Lecture 18: Linear Programming and Optimization

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November 14, 2025

Williams College

Admin

- Assignment Resubmissions open! Will give more detail next slide
- Colloquium after class today: “Programming for Everyone: Developers, Scientists, and Creators of Tomorrow”
    - Faculty candidate! Come and (optionally) give us anonymous feedback
- Questions?

Assignment Resubmission

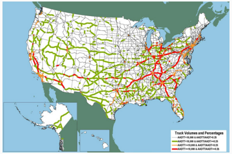
- Can bring your grade up 10 points out of 100
- Reminder: cap at 95 unless extra credit
- Submit your new version under (e.g.) "Assignment 1 Resubmission" on gradescope
- Please *say where you fixed it* by typing RESUBMIT at the beginning of any problem/code you want me to look at
- For code, just need to git push the new version (and write RESUBMIT on the relevant parts of the pdf)
- Reminder that honor code/etc. is still in effect; no LLMs allowed

Linear Programming and Optimization

What is an algorithmic problem?

- Constraints
- Objective
- What if we had a *single tool* that could solve *any* problem with certain kinds of constraints and objectives?

Next section of the course



- Framework to phrase large varieties of algorithmic problems

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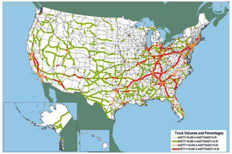

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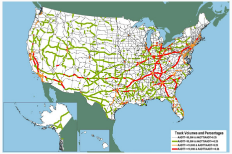

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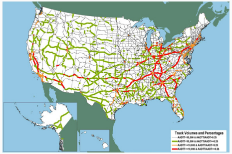

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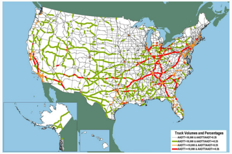

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- This topic is much older and much much broader than anything else we’ve covered
- Focus for this class: using linear programming and integer linear programming (and their solvers) to obtain optimal solutions to difficult problems. (Won’t be focusing on structure, mathematical properties.)

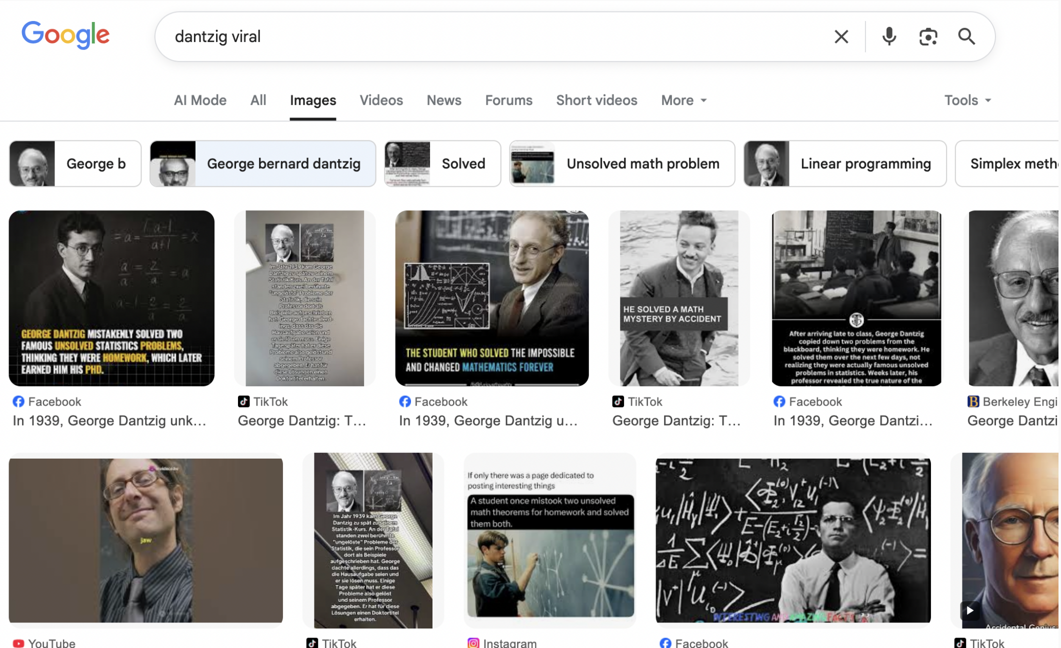
Context

> *I have a strong interest in the question of where mathematical ideas come from, and a strong conviction that they always result from a fairly systematic process—and that the opposite impression, that some ideas are incredible bolts from the blue that require "genius" or "sudden inspiration" to find, is an illusion.*
>
> *Timothy Gowers*

