Lecture 15: Suffix Arrays 2

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Williams College

Admin

- Suffix Array “checkin” out
    - By Thursday at 10pm: Submit one function (suffix array construction)
    - I would *highly* recommend: before Thursday, go through the C++ code linked on the website, and map each step to what we do today on the board/slides
- Any questions?

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- Idea: sort one character at a time using $O(n)$ time counting sort
- Specifically, proceed *backwards*: sort by last character, then (stable) sort by second to last character, and so on

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- Let’s do this for `CACATACACAGACACAC$`
- Correct answer:
  - `17 15 13 11 5 7 1 9 3 16 14 12 6 0 8 2 10 4`

Suffix Array Construction: Detailed Description

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- Will also use two temporary arrays $pn[]$, $cn[]$ as we move items around, and a temporary array $cnt[]$ to count items in the counting sort.

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- Go through each suffix one at a time; look at its first character and place it into *p* using *cnt*

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- We can radix sort this pair in $O(n)$ time: first counting sort all suffixes by the second 2^k characters, then counting sort by the first 2^k characters
- Then we're sorted by 2^{k+1}! If $2^{k+1} < n$, then increment k and go again; otherwise we're done

Doubling Step: First, sort by second 2^k characters

Our current status: $p[]$ contains all suffixes sorted by their first 2^k characters. We want to sort them by the *second* 2^k characters

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- (I love this step)

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- Use counting sort to place each suffix of *pn*[] in the correct place by its first 2^k characters:
  - Use *cnt*[] to count the number of suffixes strictly before each class
  - Go through each suffix in *pn*[]; *c*[] gives its class. Look up the class in *cnt*[], decrement it, and place it in *p*[].

Doubling Step: finally, update $c[]$

Our current status: $p[]$ contains all suffixes sorted by their first 2^{k+1} characters. We want to update $c[]$ so we can use it in the next iteration. We'll place the new values into $cn[]$; then swap $c[]$ and $cn[]$. To begin, $cn[p[0]] = 0$: the first prefix in alphabetical order has class 0.

- Remember that we sorted by: $c[i], c[i+2^k]$

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- Remember that we sorted by: $c[i], c[i + 2^k]$
- Go through each suffix j in order of p; let j' be the next suffix in p (so $j = p[i]$ and $j' = p[i + 1]$). If $c[j] = c[j']$ and $c[j' + 2^k] = c[j' + 2^k]$, then we keep $cn[j'] = cn[j]$; otherwise, $cn[j] = cn[j'] + 1$.

Putting it all Together

First, counting sort by first letter.

For $k = 1$ to $\lceil \log_2 n \rceil$:

1. Place suffixes from $p[]$ into $pn[]$, sorted by *second* set of 2^k characters. [Performed using subtraction]

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2. Place suffixes from $pn[]$ into $p[]$, sorted by *first* set of 2^k characters. [Performed using counting sort]

3. Store classes of length 2^{k+1} in $cn[]$ by iterating through $p[]$ and comparing successive classes. This is the new $c[]$ for the next loop

Each inner step requires $O(1)$ array scans; $O(n \log n)$ time!

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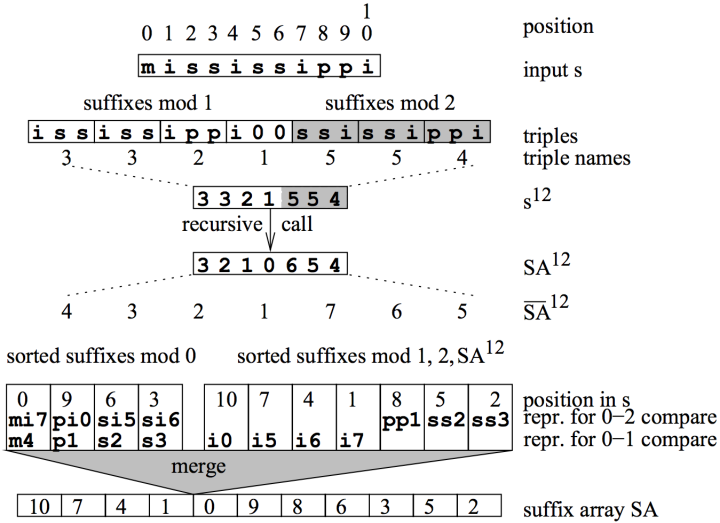
Linear-time Algorithm Diagram



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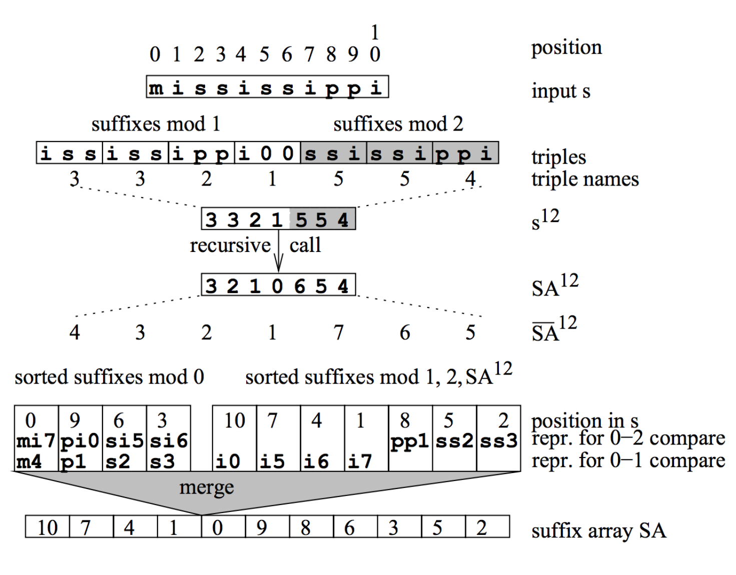

(There's a reason why we're not going to use this algorithm. But, let's go over the

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- Use the ordering of the odd-numbered suffixes to build the suffix tree for the even-numbered suffixes

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- It’s possible in $O(n)$ time with some bookkeeping!

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  - Your algorithm will run in about 40s on `timeData.txt`; SA-IS runs in about 2s.
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- Incredibly short, clean, and unintuitive

Suffix Array Uses

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- Then: binary search for P in the suffix array

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- Gives $O(m \log n)$ time overall
- Can be improved to $O(m)$ time—can search for all occurrences of a pattern in the time it takes to read the pattern!

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- With more work: searching for P in T with errors