Lecture 9: Bloom Filters and Cuckoo Filters

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October 7, 2025

Williams College

Admin

- Did everyone have a good mountain day?
- Assignment 3 out today, due next Thursday
- Cuckoo filters today; streaming Thursday
- Best plan (in my opinon)
  - Finish cuckoo filter coding part of assignment on Thursday in lab
  - Should be very doable if you *read through the lab and starter code beforehand*
  - Finish rest next Thursday
- No TA hours during reading period
- Assignment 4 next week based on lecture Thursday
- Midterm on Friday after Assignment 4; review the Tuesday before

Cuckoo Hashing Wrapup

Cuckoo Hashing [Pagh, Rodler 2005]

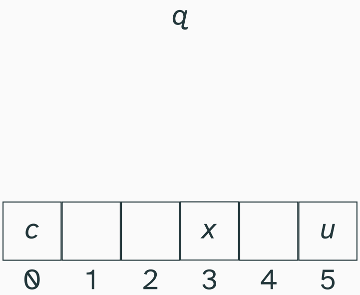
- A third method of resolving collisions
- Queries are $O(1)$ *worst case*
- Insert will still be $O(1)$ in expectation
- Comparison to linear probing and chaining?
  - Chaining: $O(1)$ worst case inserts; $O(1)$ expected queries. Not as good for query-heavy workloads!
  - Linear probing: more cache-efficient, but both inserts and queries are only $O(1)$ on average

Cuckoo Hashing Queries

c			x		u
0	1	2	3	4	5

- Query: Just check both slots

Cuckoo Hashing Queries



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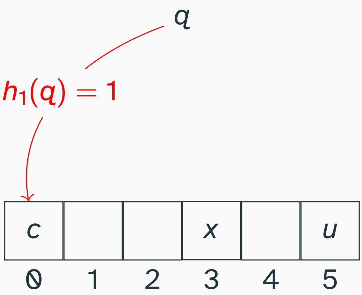

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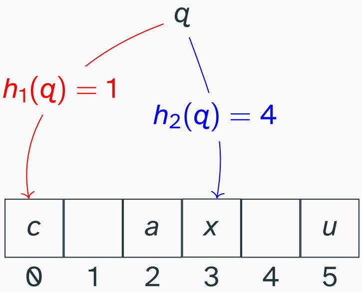

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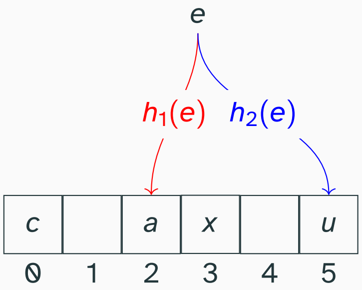
Cuckoo Hashing Inserts



- Let's say we want to insert a new item a. How can we do that?
- Easy case: if $h_1(a)$ or $h_2(a)$ is free, can just store a immediately.
- What do we do if both are full?
- Move one of the items in the way to its other slot!
- If there's an item THERE, recurse

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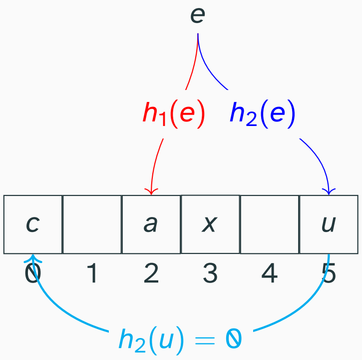

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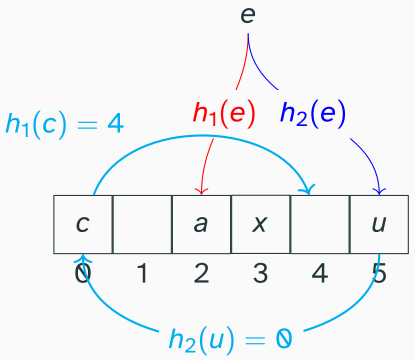

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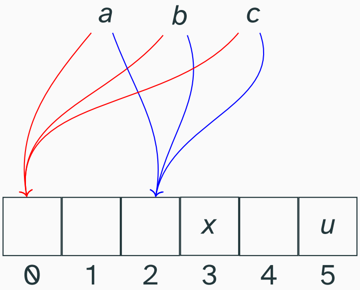
Does this *always* work?

- Recall our invariant: every item x is stored at $h_1(x)$ or $h_2(x)$
- Is there a simple example where this is impossible?
- One potential problem: three items x, y, and z all have the same two hashes. Can't maintain the invariant!
- If this occurs, our insert algorithm (so far) loops infinitely



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  - There exist slots s_1 and s_2 such that all of x, y, and z all hash to one of these two slots
  - For a given x, y, z, s_1, and s_2, how often does $h_1(x) = h_1(y) = h_1(z) = s_1$ and $h_2(x) = h_2(y) = h_2(z) = s_2$?
  - $(1/2n)^6$
  - There are $\binom{n}{3}\binom{2n}{2}$ choices of x, y, z, s_1, and s_2
  - So this happens with probability $\Theta(n^3n^2/n^6) = \Theta(1/n)$.

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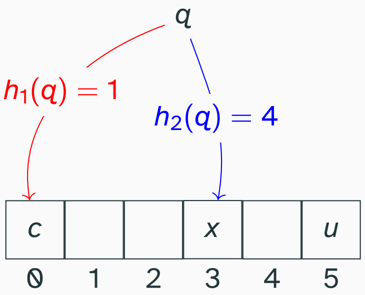
- Inserts loop very rarely if n is large (you probably will not see this happen on Assignment 3). Usually put in a maximum number of iterations, after which the insert fails, to prevent looping infinitely

Cuckoo Hashing Performance



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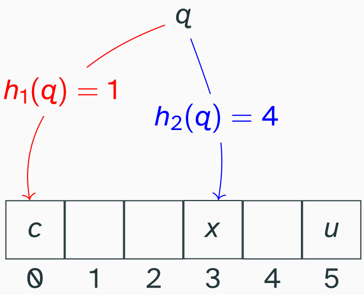

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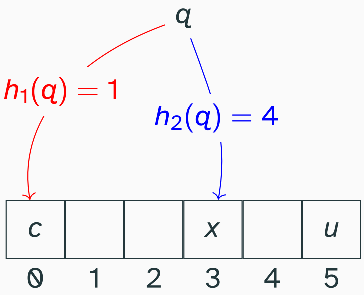

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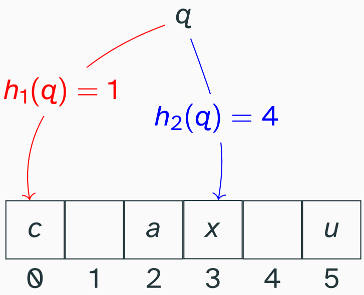

