Applied Algorithms Lec 8: Probability

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Williams College

Probability

Probability Takeaways

What I want you to know by the end of this section of the course:

1. Definition of probability/basic calculations

2. Determine if two events are independent

3. Calculate expectation

4. Linearity of expectation

5. Difference between “concentration bounds” vs expected performance

Definition of Probability

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- Formal definition probability generally applies weights to the events (in which case the definition of probability is the weight of outcomes in the event, divided by total weight of all events). We will usually have equal-weight events.

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- $\binom{10}{5}/2^{10}$ Which is $\frac{10!}{5!5!2^{10}} \approx .246$
- (In a couple lectures we'll see tools to estimate probabilities without lots of large numbers all over the place. For now, use wolfram alpha or whatever)

Probability Calculation: Final Example

- What is the probability of getting a straight flush in poker?
- Simple version: you are dealt 5 cards. You want to know the probability that they all have the same suit, and they are increasing numbers, like 3-4-5-6-7. (Ace is high only)
- 52 cards: 4 suits; 13 cards of each suit
- Let's do the anlaysis out on the board
- Reminder:

$$\Pr[\text{Event } E] = \frac{\text{\# outcomes in the event}}{\text{Total \# of outcomes}}$$

  - Number of possible hands is $\binom{52}{5}$
  - Number of straights is $4 * 9$ (4 suits; straight starts with 2-9)
  - Probability is $36/\binom{52}{5} \approx 0.0000138517$

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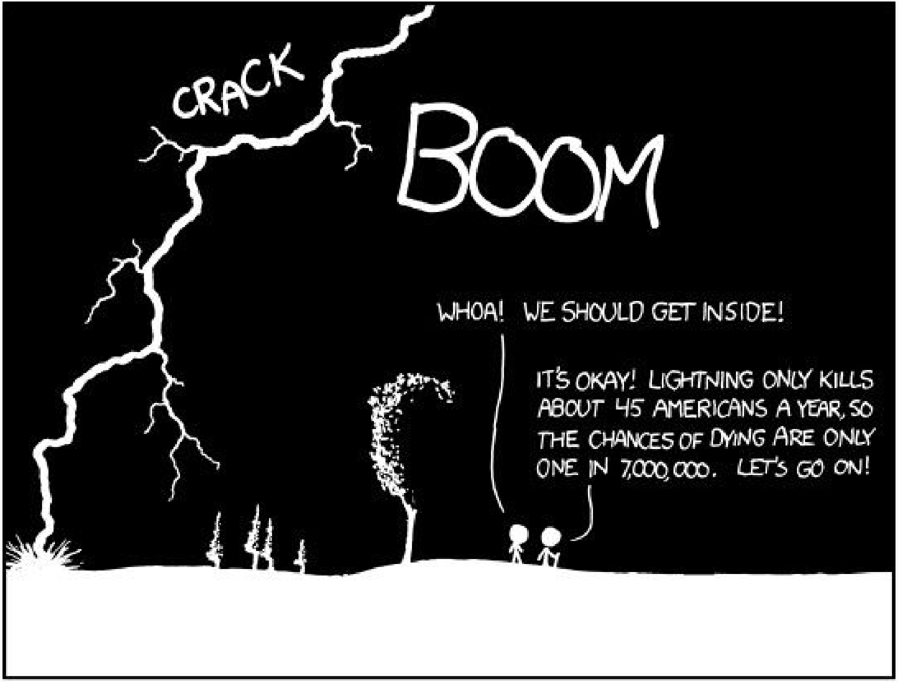
- How many outcomes are there for the fourth card? How many of them are a club?
- 49 outcomes. 10 of them are clubs. Probability: $10/49$.



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Conditional Probability Second Example

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- Probability $1/2$

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- Rephrasing the question: Let's say I have two children. They are not both girls. What is the probability that they are both boys?
- Yes, this kind of a wording trick. But similar issues can arise when analyzing algorithms.

