# Applied Algorithms Lec 5: Hirshberg's Algorithm

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Williams College

# Admin

- Cool talk right after class! On “practical” algorithms
- Assignment 1 handed in
- Assignment 2 out later today. Get started early!
- Recursive algorithm—so getting (my) help with debugging will be particularly useful

# Plan for today

- Topics for Assignment 2
- More external memory at the end if we have time

# Assignment 2: Hirschberg's Algorithm

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- In Homework 2, we're going to do the opposite: we're going to show how a space-efficient approach can actually result in smaller wall clock time
- True even though the space-efficient approach does extra computations!

# Edit Distance

- Minimum number of inserts/deletes/replaces to get from one string to another
- Useful in comp bio. Classic dynamic programming solution.

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OCCURRENCE

Replace E with A

OCURRENCE

OCURRANCE

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- If the last characters of $X$ and $Y$ match, what is $ED(X, Y)$?
  - If $X'$ and $Y'$ are $X$ and $Y$ respectively with the last character removed, then $ED(X, Y) = ED(X', Y')$

OCCURRA**N**

OCCURRE**N**

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  - **Delete:** $1 + ED(X, Y')$ (delete the last character of $Y$; match the rest to $X$)

OCCURR**A**

OCCURR**E**

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- Let $m = |X|$, $n = |Y|$.
- Build an $n + 1 \times m + 1$ table
  - $(+1$s are so we can have 0-length entries)
- Fill out the table row-by-row using our recursive method (doing lookups instead of recursive calls)

# Example DP execution

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- $O(mn)$ time (to fill out a table entry just need to look in three other table slots)
- $O(mn)$ space

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- Edit distance is an important problem. Can we do better than quadratic time?
- Probably not by more than log factors
- [Backurs Indyk 2014]: if you can solve edit distance in less than $O(nm)$ time, you can solve 3SAT in less than $2^n$ time

**Edit Distance Cannot Be Computed in Strongly Subquadratic Time (unless SETH is false)**

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**ABSTRACT**

The edit distance (a.k.a. the Levenshtein distance) between

with many applications in computational biology, natural language processing and information theory. The problem of computing the edit distance between two strings is a classical

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- Optimal # cache misses required to fill out that table

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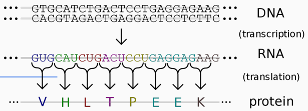
- What is the cache efficiency of this algorithm if $3n + m \leq M$?
- $O(\frac{n+m}{B})$: the only cache misses are from reading in the strings!
- WAY better than $O(\frac{mn}{B})$!

# Summary

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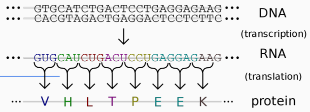


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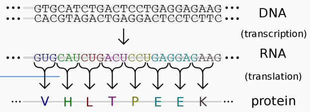


- Classic edit distance: $O(\frac{mn}{B})$ cache misses
- Improving space usage: if $3n + m \leq M$, then only $O(\frac{n+m}{B})$ cache misses
- Improve by a factor of $\min\{n, m\}$ to compute edit distance of two strings that fit in cache

# Takeaway: Improved Space Can Imply Improved Cache Efficiency

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- Actually not so bad: follow the path back!

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- How can we tell where each entry came from?

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- Redo same min computation from the normal dynamic program. (Break ties arbitrarily—for now.)

# Recovering the edits

Match
Match
**Delete**
Match
Match
Match
**Replace**
Match
Match
Match

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- Once you have the path back, can essentially read back the edits: a diagonal is a match or replace; right is a delete; down is an insert. (This is if we're putting the target string vertically—if $Y$ is being edited to become $X$, then $X$ is vertical.)

