Lecture 12: Locality-Sensitive Hashing and MinHash

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October 17, 2025

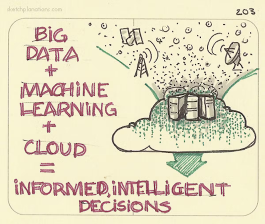
Williams College

Admin

- Midterm a week from today in class
- Practice midterm out in the next 24 hours; solutions early next week
- I created both the midterm and the practice midterm at the same time; they should be extremely similar in terms of structure and very similar in terms of topics
- No writing code on the midterm
- Review session Tuesday; bring questions
- Assignment 4 out tonight (I'm adding some extra starter code because I know you're also studying)
- I will grade everything before the midterm

Finding Similar Items

Back to Normal Inputs



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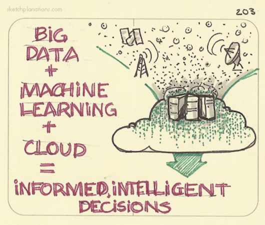

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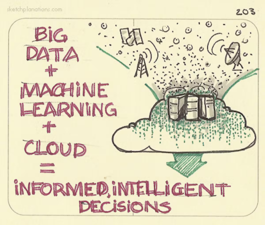

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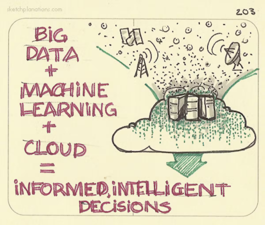

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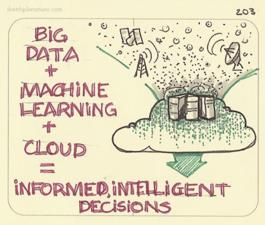

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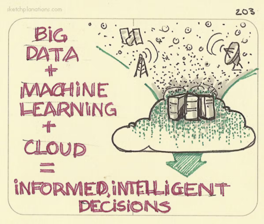

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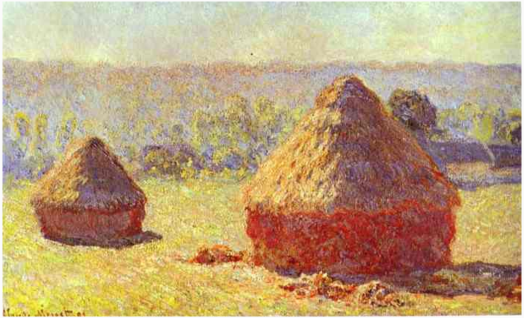
- But data is still big!
    - In particular: high-dimensional
    - Table with many columns
    - For each Netflix user, what movies have they seen
- Goal: solve a computationally difficult, but important, problem. If you've taken an ML course it's reasonably likely that you've seen some variant of this problem.

Finding Similar Pair



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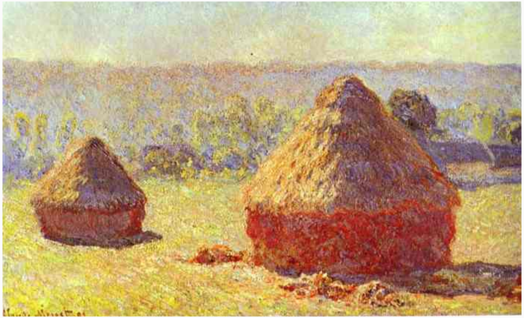