- Queries: $O(1)$ worst case operations
- Query cache performance?
  - Two cache misses per query. Is that good?
  - Kind of! Probably better than chaining. But linear probing has only $\approx$one cache miss on any query, so long as $\log n$ items fit in a cache line

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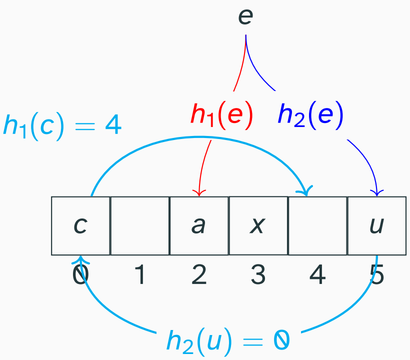

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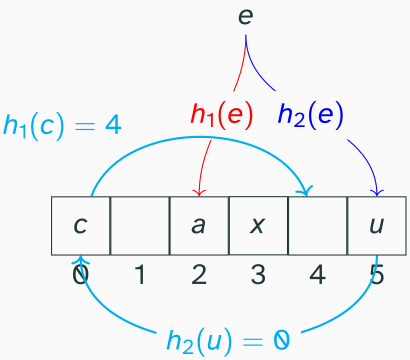

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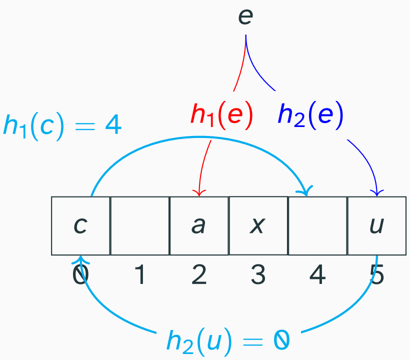

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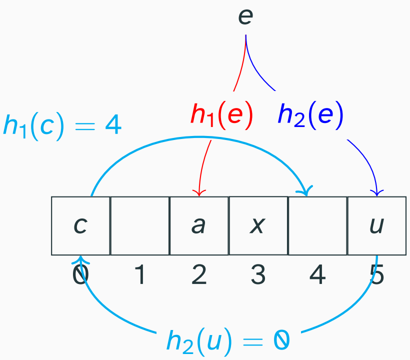

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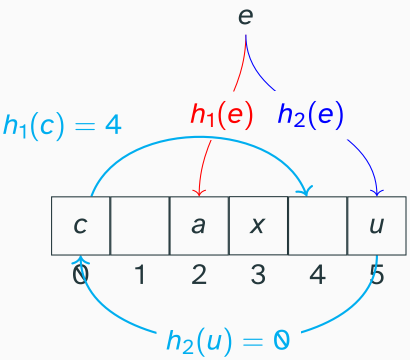

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  - Idea: half the slots are empty, so each time we go to a new slot, we should have a $\approx 1/2$ probability of being done
  - True analysis is nontrivial since these events are not independent

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  - In practice, inserts are a downside for cuckoo hashing due to poor constants
- Idea: cuckoo hashing does great on queries (though with potentially worse cache efficiency than linear probing), but pays for it with relatively expensive inserts

Filters: Goals for Today

What we want

- *Worst-case* compression

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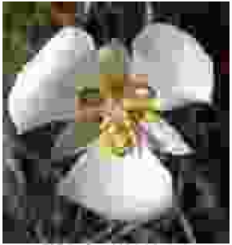
- Lossy compression with algorithmic guarantees

What we want


Text Caption

- *Worst-case* compression
- Lossy compression with algorithmic guarantees
- That is to say: we know what we're losing and what we're not

Filter

- Stores a set S of size n (basically: think of it as a (lossy) compressed dictionary)

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- All elements $x \in S$ and all queries must be from some universe U
- (Only need U to make sure that we can hash everything.)

Filter Guarantees

Guarantee (No False Negatives)

A filter is always correct when it returns that $q \notin S$.
Equivalently, if we query an item $q \in S$, then a filter will always correctly answer $q \in S$.

A filter *always* reports that every key in your dictionary exists. But it may (falsely) report that others exist as well

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Filter Guarantees

Guarantee (Bounded False Positive Rate)

A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

We want our filter to have a false positive rate $\varepsilon < 1$.

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The filters we will talk about today will work for *any* false positive rate ε, so long as $1/\varepsilon$ is a power of 2.[1]

So we can, if we want, guarantee a false positive rate of $1/2$, or $1/1024$—whatever is best for your use case.

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Guarantee 2 Sanity check

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- I store the entire set S using a standard dictionary (perhaps using a hash table). On a query q, I look it up and give the correct answer. This satisfies Guarantee 2 with $\varepsilon = 0$.

Filter example

Let's say we store all English words in the filter (S consists of all English words).

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- What happens when I query for the (non) word "fhqwhgads"?
  - With probability ε, the filter returns yes, $x \in S$
  - With probability $1 - \varepsilon$, the filter returns no, $x \notin S$

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    - Space is very small, but filter is very inaccurate!
- A filter generally requires $O(n \log 1/\varepsilon)$ bits of space.

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How can we interpret this?

- Plugging in numbers: if we have a cuckoo filter with $\varepsilon = 1/63$, the filter takes less than 1 byte of space per element being stored.
- Notice that this space does *not* depend on the size of the original elements. We can store very long strings and still require only one byte per string stored.

History and Discussion

Bloom filter

- Invented by Burton H. Bloom in 1970

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- Original publication only talked about good practical performance; theoretical analysis came later.

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Cuckoo filter

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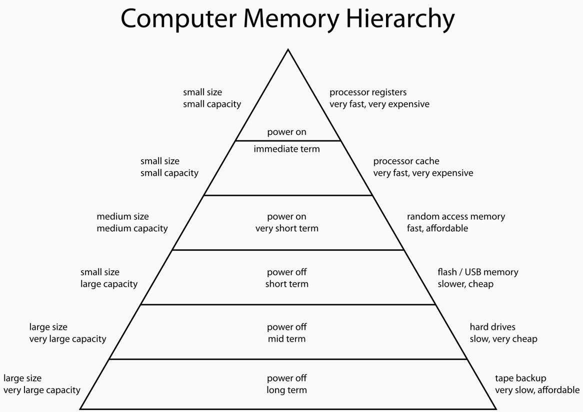
- Provides better space usage for small ε (i.e. when the compression is not too lossy)
- Requires fewer hashes; has better cache performance.

When should you use a filter?