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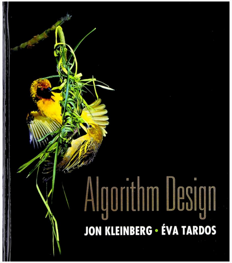
- A note on space vs time:
  - This problem was originally looked at in 1975 with the goal of limiting space to *fit the problem* on computers at that time
  - Now it's still used, but the goal is to fit the problem in cache

# Intro to Hirshberg's Original Paper

## Introduction

The problem of finding a longest common subsequence of two strings has been solved in quadratic time and space [1, 3]. For strings of length 1,000 (assuming coefficients of 1 microsecond and 1 byte) the solution would require $10^6$ microseconds (one second) and $10^6$ bytes (1000K bytes). The former is easily accommodated, the latter is not so easily obtainable. If the strings were of length 10,000, the problem might not be solvable in main memory for lack of space.

# Answer: Hirschberg’s algorithm!

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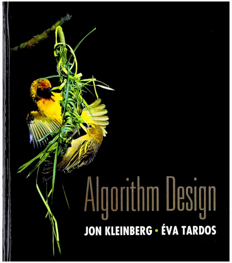


- Recursive approach that extends the dynamic program to make it space-efficient

- Can find in Kleinberg-Tardos (algorithms) textbook (woo). I'll email you the relevant pages. I also posted the original paper (a tad old but still a reasonable resource).

# (Slightly odd) Thought question

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- Can I recover just ONE piece of the optimal path?
- Specifically: find one square in the middle row that is on the optimal path?

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# Structural Lemma

**Lemma**

*Let's say that $X$ and $Y$ have edit distance $k$. Divide $X$ into two halves $X_1$ and $X_2$. Then there is some way to partition $Y$ into two parts $Y_1$ and $Y_2$ such that $ED(X_1, Y_1) + ED(X_2, Y_2) = k$.*

For example:

ADVICE and VINCENT have edit distance 5.

What parts of VINCENT match up with ADV? ICE?

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ED(ADV, V) = 2

ED(ICE, INCENT) = 3

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Proof idea: there is some optimal sequence of edits applied to $Y$ that obtain $X$. Let's apply those edits left to right. As we apply those edits, more and more of $Y$ will match $X$ (let's do an example with ADVICE and VINCENT on the board).

At some point, the beginning of $Y$ will match the first half of $X$ (that is to say: will match $X_1$). We can take that as $Y_1$, and the remainder of $Y$ as $Y_2$.

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Note: I am not showing you this lemma just to be formal. This is a useful reference for when you're coding so that you know *exactly* how subproblems fit together. Perhaps most importantly: $Y_1$ and $Y_2$ do not overlap; nor do $X_1$ and $X_2$.

**$Y_1$ and $Y_2$ do not overlap; nor do $X_1$ and $X_2$**

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**(Check this when you are coding! It will save you time. ☺)**

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- Remember: our goal is to find where the optimal sequence crosses the middle row of the table.

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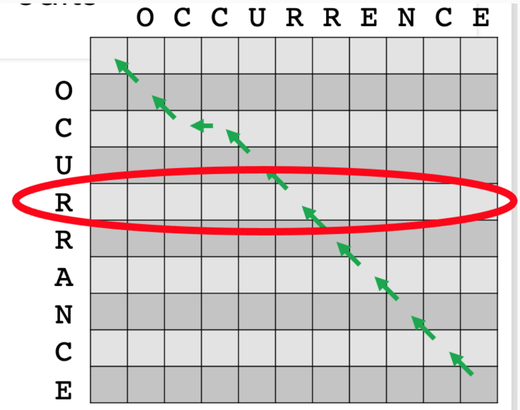
- By the above lemma, we can optimally edit $X$ into $Y$ by: first editing $X_1$ into $Y_1$, and then editing $X_2$ into $Y_2$

# Using the Structual Lemma



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- Kind of like $T(X) = T(X/2) + O(X)$

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- Let's calculate them separately: let's calculate $ED(X_1, Y_1)$ for all $Y_1$, and $ED(X_2, Y_2)$ for all $Y_2$.