- Given a set of objects
- Find the most similar pair of objects in the set

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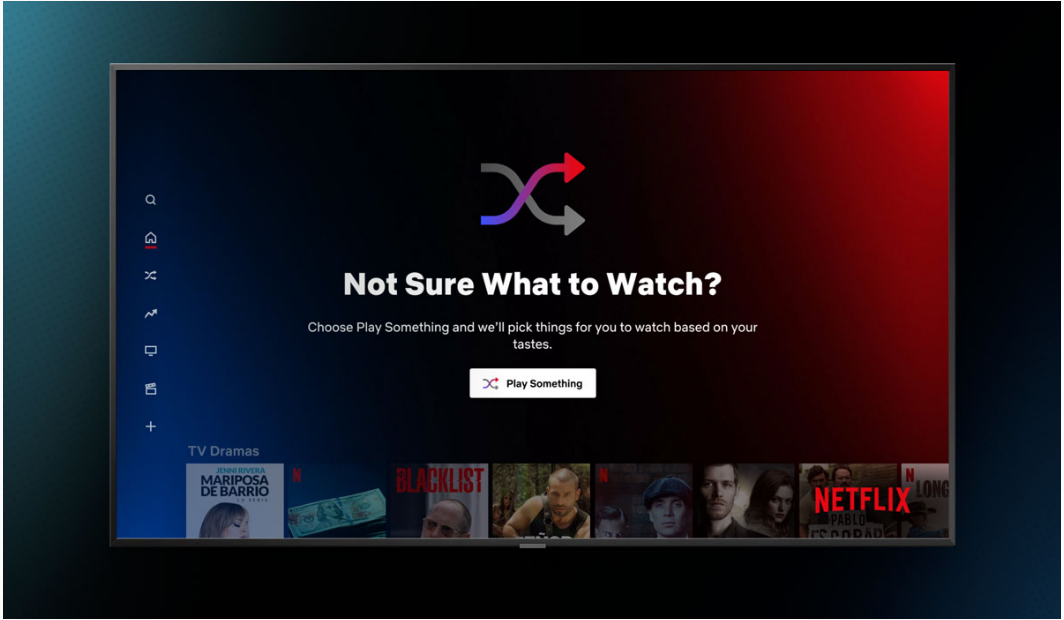
- Data deduplication, etc.
- “Give me a similar pair in this dataset” is a common query!

Similarity Search Example



Strategies for Similarity Search

First attempt: 1-dimensional data

92
44
7
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80
67

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- Given a list of numbers
- “Similarity" is the absolute value of the difference between them
- How can we find the closest numbers (i.e. ones with smallest difference)?

First attempt: 1-dimensional data

- How efficiently can we do this?

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Aside: can we do better? Yes, there's a clever $O(n)$ algorithm based on sampling.

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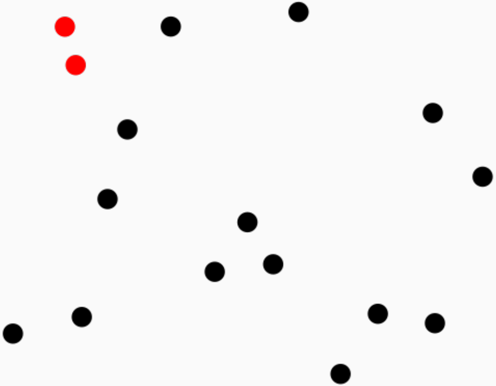

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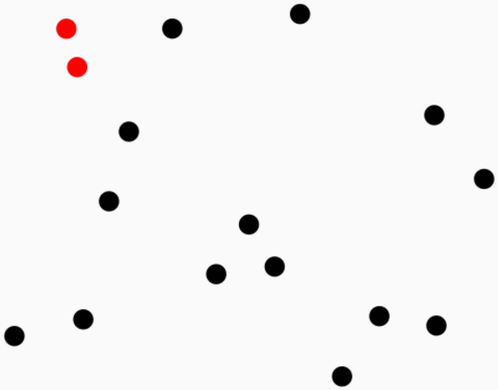

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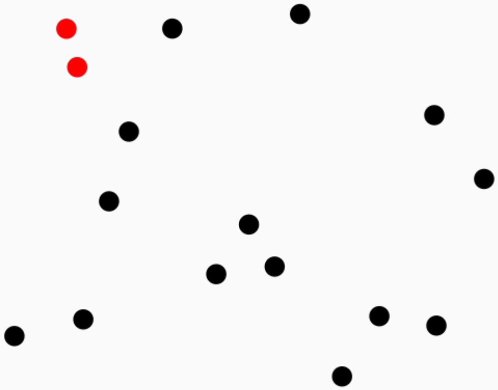

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- (Again, possible in $O(n)$ with sampling.)

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- Songs listened to, movies watched, image tags, etc.
- Words that appear in a book, k-grams that appear in a DNA sequence
- In ML applications, often want to search in the embedded space

What I mean by “dimension”


Josefine S. (Protected by ...