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    - Knowing the outcome of one does not affect the probability of the other!
- I'll generally ask you if things are independent *intuitively* rather than asking for a proof. But, let's look at a couple classic examples of independence and how this definition works with them.

Proving Independence

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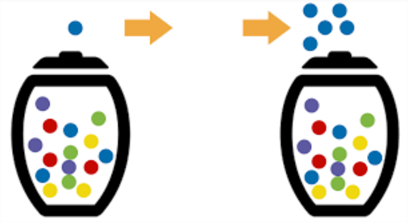
$$\Pr[E_2 \mid E_1] = 1/2$$

Again: we'll normally be looking at this intuitively.

Independence: Examples

- Let’s say I have a bag of balls, half of which are black, and half of which are white. I take a ball out of the bag and look at what color it is. Then I take another ball out of the bag and look at what color it is.
  - Event 1: The first ball is white.
  - Event 2: The second ball is black
  - Are these events independent?



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- No! If the first ball is white, there will be more black balls then white balls remaining in the bag for the next draw.

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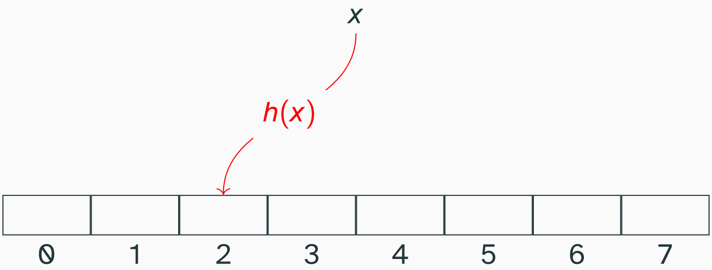

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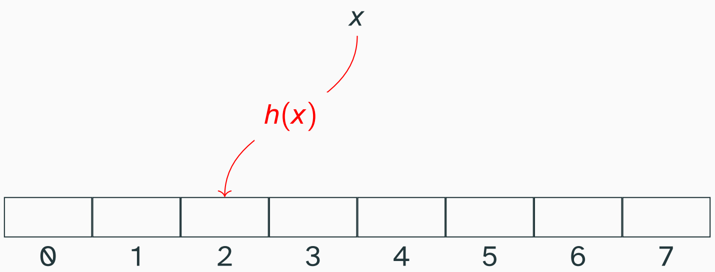

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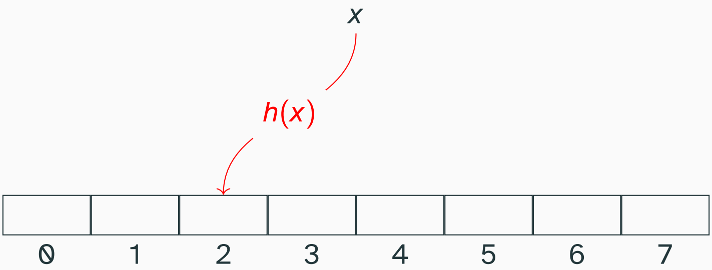

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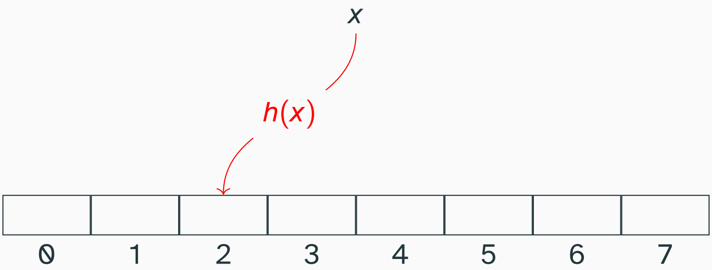