# Calculating $ED(X_1, Y_1)$ for all $Y_1$

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|---|---|---|---|---|---|---|---|---|---|---|---|
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| O | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| C | 2 | 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| U | 3 | 2 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| R | 4 | 3 | 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| R | 5 | 4 | 3 | 3 | 3 | 2 | 1 | 2 | 3 | 4 | 5 |
| A | 6 | 5 | 4 | 4 | 4 | 3 | 2 | 2 | 3 | 4 | 5 |
| N | 7 | 6 | 5 | 5 | 5 | 4 | 3 | 3 | 2 | 3 | 4 |
| C | 8 | 7 | 6 | 5 | 6 | 5 | 4 | 4 | 3 | 2 | 3 |
| E | 9 | 8 | 7 | 6 | 6 | 6 | 5 | 4 | 4 | 3 | 2 |

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- How can we do this in $O(nm)$ time and $O(n)$ space?

The values we want *are* the entries in row $m/2$ of the DP table! So we already know how to calculate these in $O(nm)$ time and $O(n)$ space

# Calculating $ED(X_2, Y_2)$ for all $Y_2$

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- Problem: this doesn't quite correspond to a table row

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# Really nice trick

**Lemma**

*Let $X^R$ be the reverse of $X$, and let $Y^R$ be the reverse of $Y$. Then $ED(X, Y) = ED(X^R, Y^R)$.*

(Proof: just apply the same edits in reverse!)

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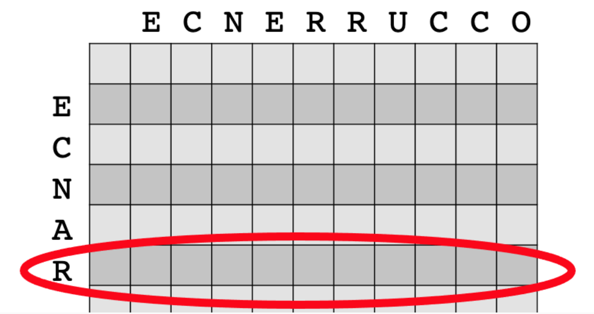
- We know how to do this from last slide! It's just the middle row of the DP table between the reversed strings

# Calculating the edit distances of the last characters



# Putting it all together

Let $X_1$ be the first half of $X$, and $X_2$ be the second half of $X$. Let $Y_i$ be the first $i$ characters of $Y$, and $Y_i'$ be the last $n - i$ characters of $Y$.

Here's how to calculate $ED(X_1, Y_i)$ and $ED(X_2, Y_i')$ for all $i$, in $O(nm)$ total time and $O(n)$ space:

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- Entry $(m - m/2, n - i)$ holds $ED(X_2, Y_i')$ by definition (and since edit distance is retained through reversal).

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- If there's a tie, *any* of them will give an optimal solution.

# Now: recurse!

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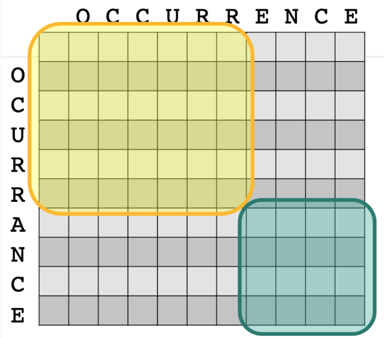
- Concatenate these two sequences to get the optimal sequence of edits for $X$ and $Y$



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- Can be more clever about it to improve the speed

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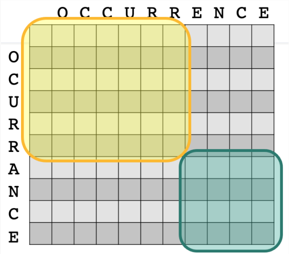
- We make two recursive calls: one with $(i, m/2)$, and the other with $(n-i, m-m/2)$
- Can prove by induction that the total time is $O(nm)$.
- Basic idea: the total cost of all recursive calls at a given level is the size of the table remaining; this decreases by a factor of 2 each time.