+ Follow

PRO

(k) The 38 movies I saw in 2010 :)

Only counting movies I saw for the first time. Faves: 4, atm. Best: "Harry Potter and the Deathly Hallows: Part I". :D
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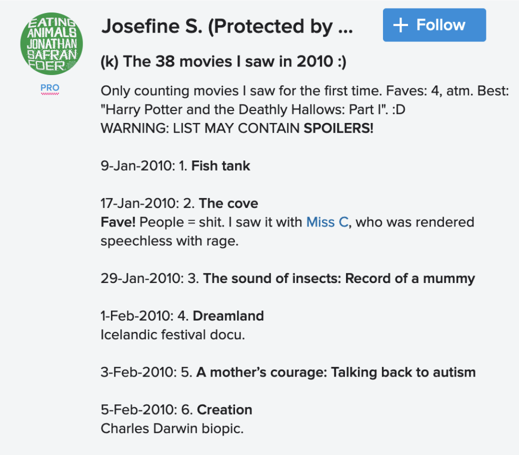
EATING ANIMALS JONATHAN SAFRAN FOER
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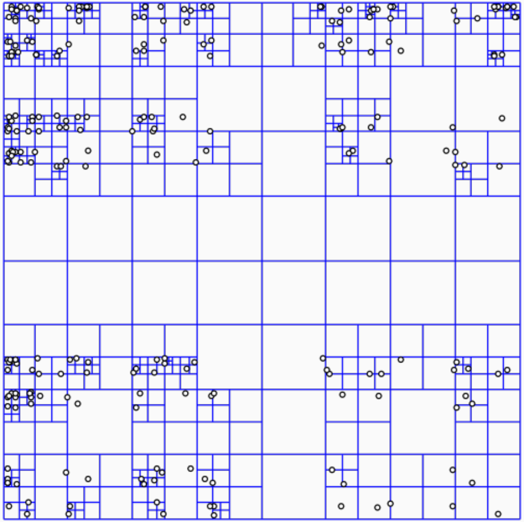
- “The tags generated for this image are: [. . .]”

How Efficient are High-dimensional Algorithms?



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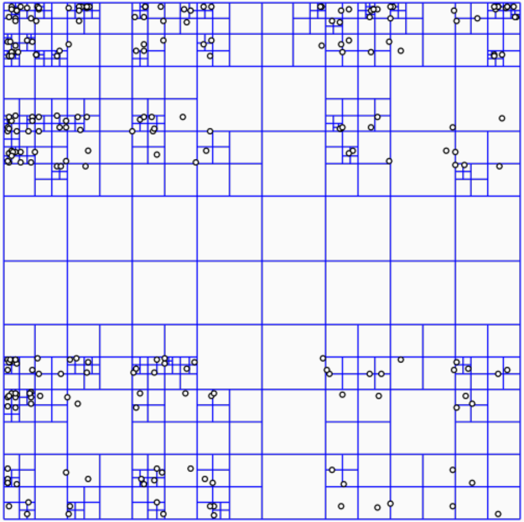

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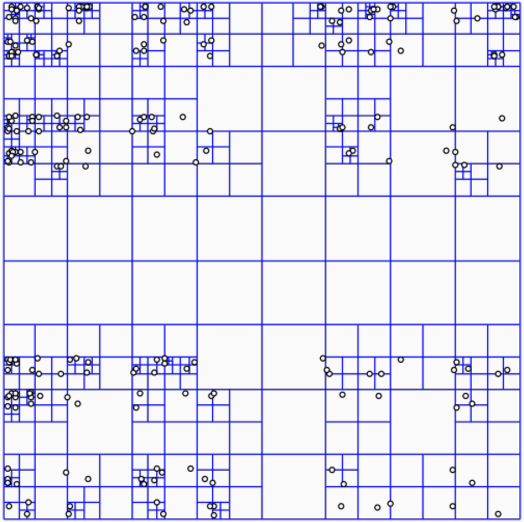

