Lecture 12: van Emde Boas Trees

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October 28, 2025

Williams College

Admin

- Midterm graded by Friday
- Lab Thursday: finish LSH
- Today: self-contained; short(?) topic
  - Bridge the gap between hashing/tables and strings
- Two-week assignment released Thursday

Strings (and, today, trees)

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- Lots of magic!
- Clever ways of looking at problems that lead to highly effective solutions

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- In CS 136 we saw how to answer this using a balanced binary search tree in $O(\log n)$ time
- This is optimal if all you can do is *compare* items

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- For example: if the items of S are 64-bit integers, then we have $M = 2^{64}$. If items of S are k-character strings, we have $M = 256^k$.

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- Know much more about the relative values of integers or strings
- Today: let's say that the items of S are taken from a bounded set $\{0, \ldots, M-1\}$
- For example: if the items of S are 64-bit integers, then we have $M = 2^{64}$. If items of S are k-character strings, we have $M = 256^k$.
- In this case, we will show how to get predecessor and successor in $O(\log \log M)$ time.
    - For a w-bit integer, get $O(\log w)$ time
    - For a k-character string, get $O(\log k)$ time

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- We'll only look at insert, successor. Can generalize to predecessor queries and deletes.
- Let's not worry about space today (we'll wind up with $O(M)$ space). Some techniques to achieve $O(n)$ space.
- Also, let's assume that $\log_2 \log_2 M$ is an integer (M is 2 to a power of 2; like 2^8 or 2^{64})

First attempt at Insert, Successor

0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1
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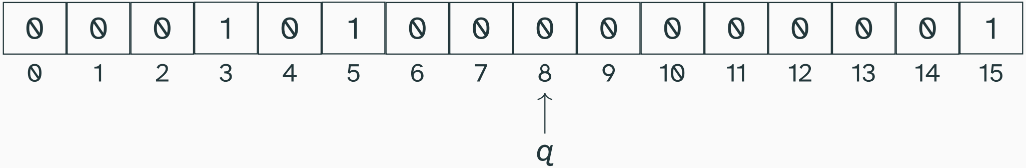

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  - Time for successor?
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- Insert is really fast. Can we try to speed up successor?

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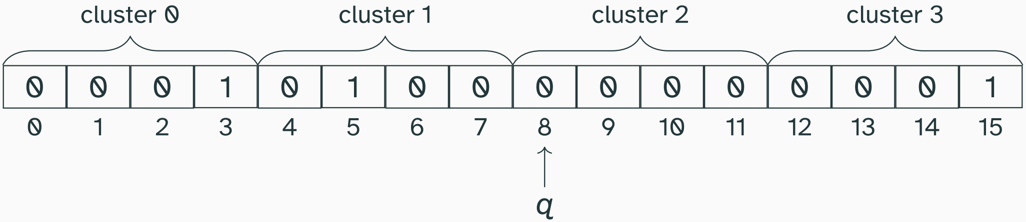
  - How can we find the *next nonempty cluster*?



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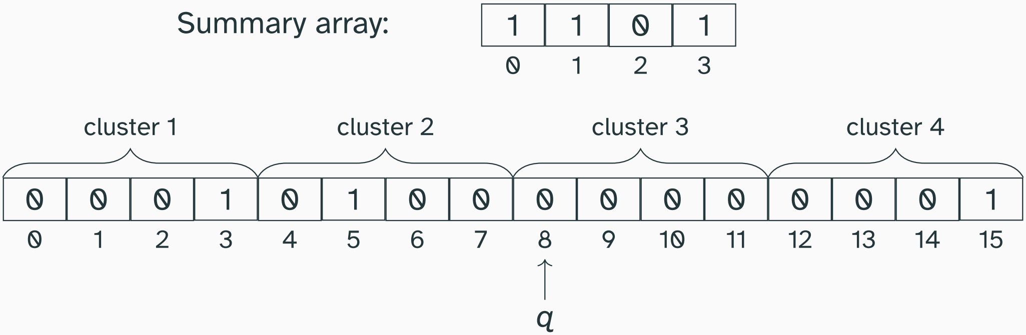

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Summary array:

1	1	0	1
0	1	2	3

cluster 1				cluster 2				cluster 3				cluster 4			
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- Where can we improve?
- All our time is spent doing *array scans* for successor queries within a cluster...
- But we know how to do better-than-linear successor queries! Let's recurse.

Recursing: van Emde Boas Tree (almost)

If $M = 1$, just store the array.

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- Let's draw a picture of it on the board

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- Solves to $O(\log M)$ insert time (too slow!)

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- Solves to $O((\log M)^{\log_2 3}) = O(\log^{1.585} M)$ insert time (way too slow!)