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- Answer: improved cache efficiency!
- If all work *fits into cache*, we only have the cache misses to set up the problem
- The space-inefficient approach may incur many cache misses to fill up the table.

# Implementation Tips

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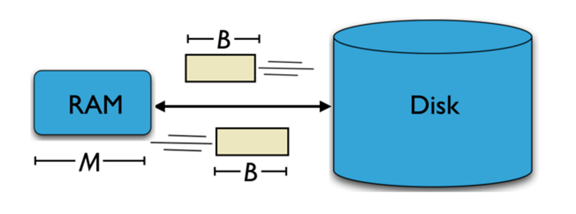
- When you make your recursive calls, your solutions *almost definitely* should not overlap. (Each character in a string should be a part of exactly one recursive call.)

# Sorting in External Memory

# External Memory Model Reminder




- Cache line of size $B$
- Cache can hold at most $M$ elements
- Computing on elements in cache is free! Only cost is to bring things in and out of cache

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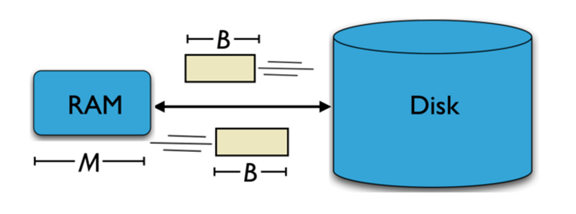

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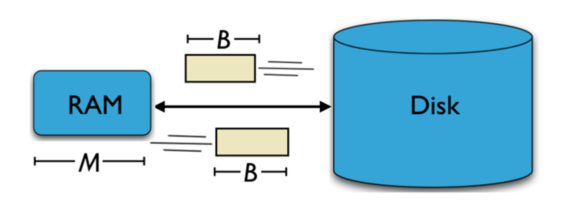


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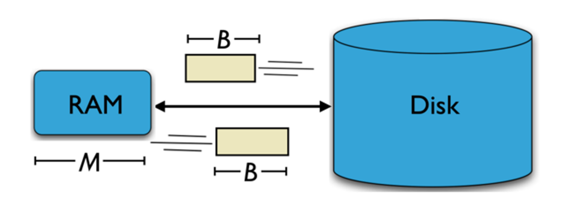
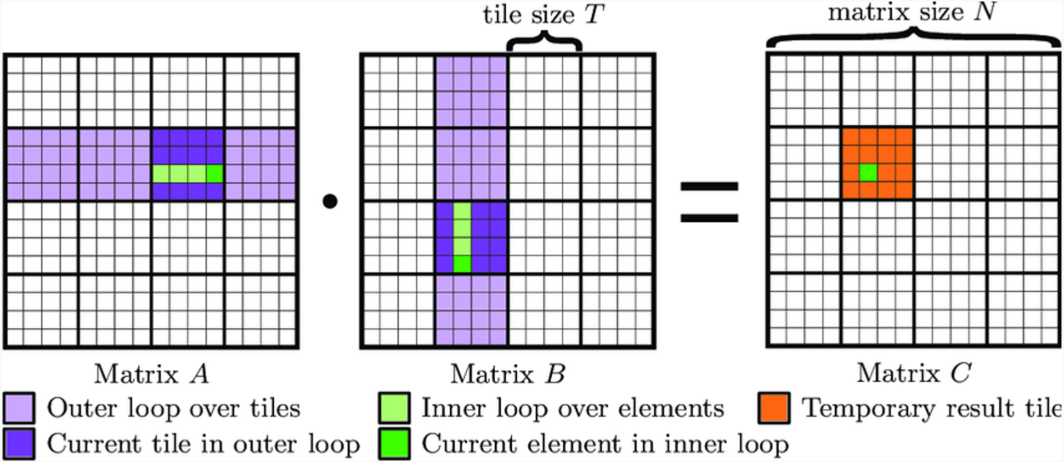



- Let's say I want to find the maximum element in an array using a linear scan. How much time does that take?
    - $O(n/B)$ cache misses: if I have a cache miss accessing $A[i]$, my next cache miss is accessing $A[i + B]$.