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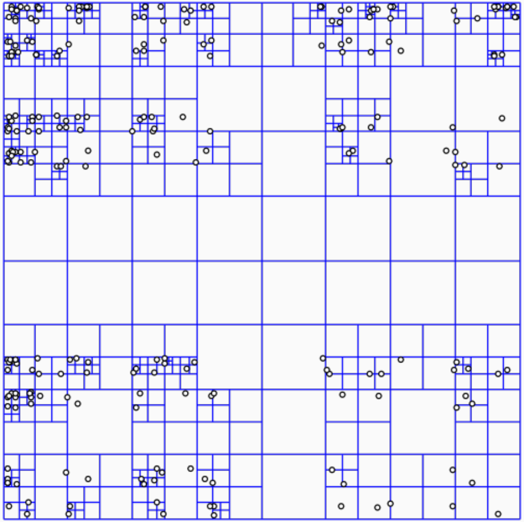

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- Worse than trying all pairs if $> \log n$ dimensions

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- Well-known phenomenon

- Applies to similarity search, machine learning, combinatorics
  - Approximation techniques, like those we learn about today, are underused to a slightly shocking extent—even in ML people sometimes keep dimensionality low to avoid this issue, affecting the quality of results

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  - Assume that the close pair is much closer than any other (*approximate* closest pair)
  - Use hashing! ...A special kind of hashing
- For many of these problems, random inputs are worst-case inputs
  - Worst case behavior actually occurs for many common use cases; guarantees (even approximate) can be very valuable

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- Normally, hashing spreads out elements.
- This is key to hashing: no matter how clustered my data begins, I wind up with a nicely-distributed hash table
- Locality-sensitive hashing tries to act like a normal hash for items that are dissimilar, but *wants* collisions for similar elements
- Similar items are likely to wind up in the same bucket. But dissimilar items are not

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- High level idea: close items are likely to collide. Far items are unlikely to collide.
- Generally want p_2 to be about $1/n$; then we get a normal hash table for far (i.e. similarity $\leq cr$) elements.

Why Locality-Sensitive Hashing Helps

	(101, 37, 65) (103, 37, 64)	(91, 84, 3)		(100, 18, 79)
0	1	2	3	4

Ideally, close items hash to the same bucket.

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We'll put numbers on this later

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- How can we increase this probability?
- Repetitions! Maintain many hash tables, each with a different locality-sensitive hash function, and try all of them.

LSH with Repetitions

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- Need a way to measure *set similarity*

Set Similarity (Example Created in 2020)

User 1	User 2
Post Malone	Ariana Grande
Ariana Grande	Khalid
Khalid	Drake
Drake	Travis Scott
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Very similar!

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- When are two sets similar?
- Let's look at our two sets. Similar if they have a lot of *overlap*
- I.e. : lots of artists in common, compared to total artists in either list

Jaccard Similarity

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- Similarity measure for sets A and B

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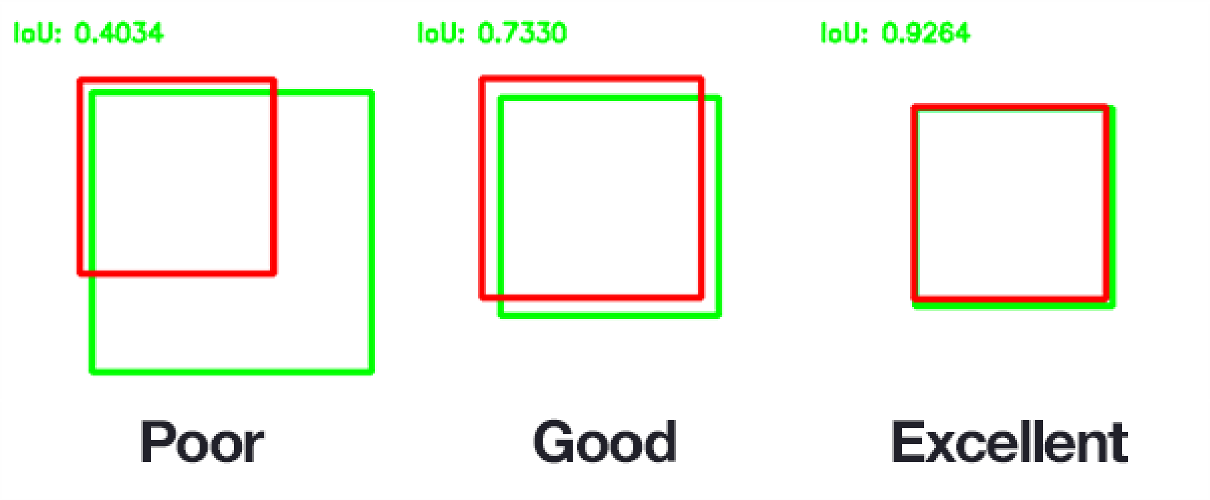
- Intuitively: Jaccard similarity says what fraction of two sets overlaps.