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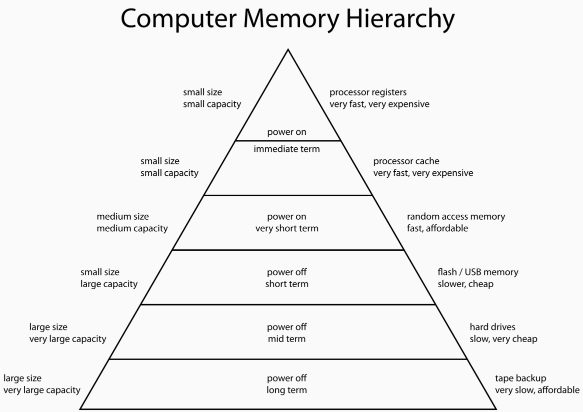

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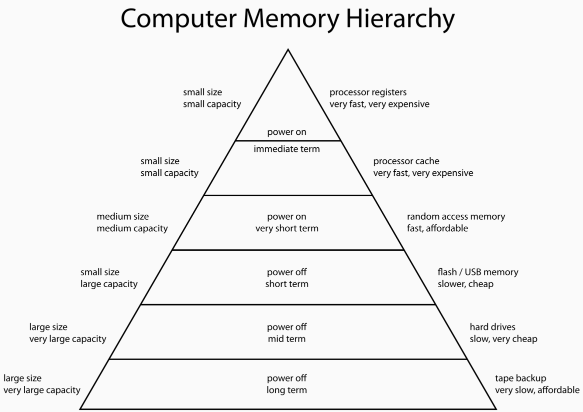

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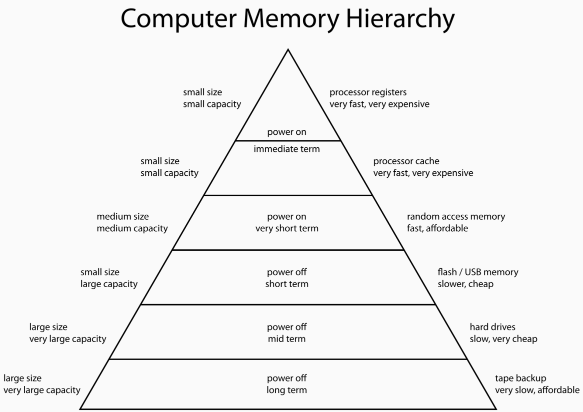

1st example: avoiding cache misses

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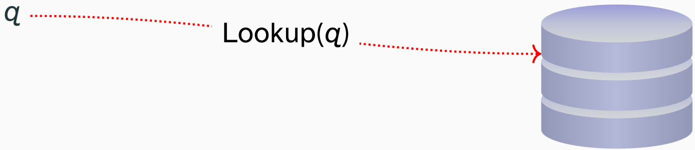
- Large enough that it doesn't fit in L3
  - Maybe it doesn't even fit in RAM
- Frequently query items not in the table

Common filter usage



Queries to the entire dataset are very expensive!

Queries are often “unnecessary”

Many workloads involve mostly “negative” queries: queries to keys not stored in the table. (query $q \notin S$)

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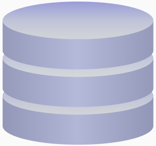
- Checking if key already exists before an insert (deduplication in general)
- Check for malicious URLs
- Table with many empty entries

Classic filter usage: succinct data structure that will allow us to “filter out” negative queries.

Common filter usage

Filters can be used to “filter out” negative membership queries, improving performance.

Common filter usage

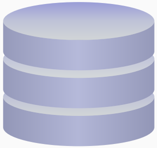


Filters are so small that they can fit in local memory.

Common filter usage

q

0	1	0	0	1	0	0	1



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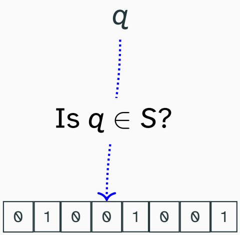
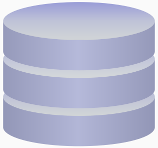
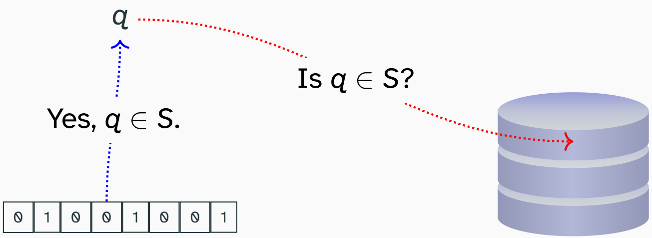




Fast in-memory query.

Common filter usage



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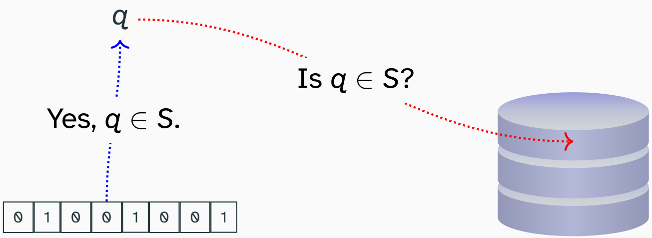

If filter reports $q \in S$, access the table.

If $q \notin S$ (false positive), still do an unnecessary access.

Common filter usage

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Always correct! Don't need to access table.

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- Greatly reduces number of remote accesses, thereby reducing time.

When should you use a filter?

2nd example: Approximately storing a set

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- Example: approximate spell checker

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- Using only a byte or so per item, can do almost as well as storing a full dictionary! (Roughly 98% accuracy.)

Bloom Filters

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- Assume $1/\varepsilon$ is a power of 2; round m up to the nearest integer

Building a Bloom Filter

- Begin with $A[i] = 0$ for all i. (Basically, just calloc the bit array.)
- Then add the items one at a time by setting *all* their slots to 1:

```
for each x in S:
    for i = 1 to k:
        A[h_i(x)] = 1
```

Building a Bloom Filter

0	0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7	8

Inserting two elements x and y into a Bloom filter with $\varepsilon = 1/8$. We have three hash functions, and (rounding up) the array is of length $m = 9$ bits.

