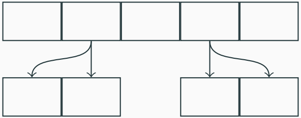
- Recurse!

Handling Big Buckets



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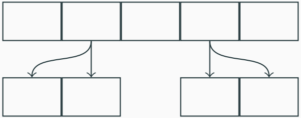
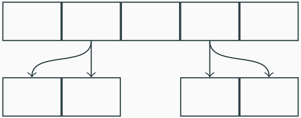


- Recurse!
- Take all items in any really large bucket, rehash them into subbuckets
- Might need to repeat

Handling Big Buckets



- Recurse!
- Take all items in any really large bucket, rehash them into subbuckets
- Might need to repeat
- This option can shave off small but significant running time

Heuristics in Practice

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- If there's enough structure, can use entirely different methods

Conclusion

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