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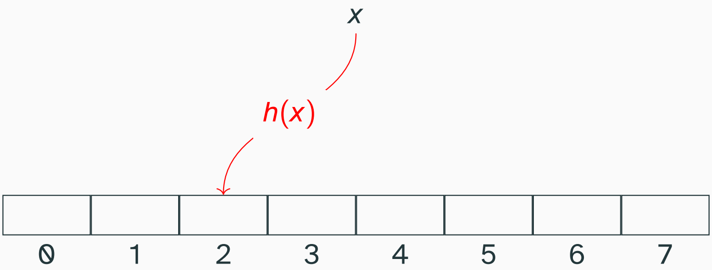

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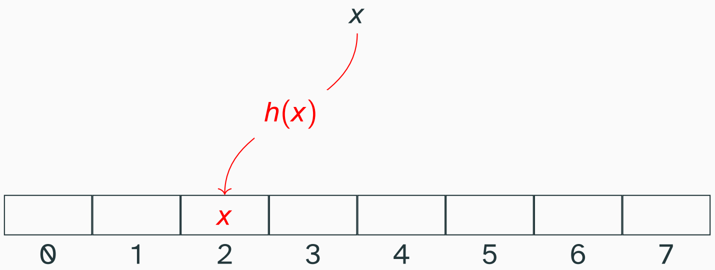

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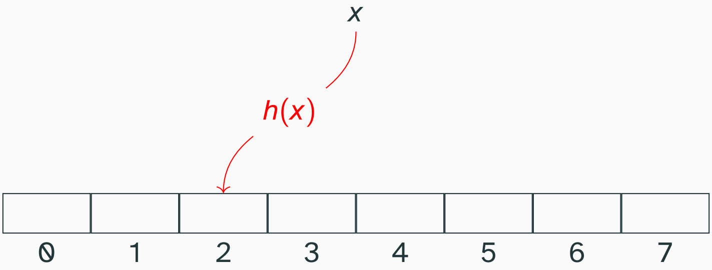

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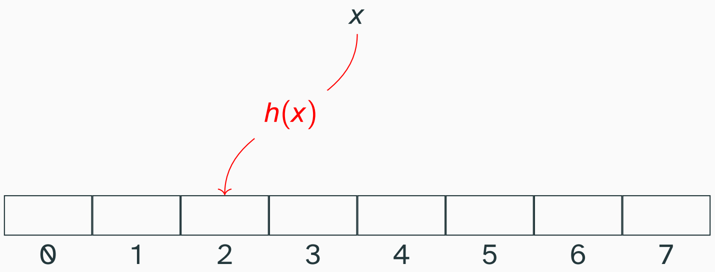

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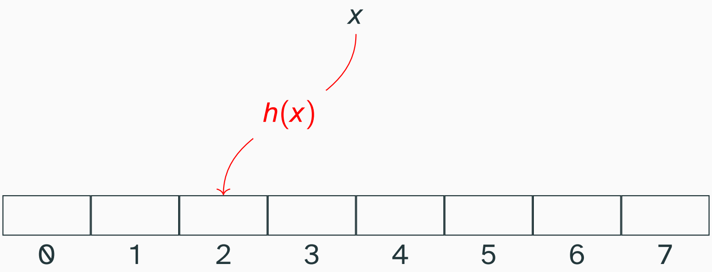

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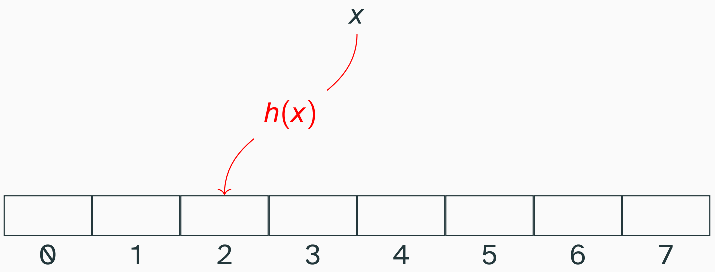