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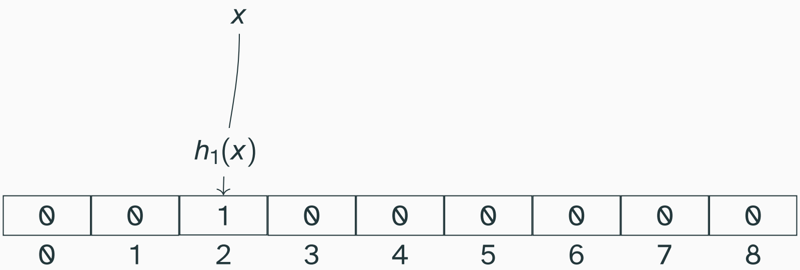

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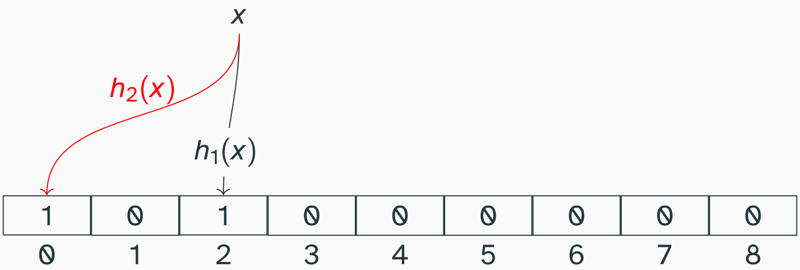

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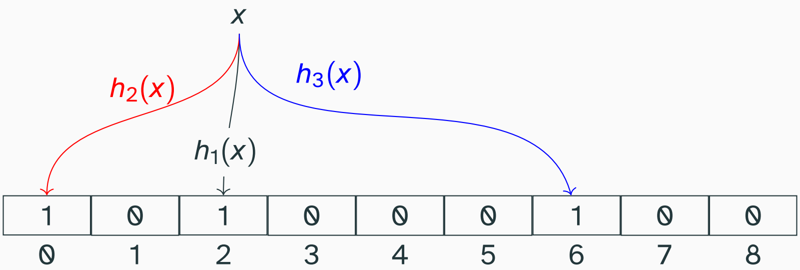

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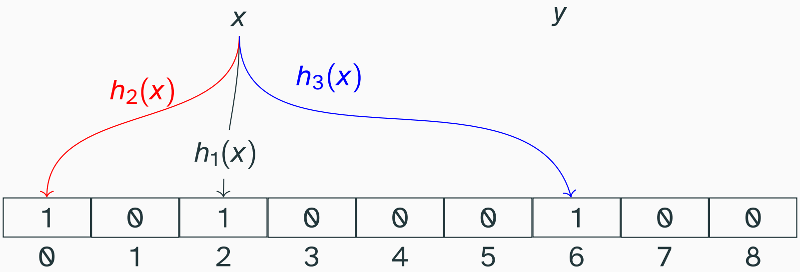

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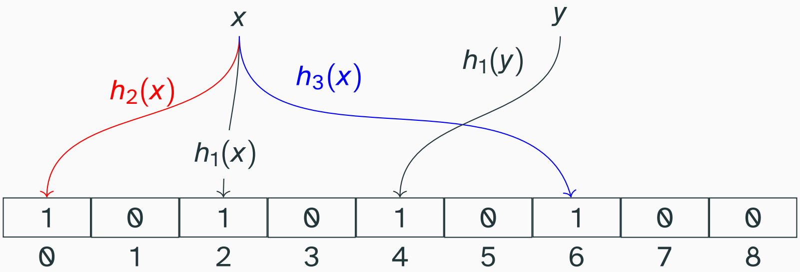

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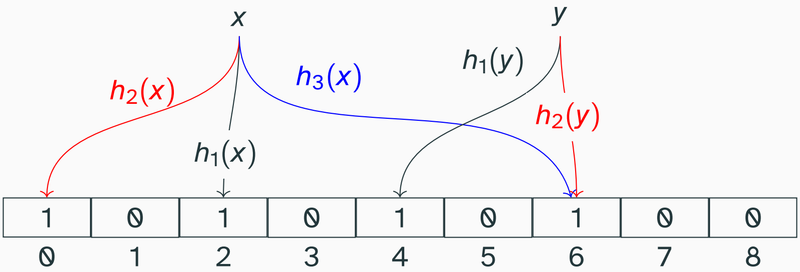

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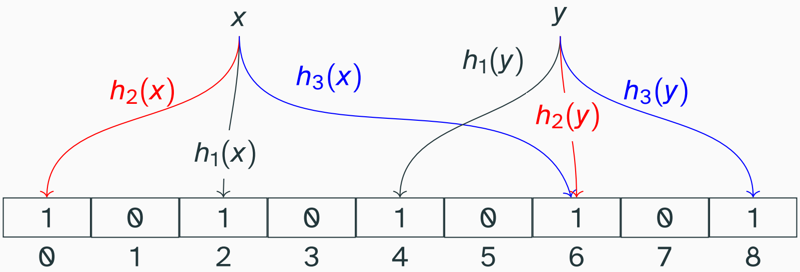

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A Bloom filter storing a set S using hashes $h_1, \ldots h_k$ satisfies $A[h_i(x)] = 1$ for all $x \in S$ and all $i \in \{1, \ldots, k\}$.

Querying a Bloom filter

On a query q, we check all the hash slots to see if any stores `0`:

```
for i = 1 to k:
    if A[h_i(q)] == 0:
        return false //q is not in S

// we have A[h_i(q)] = 1 for all h_i
return true //q is in S
```

Query example

q

1	0	1	0	1	0	1	0	1
0	1	2	3	4	5	6	7	8

An example query to an element not in the set; $k = 3$.

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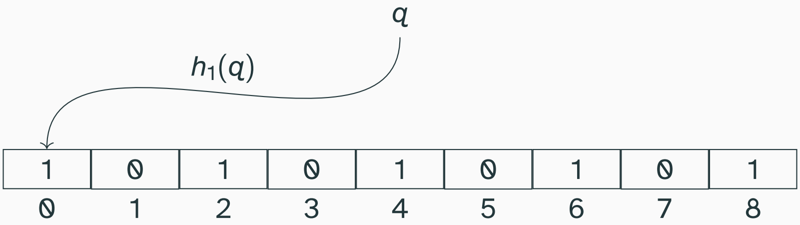

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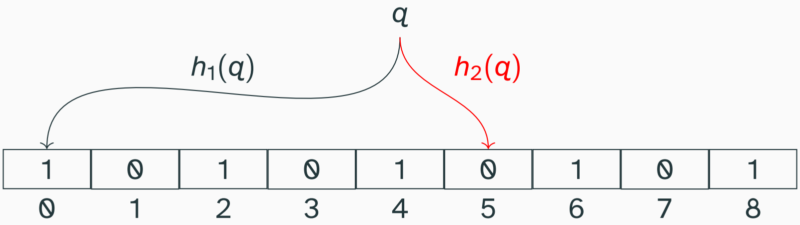