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- Can we get rid of some of them? Let’s focus on successor

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- Finding the minimum element doesn't require a whole successor call! Let's just store the minimum element in each cluster. Then finding the minimum element is $O(1)$.

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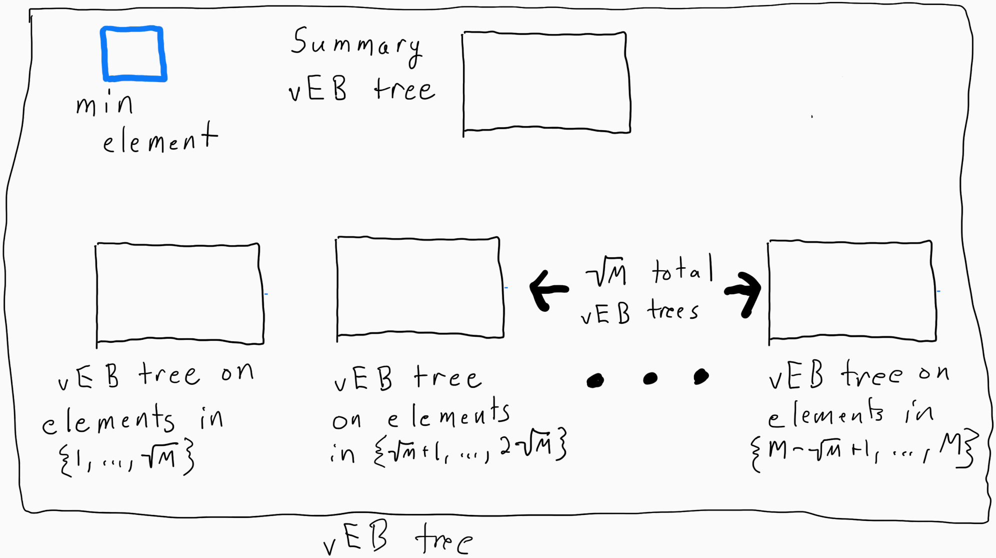
- Recurrence for both: $T(M) = 2T(\sqrt{M}) + O(1)$; solves to $T(M) = \log M$.

vEB Tree



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Summary vEB tree
vEB tree on elements in {1, ..., √M}
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√M total vEB trees
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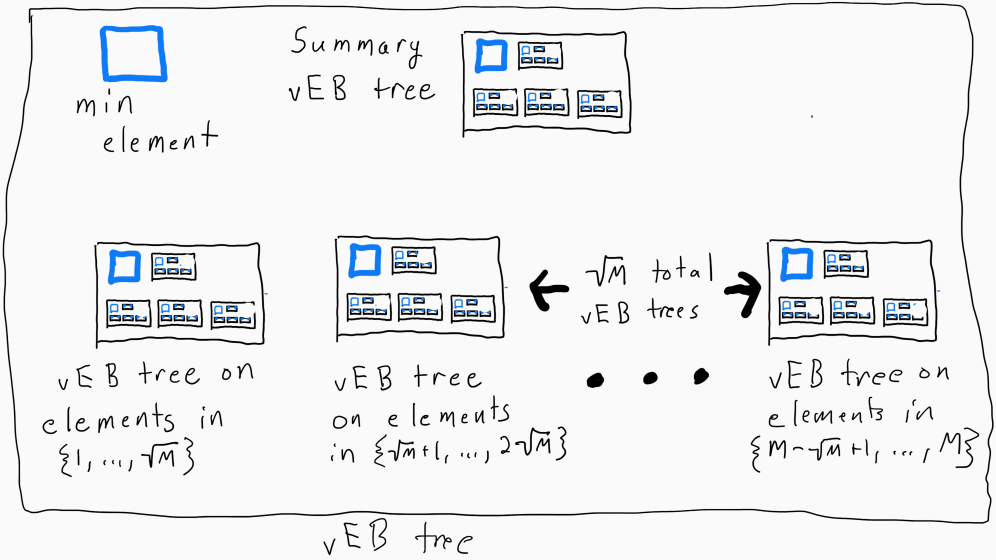
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- *Hint:* Can we store something to help us determine if q has a successor in its cluster without a recursive query?
    - Store the *max element* in each cluster!

vEB Tree: Store the Max and Min in each cluster

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- If q is less than the max, find successor(q) in that cluster and return it