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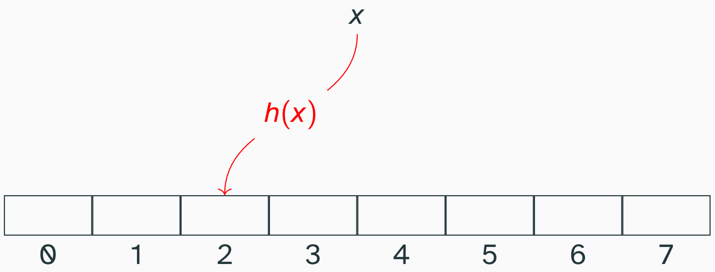

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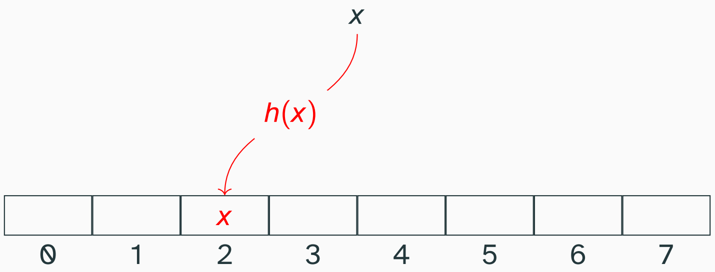

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- Yes! We assumed that the hashes of distinct elements are independent. Each event happens with probability $1/m$, independently

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- No! Since the first bucket contains no elements, the remaining elements are *slightly* more likely to hash to the second bucket
- For intuition: let's say there are only two slots in the hash table. Then these are not independent at all—in fact, if the first occurs, the second *cannot* occur

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- What is the probability of flipping 10 heads in a row when flipping a fair coin?
  - All 10 are independent, so $1/2^{10}$.

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- OK, that's a formula for the probability, but it doesn't tell me much. Is this large or small? How does it change?

Useful formulas for probability ($e = 2.71\ldots$)

Two useful approximations for simplifying exponents (presented as inequalities, but really quite tight even for moderate n):

Lemma

$$(1 + 1/n)^n \leq e \qquad\qquad (1 - 1/n)^n \leq 1/e$$

Example: $(1.1)^{10} = 2.593\ldots$

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With probability we often use choose (a.k.a. binomial) notation, but it's similarly inscrutible. These inequalities can help approximate it:

Lemma

$$\left(\frac{x}{y}\right)^y \leq \binom{x}{y} \leq \left(\frac{ex}{y}\right)^y$$

Example: $\binom{n}{10} = \Theta(n^{10})$

Why independence is useful (simplifying from before)

- Let's say you're in a class of n students. Every day the professor asks a student to explain the previous night's reading (the student is chosen by rolling an n-sided die). What is the probability that you *won't* be chosen after all k lectures in the course?
- Probability (not being chosen on one day) is $(1 - 1/n)$
- Probability (not being chosen after k days) is $(1 - 1/n)^k$
- Can simplify (assuming n is large):

$$(1 - 1/n)^k = ((1 - 1/n)^n)^{k/n} \approx e^{k/n}.$$

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- (So: one more probability definition, then we can talk about algorithmic performance)

Random Variable and Expectation

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- S is a *random variable.* Like normal variables, it is used to represent a value. Here, the value is the outcome of a random process.

Random Variable

- Let's say I draw four cards from a deck of cards. Let S be a random variable indicating the number of clubs I draw.

- One more comment about S. Since each card is a club with probability (about) $1/4$, and we draw 4 cards, it seems like S should generally be around 1. Can we formalize this intuition?

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- Another example: quicksort is $O(n^2)$ in the worst case, but if you pick pivots randomly, it is $O(n \log n)$ in expectation

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- It is a weighted average of the outcomes: the outcome is i, and it is weighted by the probability that i occurs.

Expectation example

Let's say I roll a 20-sided die, and I give you money equal to the number that shows up on top. I charge $10 to play this game. Should you play it?

Split into pairs and try to calculate the expected outcome.