Jaccard Similarity Intuition 1



Jaccard Similarity Intuition 2

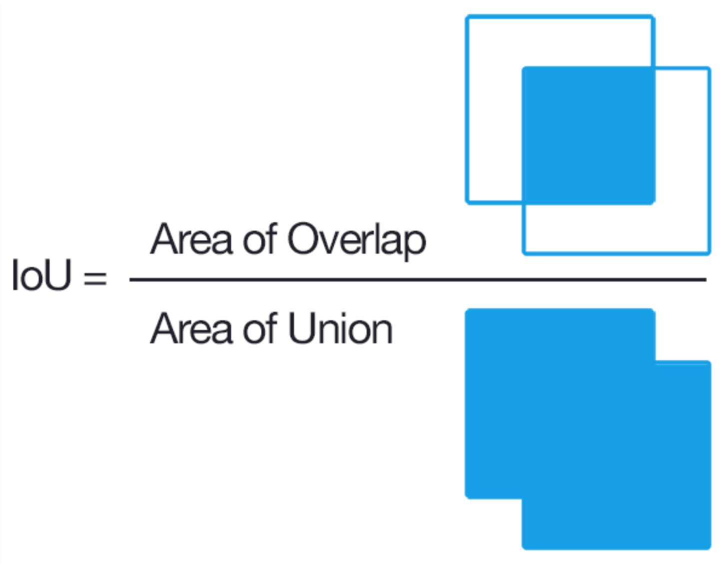
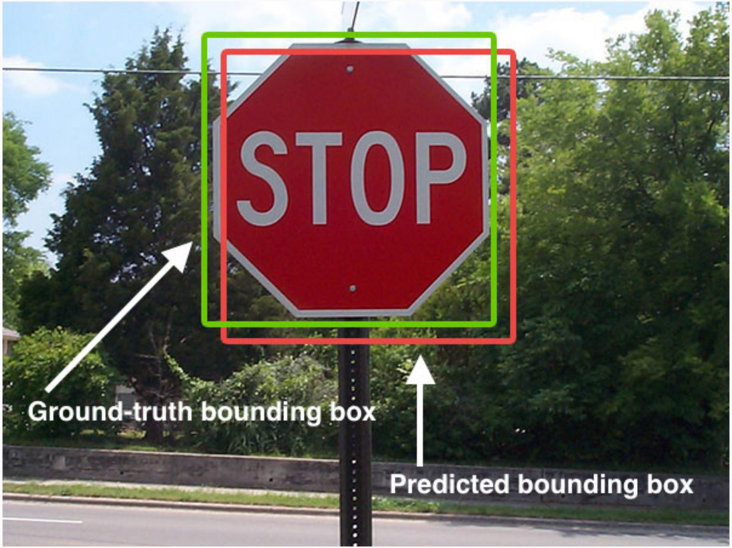


Image Search Example



Jaccard Example 1 (Example Created in 2020)

User 1	User 2
Post Malone	Ariana Grande
Ariana Grande	Khalid
Khalid	Drake
Drake	Travis Scott
Travis Scott	

- Similarity: $|A \cap B| / |A \cup B|$.

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- $|A \cap B| = 4$

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- $|A \cap B| = 4$
- $|A \cup B| = 5$

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- Similarity: $|A \cap B|/|A \cup B|$.
- $|A \cap B| = 4$
- $|A \cup B| = 5$
- Jaccard Similarity: $4/5 = .8$

Set Similarity (Example Created in 2020)

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- Similarity: $|A \cap B| / |A \cup B|$.
- $|A \cap B| = 1$

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Ariana Grande	
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- Similarity: $|A \cap B|/|A \cup B|$.
- $|A \cap B| = 1$
- $|A \cup B| = 5$
- Jaccard Similarity: $1/5 = .2$

Set Similarity (Example Created in 2020)

User 1	User 2
Post Malone	Ariana Grande
Ariana Grande	Ed Sheerhan
Khalid	Drake
Drake	Travis Scott
Travis Scott	Taylor Swift

- Similarity: $|A \cap B| / |A \cup B|$.