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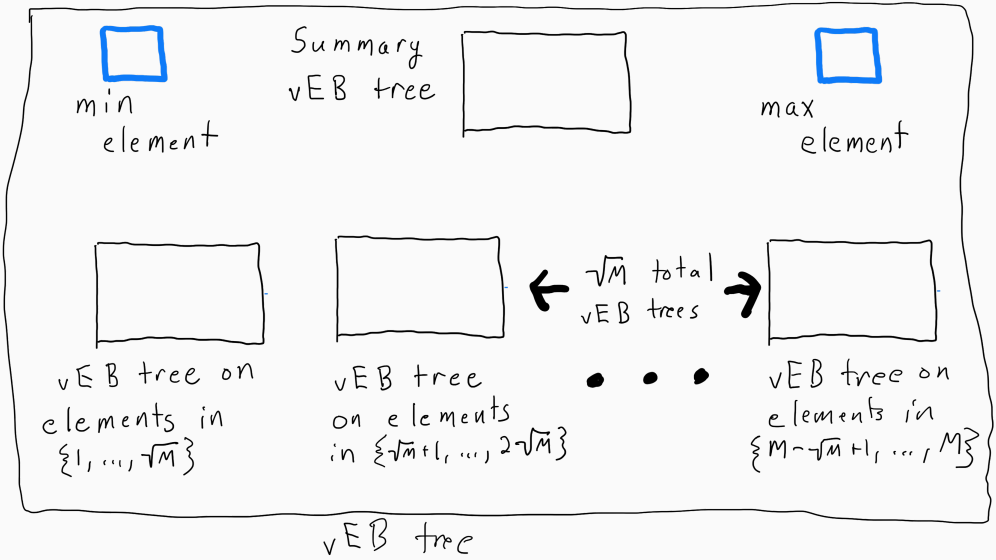
- Example on board: store $3, 5, 15$ from universe $\{0, \ldots 15\}$; query for element 8.

vEB Tree



min element
Summary vEB tree
max element
$\sqrt{M}$ total vEB trees
vEB tree on elements in $\{1, \ldots, \sqrt{M}\}$
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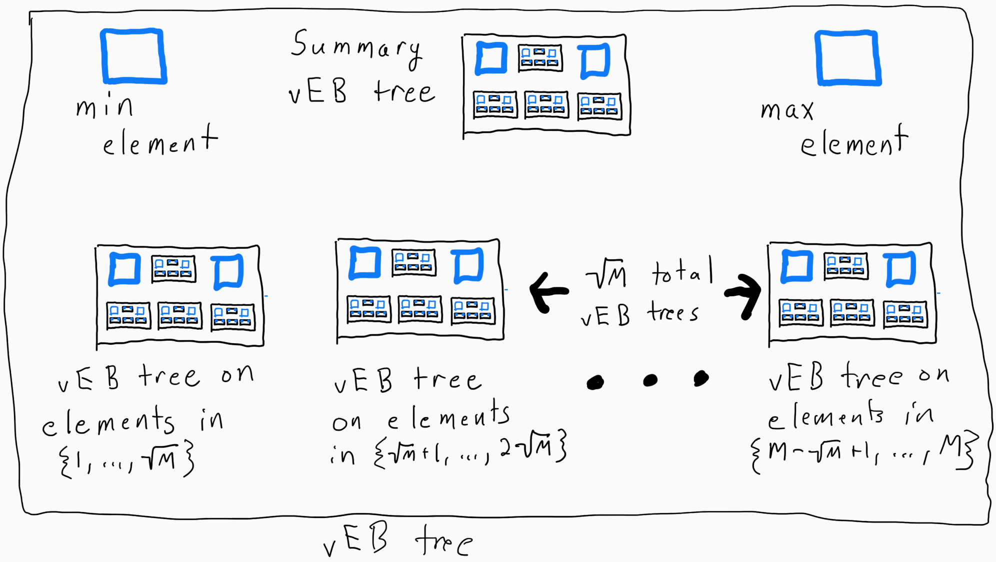

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- Done!

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- Otherwise:
  - Check if x is less than the minimum m.
  - If so, set x to be the minimum, and insert m into x's cluster.
  - Do the same for the maximum.
  - Otherwise, insert x into its cluster.

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- Find x's cluster c. If x is smaller than the maximum value in that cluster, query vEB tree c for the successor of x.
- Otherwise, query the summary vEB tree for the successor of c; call it c'. Return the minimum element of c'.

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- $T(M) = T(\sqrt{M}) + O(1)$ gives $O(\log\log M)$ query time
- Insert does $O(1)$ work and makes one recursive call of size $\sqrt{M}$; also $O(\log\log M)$ time

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- Deletes?
- Can make deletes work pretty easily with what we have.

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- Possible to get $O(n)$ space deterministically using another, more complicated data structure (y-fast tries)

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- Takeaway: unless M is very large or n is very small, vEB trees are quite a lot faster
- But, they're probably a bit more complicated