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- $\mathrm{E}[X] = \frac{20 \cdot 21}{2 \cdot 20} = 10.5$
- So you'll win $.50 on average; you should probably play the game

Linearity of Expectation

- Consider a random variable that can be represented as the sum of other random variables (we'll see an example in a moment). Linearity of expectation means that if I sum the expectations of the parts, I get the expectation of the whole

Theorem (Linearity of Expectation)

For any random variable $X = X_1 + X_2 + \ldots + X_n$,

$$E[X] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

True even if the X_i are not independent!!!

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- Let's say I flip a coin 100 times. How many heads will I see on average?
- Let's figure this out on the board using linearity of expectation
- X = number of heads I see in 100 flips.
- $$X_i = \begin{cases} 1 & \text{if the } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$
- So then $X = X_1 + X_2 + \ldots X_{100}$
- We can see that $\mathrm{E}[X_i] = 1/2$.
- $\mathrm{E}[X] = 50$ by linearity of expectation. This would have been very difficult to calculate directly using the definition!

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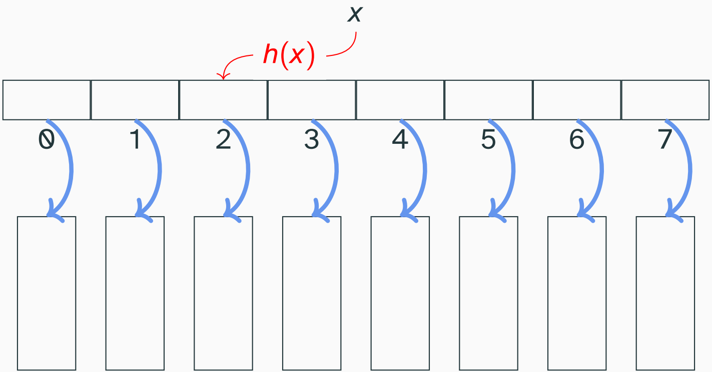
- Downside: expectation only discusses average (we'll come back to this)

Classic Hash Tables

Collisions

- (We discussed last class)
- Several items might hash to the same location. How can we resolve this?
  - Chaining
  - Linear Probing

Chaining



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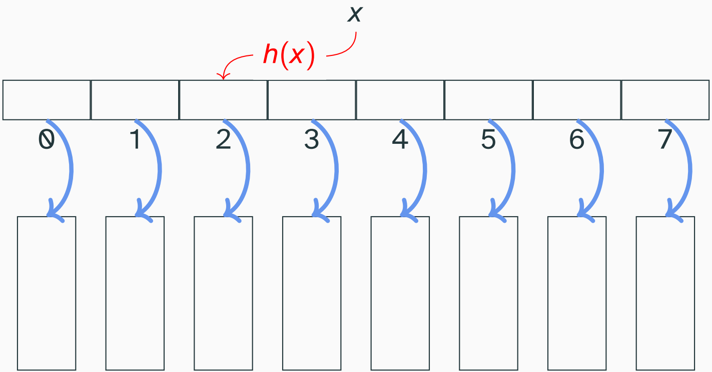

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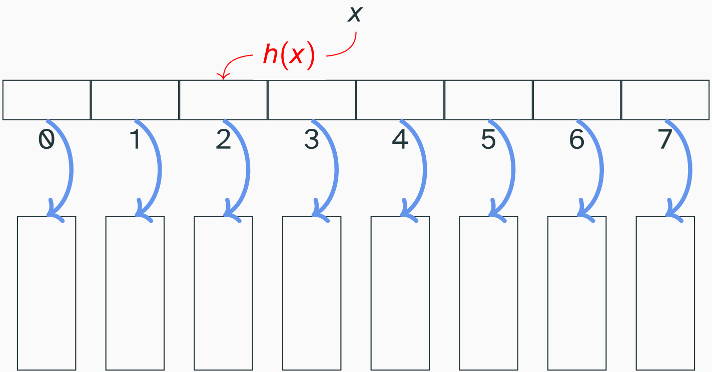