History

- Starts with a legend

(You may have seen heard this story before)



History

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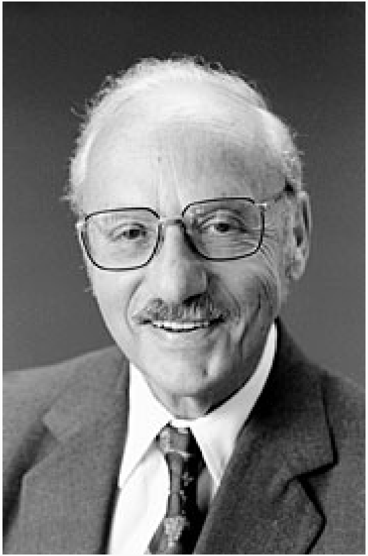
George Dantzig



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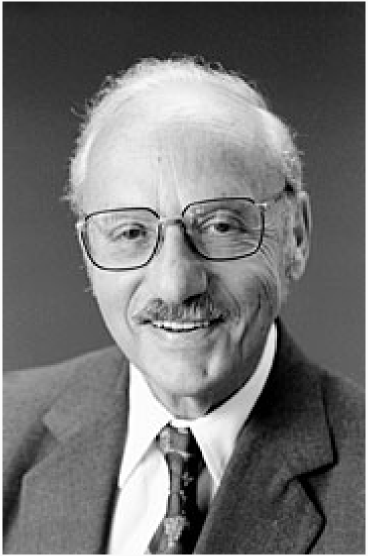

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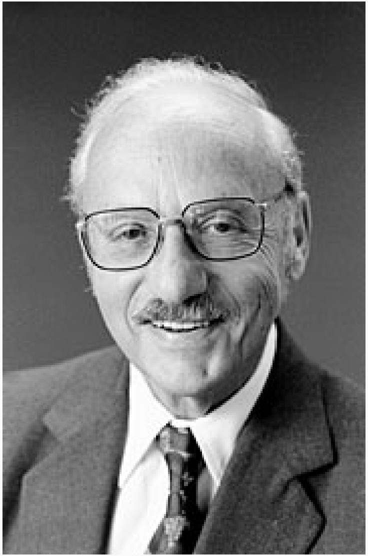

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- Invented the simplex algorithm

Where we’re headed

- A “big hammer” to solve an extremely wide variety of algorithmic problems

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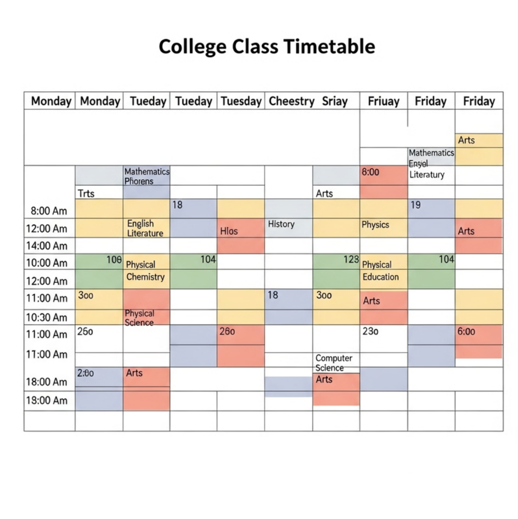

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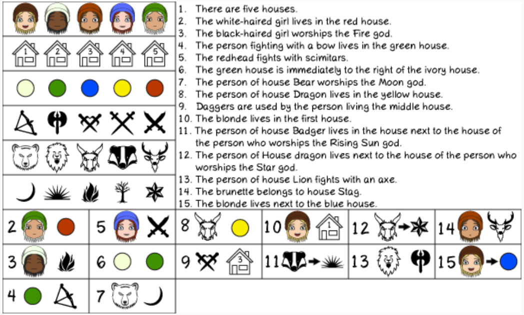

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- (We'll discuss what we mean by linear in a moment.)

Goal: achieve the best possible objective function value while satisfying the constraints

Why linear programming

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- 2004 survey: 85% of fortune 500 companies report using linear programming

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- Example 3: $|\sqrt{x_3} - x_7| \geq 5$ is not

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- Note that variables need not be integer or positive