# Blocked Matrix Multiplication

- Decompose matrix into blocks of length $T$ (where $T^2 = M/3$)
- Do a normal $n/T \times n/T$ matrix multiplication
- Last class, saw: $O(n^3/B\sqrt{M})$ total cache misses.



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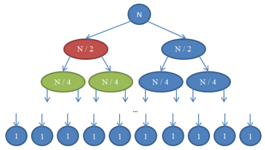


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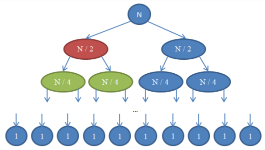


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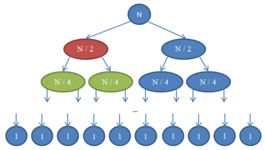


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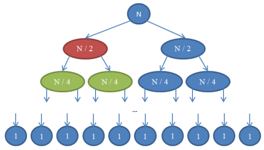


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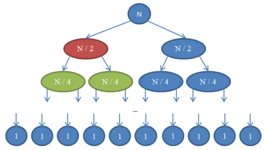


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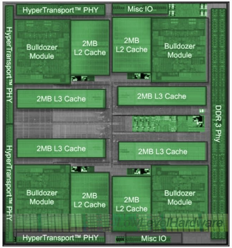
- Seems pretty good! Can we do better?

# Using the cache

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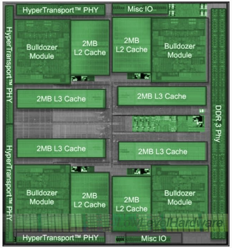


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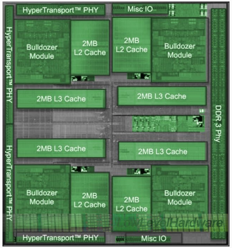


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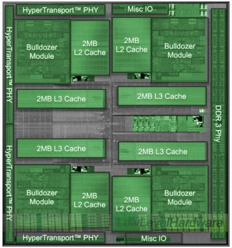


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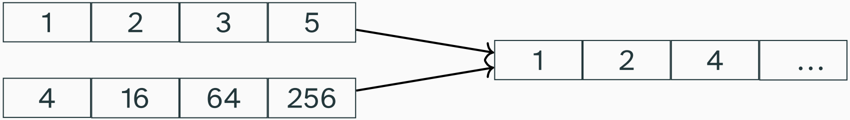
- Each time we bring a cache line into cache, how many more things can we compare it to?

# Merge sort reminder

- Divide array into two equal parts
- Recursively sort both parts
- Merge them in $O(n)$ time (and $O(n/B)$ cache misses)



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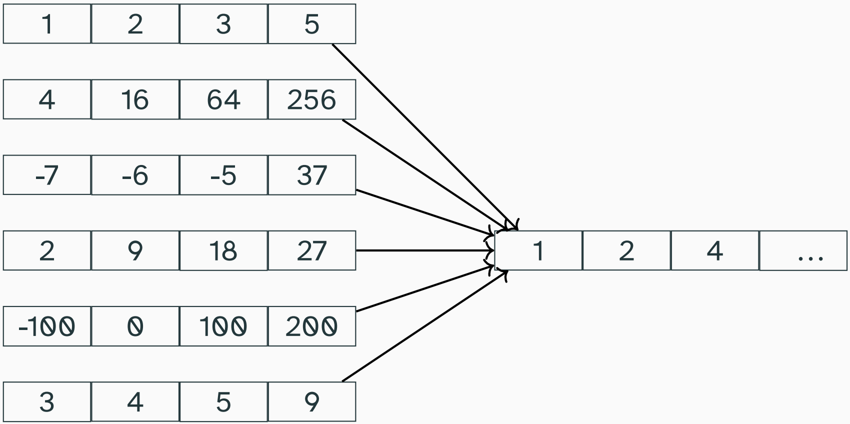
# Diagram of $M/B$-way merge sort