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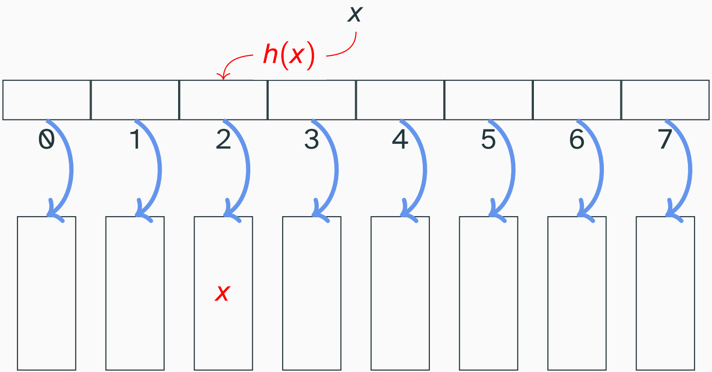

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- Chaining has $O(1)$ expected query time!

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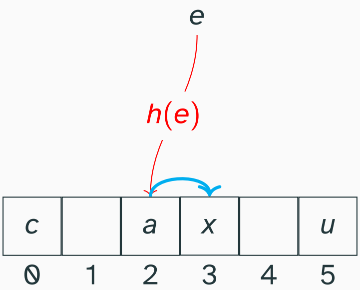

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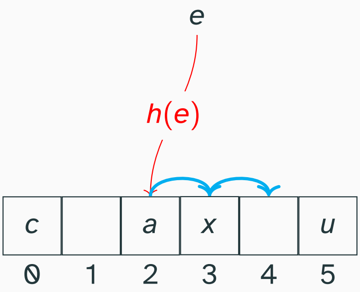

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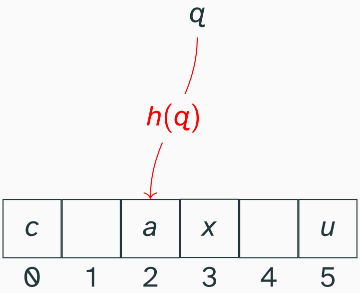
- Answer: look in slot $h(q)$. If it has q we are done; if it is empty we are done. Otherwise, continue to slot $h(q) + 1$



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- Disadvantages?
  - Performance is terrible if c is close to 1—that is to say, if A is nearly full
- This is probably the most common choice in practice: it's simple and cache-efficient

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- But most hash tables spend *more* time inserting then query
- It would be really nice to get the reverse: $O(1)$ expected insert time, $O(1)$ worst-case query time

Cuckoo Hashing [Pagh, Rodler 2005]

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  - Chaining: $O(1)$ worst case inserts; $O(1)$ expected queries. Not as good for query-heavy workloads!
  - Linear probing: more cache-efficient, but both inserts and queries are only $O(1)$ on average

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  - Check both hash slots. Immediately get $O(1)$ time

Cuckoo Hashing Inserts

c			x		u
0	1	2	3	4	5

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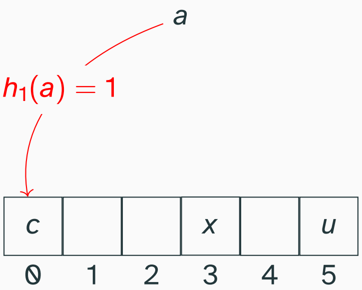

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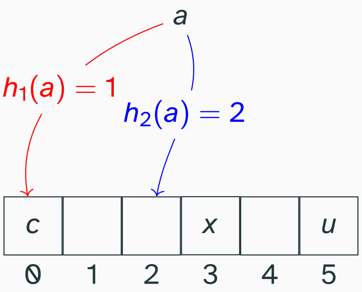

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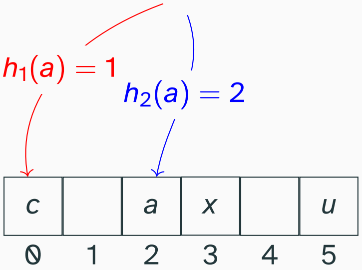