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q

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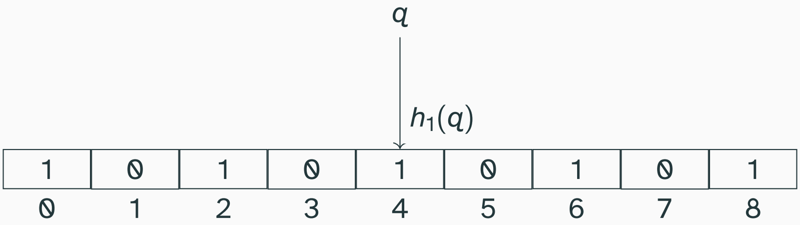

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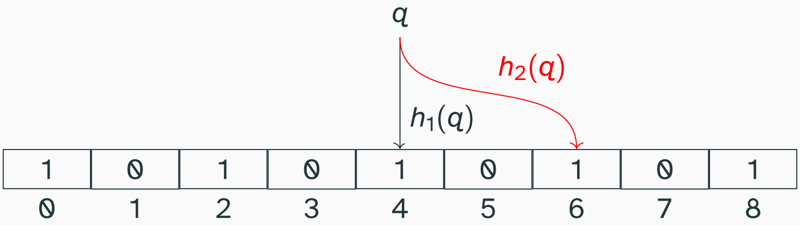

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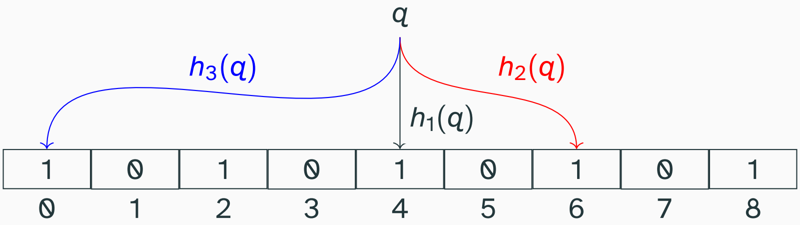

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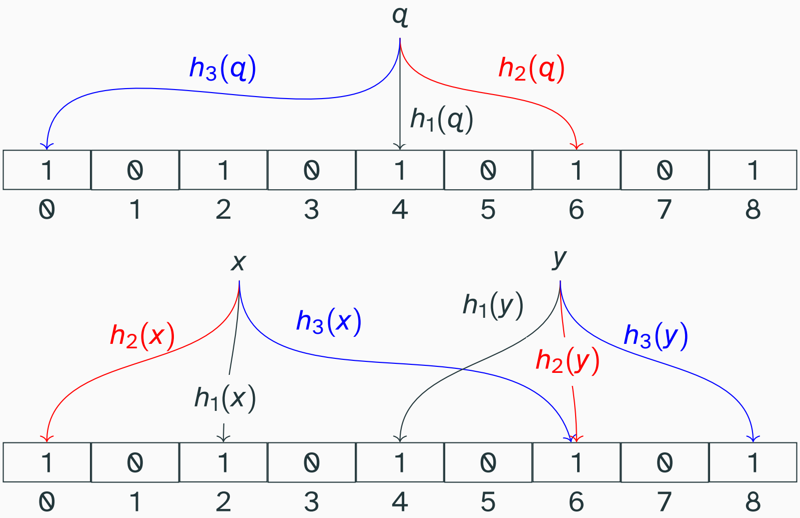

In pairs: is it possible to insert a new item into a Bloom filter?

Is it possible to delete an item?

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- Can we delete?
  - No. If we flip a bit from 1 to 0, it may cause a false negative, violating Guarantee 1.

Bloom filter analysis

- Assume our hashes h_i are perfectly uniform random: any $x \in U$ is mapped to any hash slot $s \in \{0, \ldots, m-1\}$ with probability $1/m$; independently of any other hash.

- Let's strategize: what about the Bloom filter can we use to prove that Guarantee 1 and Guarantee 2 hold?

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If we query an item $q \in S$, then a filter will always answer $q \in S$.

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- By the Bloom filter Invariant, if $q \in S$, then $A[h_i(q)] = 1$ for all $i \in \{1, \dots k\}$.
- This means that the query algorithm always returns "$q \in S$."

Guarantee 2 (False positive rate)

Guarantee (Bounded False Positive Rate)

A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

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- You're looking for "bilingual palindromes": strings whose reverse is a word in another language
- Most words are not bilingual palindromes, so a filter can significantly speed up queries

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- k hash functions denoted by $h_1, h_2, \ldots, h_k$ (k is a constant)
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- a hash table T of m slots, where each slot has room for $\log_2(1 + 1/\varepsilon)$ bits.

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- Also assume that $1/\varepsilon + 1$ is a power of 2, and m is a power of 2.

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Question: with this invariant, how can we query to avoid false negatives?

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- If we cuckoo more than $\log n$ elements, we rebuild the filter.

Cuckoo Filter Insert First Attempt

00	01	00	00	11	00	10	00
0	1	2	3	4	5	6	7

A cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Cuckoo Filter Insert First Attempt

x

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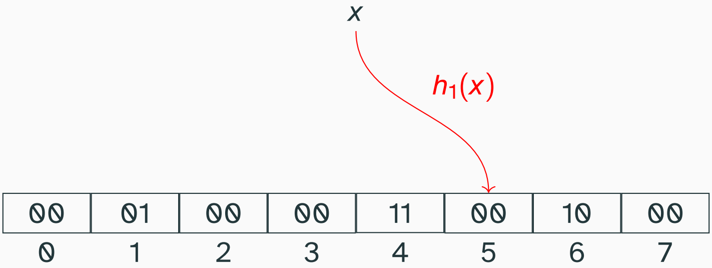

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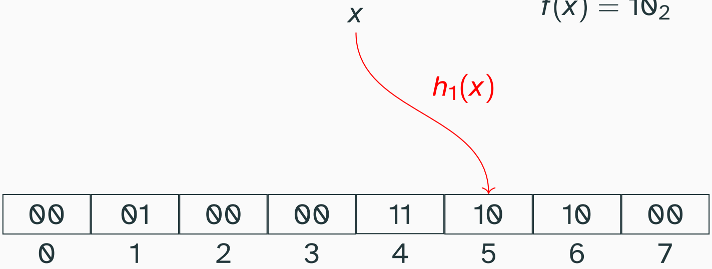
Cuckoo Filter Insert First Attempt