Example of a Linear Program

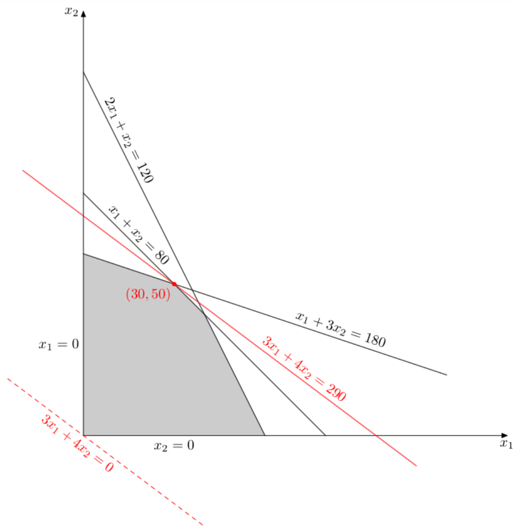
Objective:

$$\max \quad 3x_1 + 4x_2$$

Subject to:

$$\begin{aligned}
2x_1 + x_2 &\leq 120 \\
x_1 + 3x_2 &\leq 180 \\
x_1 + x_2 &\leq 80 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}$$

Feasibility



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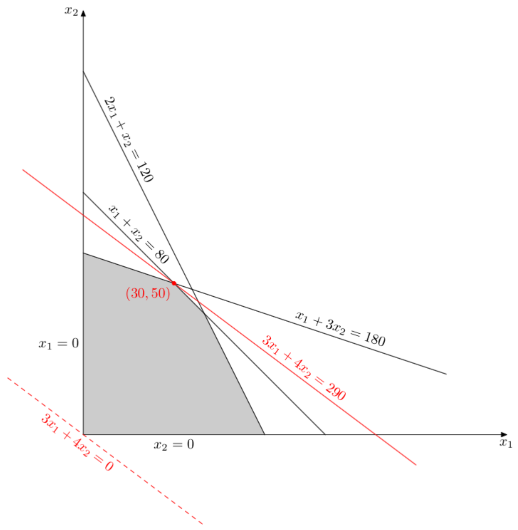

- An LP is *feasible* if there exists an assignment of variables that satisfies the constraints
- Nontrivial result: feasibility is not trivial to determine. In the worst case, it is as difficult as solving the entire LP.

Matrix Representation

Objective:

$$\begin{bmatrix} 3 & 4 \end{bmatrix}$$

Subject to:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 120 \\ 180 \\ 80 \\ 0 \\ 0 \end{bmatrix}$$

- Can represent with a matrix and vector
- Useful! Can use linear algebra tehniques and optimizations for these problems.
- I don't plan to use this representation again in this class

Visual representation

- We can plot these inequalities
- Works best for instances with 2 or 3 variables
- We'll use extensively as it gives good intuition

Plotting an LP

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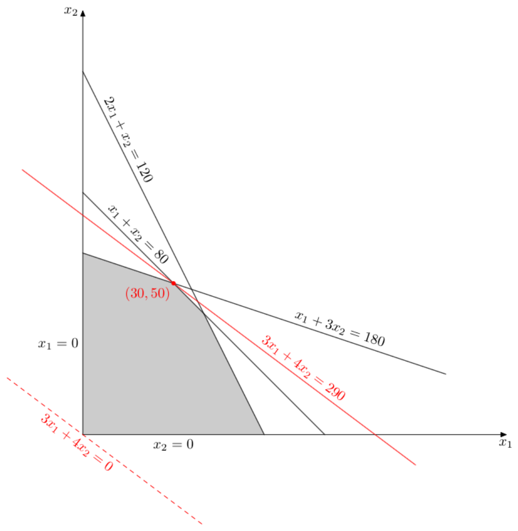

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- Our goal: use this tool to solve computational problems

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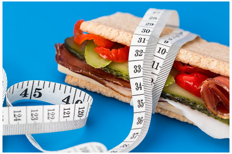
- Essentially a reduction: similar to using Network Flow to solve problems

Solving Problems with Linear Programming

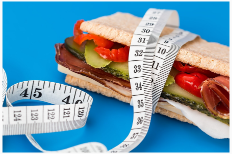
Optimization Problems



Example 1: Diet

- You need to eat 46 grams of protein and 130 grams of carbs every day

Optimization Problems



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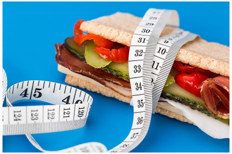

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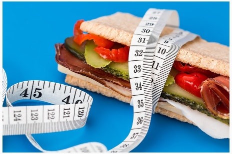

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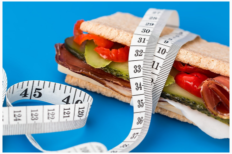

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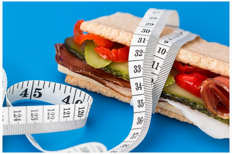

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In Pairs: try to phrase this problem as a linear program. What variables do we want? What is the objective function? What are the constraints?