Set Similarity (Example Created in 2020)

User 1	User 2
Post Malone	Ariana Grande
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- Similarity: $|A \cap B| / |A \cup B|$.
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Set Similarity (Example Created in 2020)

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- Similarity: $|A \cap B| / |A \cup B|$.
- $|A \cap B| = 3$
- $|A \cup B| = 7$
- Jaccard Similarity: $3/7 = 0.428$

Jaccard Similarity: Properties

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- Jaccard similarity ignores items not in either set. So we learn nothing if neither of us like an artist. (Is this good?)
- Still works if one list is much longer than the other. (Generally, they'll have small similarity)

Locality-Sensitive Hash for Jaccard Similarity

- Want: items with high Jaccard Similarity are likely to hash together

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- Classic method: MinHash

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- (AltaVista was probably the most popular search engine before Google, they wanted to detect similar web pages to eliminate them from search results)
- Now used for similarity search, database joins, clustering—LOTS of things.

AltaVista in 2001

Bit Vectors

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- So if I’m keeping track of different people’s favorite colors, my universe may be {red, yellow, blue, green, purple, orange}
- If someone likes red and blue, we can store that information as `101000`.
- Effective if universe is fairly small; use a list for larger universe

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- How can we determine $A \cap B$?
    - This is exactly A & B in C-style notation
- What about $A \cup B$?
    - This is exactly A | B in C-style notation
- We want the size of these sets—need to count the number of 1s in A & B, or A | B.

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- First set is 101000 (same as {red, blue}). blue is in the set, so the hash value is blue.
- Second set is 110010 (we could also write {red, yellow, purple}). blue is not in the set; nor is orange; nor is green. purple is, so purple is the hash value

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For the sake of space let's do 8 bits

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  - On your own: what is the minhash of x for this permutation?

- The minhash of x is 5.

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- Not useful yet—output is too small! Almost all items will have one of the first few items in the permutation, so will hash to the first few buckets
- Let's do some analysis to look at this issue in more detail

Analysis of Basic MinHash

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- Any index in $A \cup B$ is equally likely to be first. If the index is in $A \cap B$, they hash together; otherwise they do not
- Therefore: probability of hashing together is $|A \cap B|/|A \cup B|$.

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- If two items have similarity at least r, they collide with probability at least $p_1 = r$
- If two items have similarity at most cr, they collide with probability at most $p_2 = cr$

Analysis: Phrased as bit vectors

- What is the probability that $h(A) = h(B)$?
- Let's look at the permutation that defines h. We can ignore any index that is 0 in both A and B.
- Look at the first index in the permutation that is 1 in A or B
  - If this index is in *both* A and B, then $h(A) = h(B)$
  - If this index is in only one of A or B, then $h(A) \neq h(B)$
- Any index that is 1 in $A|B$ is equally likely to be first. If the index is in $A\&B$, they hash together; otherwise they do not
- Therefore: probability of hashing together is (number of 1s in $A\&B$)/(number of 1s in $A|B$).

Analysis Example

- Let's say we have $A = \{\text{red, blue, green}\}$ and $B = \{\text{red, orange, purple, green}\}$.

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- If red or green appears before blue, orange, and purple then they hash together
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- Probability that red or green is first out of {red, blue, green, orange, purple} is $2/5$.

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- When do A and B hash together?
- If red or green appears before blue, orange, and purple then they hash together
- If blue or orange or purple appear before red and green, then they don't hash together
- Probability that red or green is first out of {red, blue, green, orange, purple} is $2/5$.
- Therefore, A and B hash together with probability $2/5$.

Making Sure We Find the Close Pair

- To find the close pair, compare all pairs of items that hash to the same value

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Making Sure We Find the Close Pair

- To find the close pair, compare all pairs of items that hash to the same value
  - (We'll talk about how to do this in a moment)
- Let's say our close pair has similarity .5. How many times do we need to repeat?
- Each repetition has the close pair in the same bucket with probability .5. So need 2 repetitions in expectation.

An Aside on Expectation

Lemma

If a random process succeeds with probability p, then in expectation it takes $1/p$ iterations of the process before success.

Examples:

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- We need to roll a 6-sided die 6 times before we see (say) a three.