$f(x) = 10_2$



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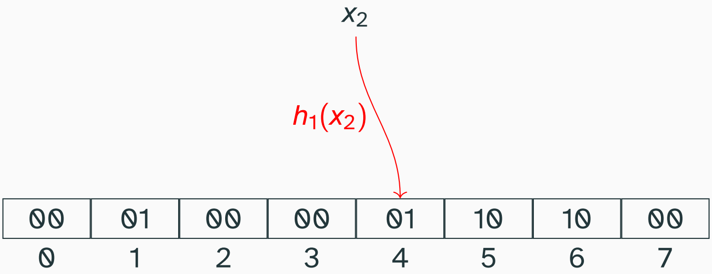
Cuckoo Filter Insert First Attempt

x_2

00	01	00	00	01	10	10	00
0	1	2	3	4	5	6	7

A cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Cuckoo Filter Insert First Attempt



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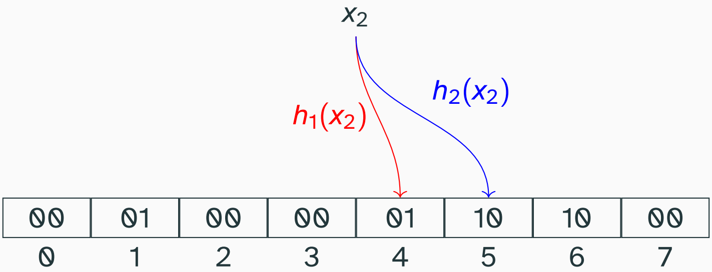

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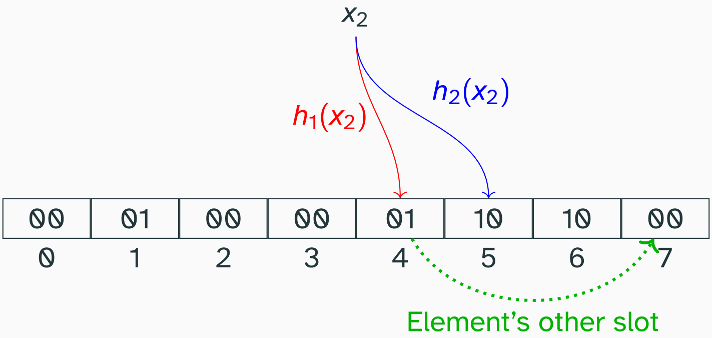

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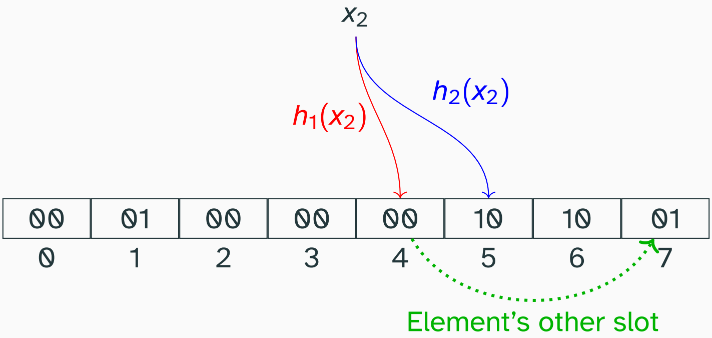

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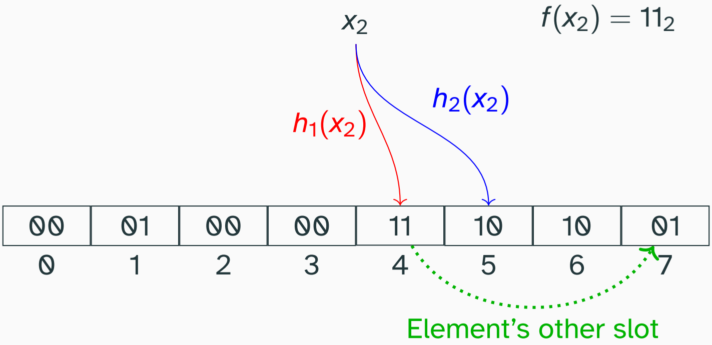

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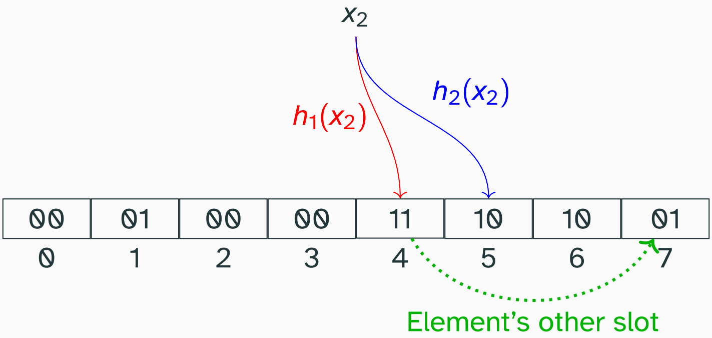

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Is our invariant maintained?

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- Calculate $h(\phi)$
- Its other slot is $s \wedge h(\phi)$. (This is XOR in C)
- If that other slot is empty we can store ϕ in it (woo)! Otherwise, take the fingerprint stored there and cuckoo it to its other slot.

Cuckoo Filter Insert Example With Partial-Key Cuckoo Hashing

00	01	00	00	11	00	10	00
0	1	2	3	4	5	6	7

A cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

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x

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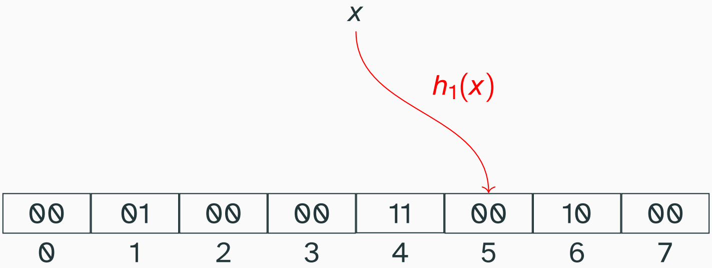

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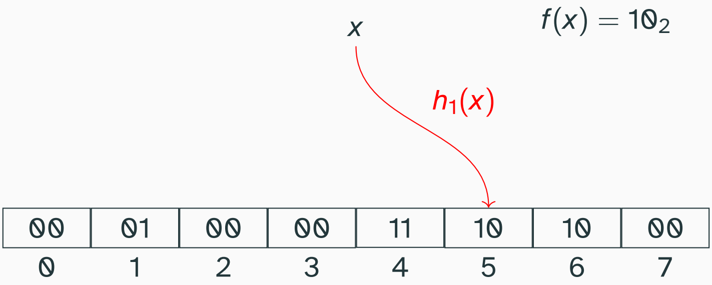