Diet Problem

How can we phrase this as a linear program? (I'll write the problem on the board; you should think about how you would do it.)

- Let p be the amount of peanuts, r be the amount of rice, and c be the amount of chicken you buy.

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Diet Problem Constraints

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$\min 1.61p + .79r + .7c$

- Protein: $25.8p + 2.5r + 13.5c \geq 46$

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- $p \geq 0, r \geq 0, c \geq 0$

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Solution: $p = 0, r = 2.9216..., c = 2.86636...$

So we want to buy about 293g of rice, and 287g of chicken, for total cost $4.32

Diet Problem Solution

A mathematically optimal meal

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$$\begin{array}{ccc} r - c \leq .5 & c - p \leq .5 & r - p \leq .5 \\ c - r \leq .5 & p - c \leq .5 & p - r \leq .5 \end{array}$$

Example 3: Facility Location (Harder Problem)

- Given coordinates for n roommates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

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- Given coordinates for n roommates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Goal: find location for a router that minimizes the average distance to each roommate

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- Given coordinates for n roommates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
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Example 3: Facility Location

Objective:

Constraints:

- Given roommates at $(3, 4)$ and $(13, 5)$
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Example 3: Facility Location

Objective:

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$$(x-3)+(y-4) \leq 10$$
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Can't make objective function. Idea: add new variables!

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$$
\begin{aligned}
(x-3)+(y-4) &\leq d_1 \\
(-x+3)+(y-4) &\leq d_1 \\
(-x+3)+(-y+4) &\leq d_1 \\
(x-3)+(-y+4) &\leq d_1 \\
(x-13)+(y-5) &\leq d_2 \\
(-x+13)+(y-5) &\leq d_2 \\
(-x+13)+(-y+5) &\leq d_2 \\
(x-13)+(-y+5) &\leq d_2 \\
d_1 &\leq 10 \\
d_2 &\leq 10
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$$

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Example 3: Facility Location (Equations Simplified)

Objective: $\min d_1 + d_2$

Constraints:

$$x + y - d_1 \leq 7$$
$$-x + y - d_1 \leq 1$$
$$-x - y - d_1 \leq -7$$
$$x - y - d_1 \leq -1$$
$$x + y - d_2 \leq 18$$
$$-x + y - d_2 \leq -8$$
$$-x - y - d_2 \leq -18$$
$$x - y - d_2 \leq 8$$

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- 1st: if there exists a feasible LP solution has values d_1, d_2, x, y then there exists a router placement at (x, y) with distance *at most* d_1 and d_2 from roommates 1 and 2, with $d_1 \leq 10$ and $d_2 \leq 10$

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- 2nd: any placement of a router at location (x, y), with distance $d_1 \leq 10$ and $d_2 \leq 10$ from the first and second roommate respectively corresponds to a feasible LP solution with variables d_1, d_2, x, y
- If we can prove these claims then solving this LP solves the router placement problem: we get the min total distance placement

Proving Correctness

Lemma

If there exists a feasible LP solution with variables d_1, d_2, x, y then a router at (x, y) has distance at most d_1 and d_2 from roommates 1 and 2, with $d_1 \leq 10$ and $d_2 \leq 10$

Proof: Router at (x, y) has distance $\hat{d}_1 = |x - 3| + |y - 4|$ from roommate 1. Because the LP soln is feasible, we have:

$$(x - 3) + (y - 4) \leq d_1 \qquad (-x + 3) + (y - 4) \leq d_1$$

$$(-x + 3) + (-y + 4) \leq d_1 \qquad (x - 3) + (-y + 4) \leq d_1$$

Since $\hat{d}_1$ is equal to the left side of one of these equations, $\hat{d}_1 \leq d_1$. Furthermore, since the LP solution is feasible, $d_1 \leq 10$, so $\hat{d}_1 \leq 10$.

Same argument works for roommate 2

Proving Correctness

Lemma

Any placement of a router at location (x, y), with distance $d_1 \leq 10$ and $d_2 \leq 10$ from the first and second roommate respectively corresponds to a feasible LP solution with variables d_1, d_2, x, y

Proof summary: We have $d_1, d_2 \leq 10$ by definition. We need to show the roommate constraints are satisfied. Let's focus on d_1. We have $d_1 = |x - 3| + |y - 4|$.

For any x, y we have:

$$x - 3 \leq |x - 3| \qquad -x + 3 \leq |x - 3|$$

$$y - 4 \leq |y - 4| \qquad -y + 4 \leq |y - 4|$$

Substituting, all equations for d_1 are satisfied.

Proving Correctness

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- So we can solve this problem by solving an LP

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