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Proof: the expectation is

$$\sum_{i=1}^{\infty} ip(1-p)^{i-1} = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = p\frac{1}{(1-(1-p))^2} = \frac{1}{p}.$$

Concatenations and Repetitions

Problems with this Approach

- Buckets are really big!! (After all, lots of items are pretty likely to have a given bit set.)

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- Answer: concatenate multiple hashes together.

Concatenating Hashes

- Rather than one hash h, concatenate k independent hashes $h_1, h_2, \ldots h_k$, each with its own permutation $P_1, P_2, \ldots P_k$.

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  - For each of the k independent hashes, A and B collide with probability $|A \cap B|/|A \cup B|$.
  - We only obtain the same concatenated hashes if *all* of the hashes are the same.
  - They are independent, so we can multiply to obtain probability $(|A \cap B|/|A \cup B|)^k$ of A and B colliding.

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- $P_1 = \{\text{red, green, blue, orange}\}$,$P_2 = \{\text{orange, green, blue, red}\}$,$P_3 = \{\text{red, green, blue, orange}\}$

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- Let's hash A.

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- $P_1 = \{\text{red, green, blue, orange}\}$,$P_2 = \{\text{orange, green, blue, red}\}$,$P_3 = \{\text{red, green, blue, orange}\}$
- Let's hash A.
    - First hash: red is in A.

Concatenation Example

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- Let's hash B.
  - First hash: red is in B.
  - Second hash: orange is in B.
  - Third hash: red is in B.
- Concatenating, we have $h(B) = \text{redorangered}$

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- What kind of values work for k and R?

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- How should we set k? How many repetitions R is it likely to take?

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- We can then solve $(n-1)(1/3)^k = 1$ to get $k = \log_3(n-1)$.

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When dealing with tricky log/exponent expressions, try putting everything as a power of 2.

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- So we expect about $R = n^{.26}$ repetitions. That's a lot!
- But it's not far from the state of the art.
- And way better than brute force!

Finding R and k in general

Let's say we have n points where the close pairs have similarity j_1, and all other pairs have similarity at most j_2

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  - Doable at home: show that this is the optimal value for k using the below analysis.
- Then, number of R we need in expectation is:

$$\left(\frac{1}{j_1}\right)^k = \left(\frac{1}{j_1}\right)^{\log_{1/j_2} n} = n^{\log_{(1/j_2)}(1/j_1)}.$$

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- (Our analysis shows that we'll need to hash all n items $n^{\log_{1/j_2}(1/j_1)}$ times in expectation)

Practical MinHash Considerations

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- So let's say we have $A = \{$black, red, green, blue, orange$\}$, and we're looking at a permutation $P = \{$purple, red, white, orange, yellow, blue, green, black$\}$.
- Then A hashes to redorangeblue

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  - But this works fine in practice (and is used all the time)
- We will do this on the Assignment; in fact I recommend using $\hat{k} = k$. That means that each repetition has only one permutation.
- I think it makes life very significantly easier. In the real world you want a smaller value of $\hat{k}$

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- (Not realistic case, but hard case!)

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- See if the corresponding bit is a 1 in the `Item` we're hashing.
- How can we do this?
- Most efficient way I know is not clever. Just go through each index, and check to see if that bit is set (say by calculating `x & (1 « index)` —but remember that these are 128 bits)

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- Option 2: Treat as bits. `0` to 127 can be stored in 7 bits. Store the hash as a sequence of *k* 8-bit chunks.

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- Why? Having a large number of buckets and/or repetitions leads to bad constants
- Smaller k means fewer buckets, fewer repetitions (but bigger buckets and more comparisons)
- Start with $k \approx \log_3 n$, but experiment with slightly smaller values.

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- The discussion of repetitions in the lecture is for two reasons: 1. analysis, 2. give intuition for the tradeoff by varying k

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- Answer: take the concatenated indices and put them into `MurmurHash`, then take modulo n.
- Then, we are hashing to a number from 0 to $n - 1$!

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- If a similar one is not found, repeat with new minhashes

Assignment 4: Minhashing a vector

We begin by generating a random permutation of the numbers from `0` to 127

We find the minhash of a 128-bit vector x as follows:

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4. Take mod n to get a number from 0 to $n-1$