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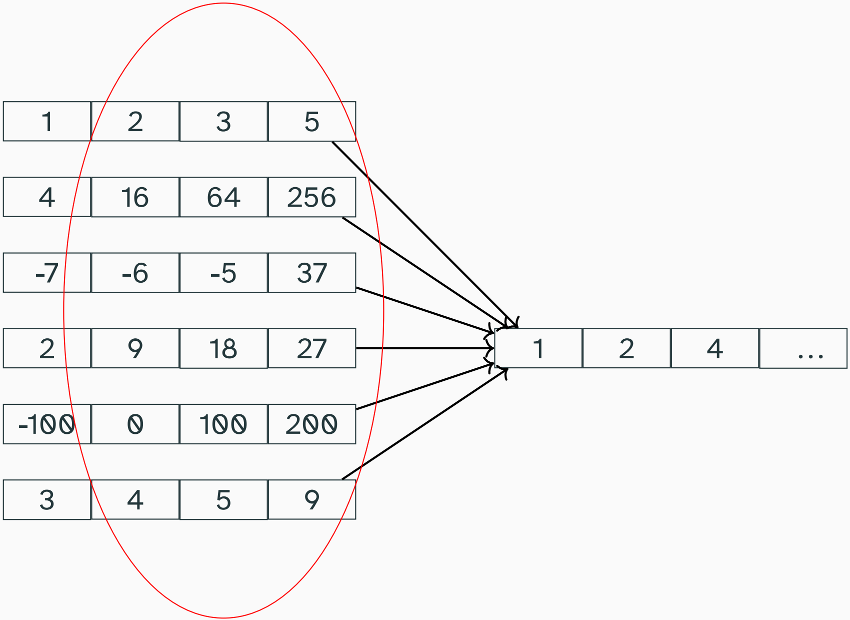

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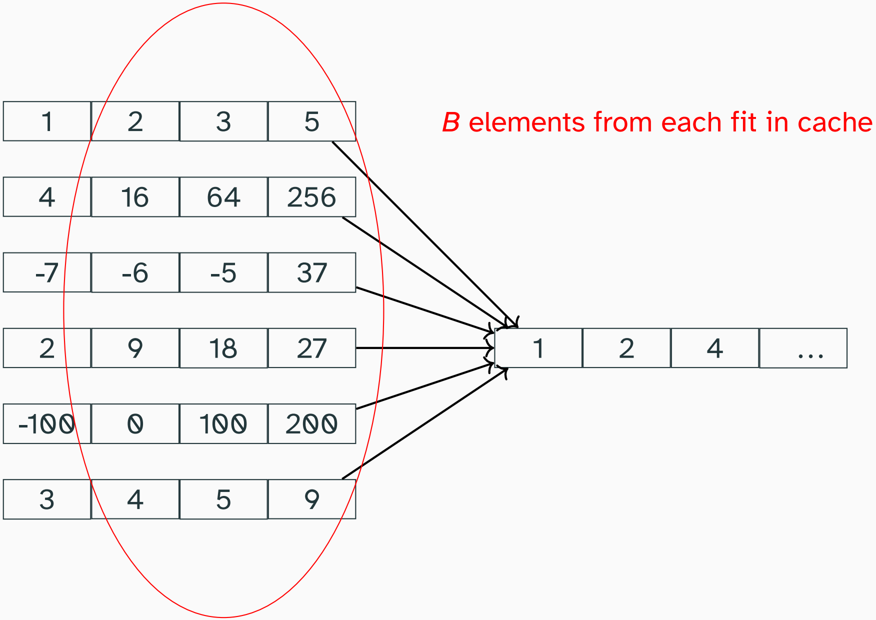

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