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Cuckoo Filter Insert Example 2

x_2

00	01	00	00	01	10	10	00
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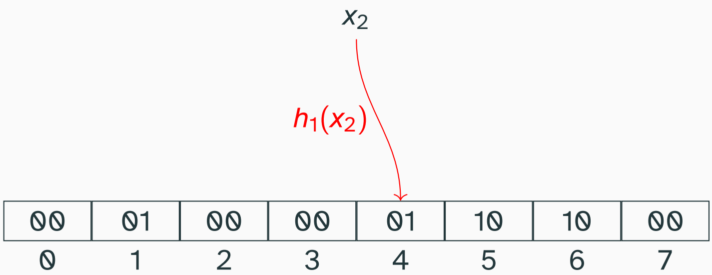

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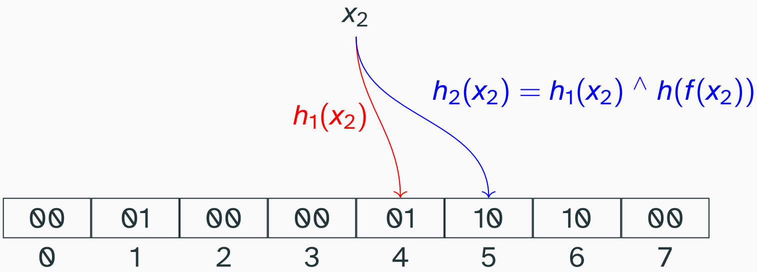

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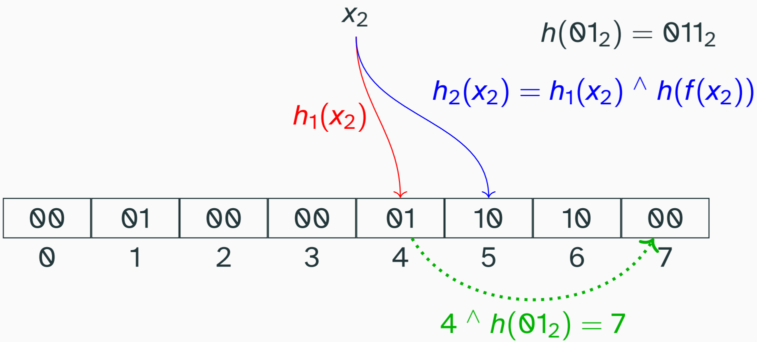

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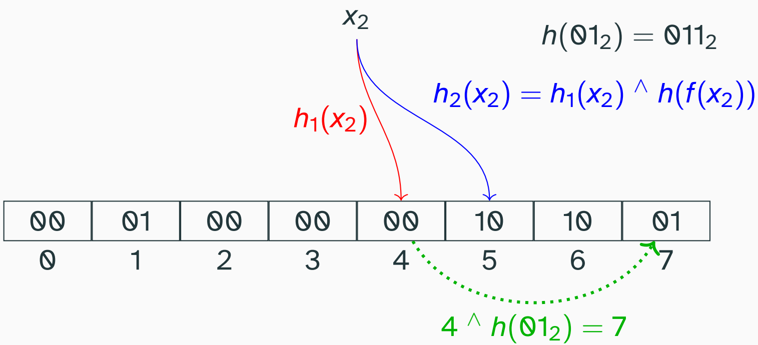

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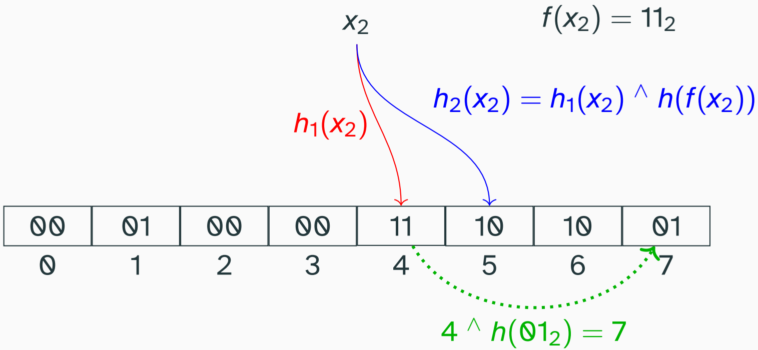

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Cuckoo Filter Invariant

Using partial-key cuckoo hashing with $k = 2$:

Invariant

For any $x \in S$, either slot $h_1(x)$ or $h_2(x) = h_1(x) \wedge h(f(x))$ stores the fingerprint $f(x)$.

For higher k:

Invariant

For every $x \in S$, there exists an $i \in \{1, \ldots, k\}$ such that $f(x)$ is stored in $T[h_i(x)]$.

Querying a Cuckoo Filter

To query an element *q*:

```
for i = 1 to k:
    if T[h_i(q)] = f(q):
        return true // q is in S
//did not find the fingerprint in any slot
return false // q is not in S
```

Querying a Cuckoo Filter: Example

00	01	00	00	01	00	10	11
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Querying a cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Querying a Cuckoo Filter: Example

q

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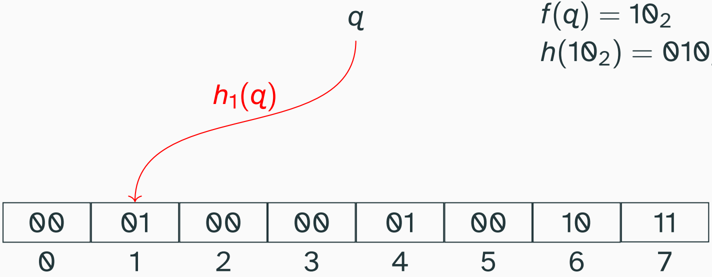

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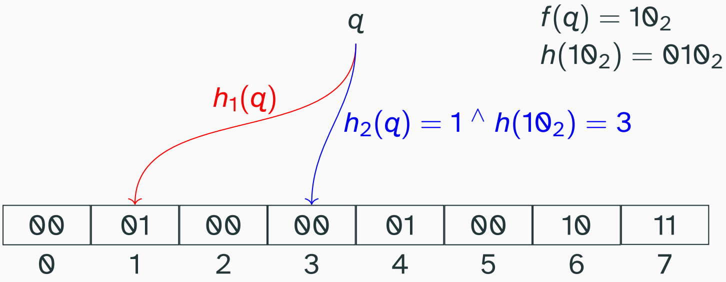