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c		a	x		u
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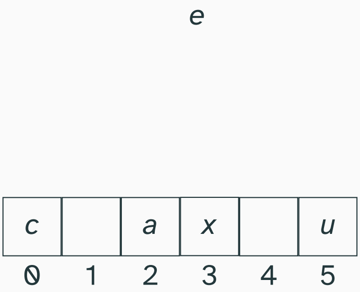

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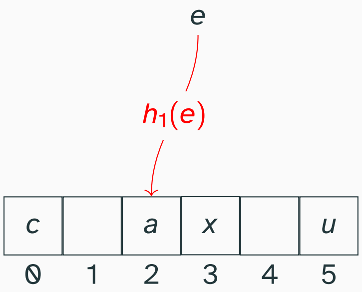

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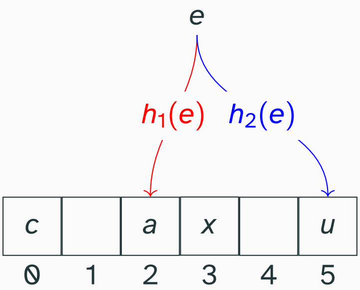

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Cuckooing!

Cuckoos kick other birds' eggs out of the nest, replacing them with their own.

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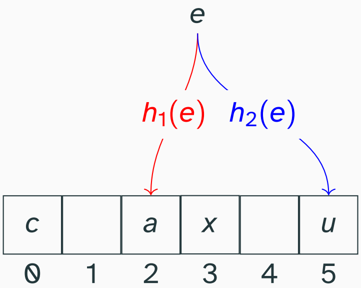

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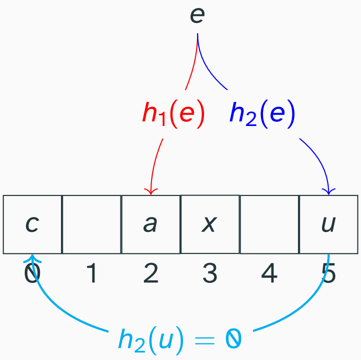

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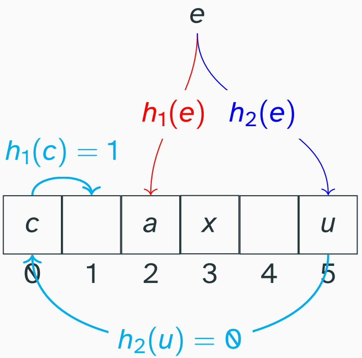

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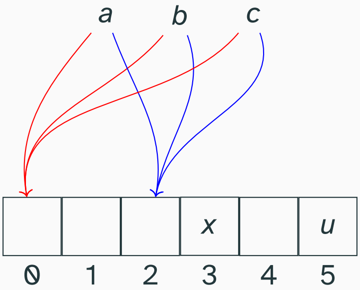

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    - So this happens with probability $\Theta(n^3 n^2/n^6) = \Theta(1/n)$.

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- Inserts loop very rarely if n is large (you probably will not see this happen on Assignment 3). Usually put in a maximum number of iterations, after which the insert fails, to prevent looping infinitely

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  - (Analysis is nontrivial since these are not truly independent, and we need to carefully avoid cases with an infinite loop)

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- Idea: cuckoo hashing does great on queries (though with potentially worse cache efficiency than linear probing), but pays for it with expensive inserts

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- Answer: some of you might, but I'm guessing many of you would not. You're just going to lose $1000.

- But expectation is good! You expect to win $9000.

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- Answer: $O(1/n)$ (this is why quicksort is not worse than merge sort even though it can be $\Theta(n^2)$: you'll never see the worst case if n is at all large)

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- With high probability is always with respect to a variable. Assume that it's with respect to n unless stated otherwise.

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- This is (a simplified version of) the analysis leading to the $O(\log n)$ worst case bounds on the last slide

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- Largely fulfills the promise of classic worst-case algorithm analysis, but applied to randomized algorithms