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q_2 $\qquad f(q_2) = 11_2$

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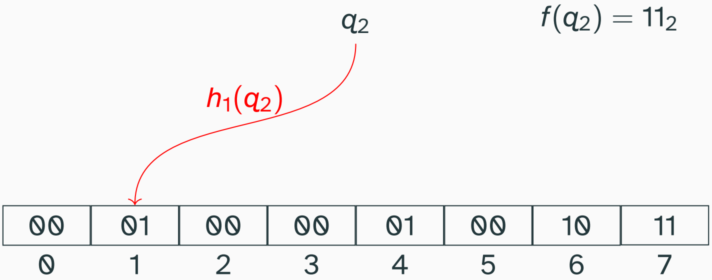

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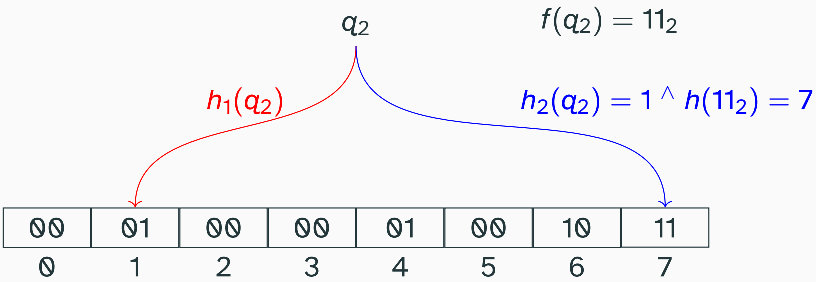

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- How about deletes?
  - Oftentimes yes—if you are careful! (Need to make sure we don’t delete another element’s fingerprint.)

Improved Cuckoo Filter Performance

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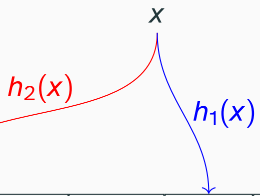
- I used a global variable to indicate what slot to cuckoo from; incrementing it each time.
- Then can set $m = 1.05n/4$, giving total space usage $1.05n \log_2(8/\varepsilon + 1) \approx 1.05n \log_2(1/\varepsilon) + 3.15n$.

Example



1000	1010	0000	0000	0101	0000	0100	0000
0000	0000	0000	0000	1001	0000	0110	0000
0000	0000	0000	0000	0010	0000	0101	0000
0000	0000	0000	0000	1001	0000	1111	0000
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A cuckoo filter with fingerprints of length 4, $k = 2$, and 4 slots per bin.

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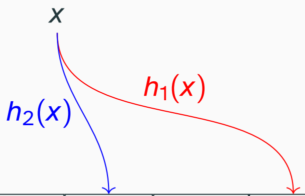

1000	1010	0000	0000	0101	0000	0100	0000
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- Good cache efficiency: only need to access the hash table 2 times, rather than $\log_2(1/\varepsilon)$.

Implementing Effective Hash Functions

Hashes we need

- h_1 which maps an arbitrary element (a string in Homework 3) to a slot in the hash table
- f which maps an arbitrary element (a string in Homework 3) to a number from 1 to 255 (we'll be doing 8-bit fingerprints)
- h which maps a fingerprint from 1 to 255 to a slot in the hash table

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- To calculate $h(i)$, for $i \in \{1, \ldots, 255\}$, just use `hashFingerprint`$[i - 1]$

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- Use mod to get them down to size

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uint32_t hash[4] = {0,0,0,0};
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- `hash` is the 128 bits of output

```
uint32_t position = hash[0] % numSlots;
uint32_t fingerprint = 1 + hash[1] % fingerprintMask;
```

Cuckoo Filter Analysis

Union Bound

Theorem

Let X and Y be random events. Then

$$\Pr(X \text{ or } Y) \leq \Pr(X) + \Pr(Y).$$

More generally, if $X_1, X_2, \ldots, X_k$ are any random events, then

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- Simple but useful tool in randomized algorithms
- *Always* works, even for events that are not independent
- Sometimes called “Boole’s inequality”

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- The probability that at least one student has ID 1 is

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This is messy! And it would be even worse if the IDs were not independent!

The union bound lets us avoid this work.

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- This gives us an upper bound of $10/100 = 10\%$.

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- We will analyze *without* partial-key cuckoo hashing (we'll assume independent h_1 and h_2)

First Guarantee: No False Negatives

Guarantee (No False Negatives)

A filter is always correct when it returns that $q \notin S$.
Equivalently, if we query an item $q \in S$*, then a filter will always correctly answer* $q \in S$.

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For every $x \in S$*, there exists an* $i \in \{1, \ldots, k\}$ *such that* $f(x)$ *is stored in* $T[h_i(x)]$.

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Invariant

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- We can see that the invariant means that there are no false negatives.

Second Guarantee: False Positive Rate

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A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

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- A query $q \notin S$ is a false positive if, for some h_i, $T[h_i(q)] = f(q)$.
- Let's examine each hash h_1 and h_2 individually.

Second Guarantee: False Positive Rate

- Let's start with h_1. What is the probability $T[h_1(q)]$ contains a fingerprint?

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- $1/2$, because we are storing n elements in $2n$ slots.
- If $T[h_1(q)]$ contains a fingerprint, the probability that $f(x) = f(q)$ is ε.
- Therefore, the probability that $T[h_1(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.

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- Same exact analysis: probability that $T[h_2(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.
- Are these events independent?
- No! If h_1 does not have a collision, we're slightly more likely to have an element collide under h_2

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- q is a false positive if either $T[h_1(q)]$ contains a fingerprint $f(x_1)$ such that $f(x_1) = f(q)$, or $T[h_2(q)]$ contains a fingerprint $f(x_2)$ such that $f(x_2) = f(q)$

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- By union bound, one or the other happens with probability at most $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

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- Answer: some of you might, but I'm guessing many of you would not. You're just going to lose \$1000.

- But expectation is good! You expect to win \$9000.

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- Answer: $1/2^k$
- Quicksort has expected runtime $O(n \log n)$. What is the probability that the running time is more than $O(n \log n)$?
- Answer: $O(1/n)$ (this is why quicksort is not worse than merge sort even though it can be $\Theta(n^2)$: you'll never see the worst case if n is at all large)

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- With high probability is always with respect to a variable. Assume that it's with respect to n unless stated otherwise.

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- So I need $1/2^k = O(1/n)$. Solving, $k = \Theta(\log n)$.
- This is (a simplified version of) the analysis leading to the $O(\log n)$ worst case bounds on the last slide

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- Expectation states how well the algorithm does on average. Could be much better or worse sometimes!
- "With high probability" gives a guarantee that will almost always be met: if n is large it becomes vanishingly unlikely that the bound will be violated.
- Largely fulfills the promise of classic worst-case algorithm analysis, but applied to randomized algorithms