Lecture 10: Cuckoo Filter Analysis and Streaming

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Williams College

Admin

- I am 90% sure we'll skip "HyperLogLog counting." I'll remove it [illegible] assignment writeup
- Assignment 3 due next Thursday. Goal: finish up Cuckoo filter; implement Count Min Sketch
- No TA hours during reading period. I think I'll be able to still have office hours; I'll send an email the day of.

Implementing Effective Hash Functions

Hashes we need

- h_1 which maps an arbitrary element (a string in Assignment 3) to a slot in the hash table
- f which maps an arbitrary element (a string in Assignment 3) to a number from 1 to 255 (we'll be doing 8-bit fingerprints)
- h which maps a fingerprint from 1 to 255 to a slot in the hash table
- This section is about how we do this: specifically, it's about the function `murmurhash` in the code

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- To calculate $h(i)$, for $i \in \{1, \ldots, 255\}$, just use $\texttt{hashFingerprint}[i-1]$

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- Use mod to get them down to size

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  - MurmurHash returns 128 bits, which don't fit in a word. Hash is just a 128-bit length array where it can store the bits
- We use `hash[0]` for $h_1()$ and `hash[1]` for $f()$

```
uint32_t position = hash[0] % numSlots;
uint32_t fingerprint = 1 + hash[1] % fingerprintMask;
```

Cuckoo Filter Analysis

Union Bound

Theorem

Let X and Y be random events. Then

$$\Pr(X \text{ or } Y) \leq \Pr(X) + \Pr(Y).$$

More generally, if $X_1, X_2, \ldots, X_k$ are any random events, then

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- *Always* works, even for events that are not independent
- Sometimes called "Boole's inequality"

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- Can you upper bound the probability that some student has ID 1?

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This is messy! And it would be even worse if the IDs were not independent!

The union bound lets us avoid this work.

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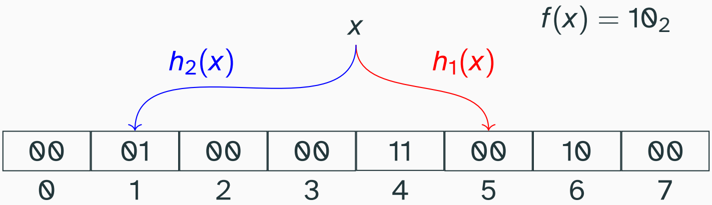
- From Union bound: The probability that *any* student has ID 1 is at most the sum, over all 10 students, of 1/100.
- This gives us an upper bound of $10/100 = 10\%$.

Analysis of Cuckoo Filters



Some assumptions going in (part 1):

- all hash functions h_i are uniformly random: any $x \in U$ is mapped to any hash slot $s \in \{0, \ldots, m-1\}$ with probability $1/m$.

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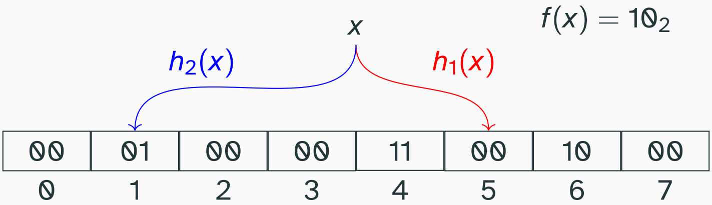

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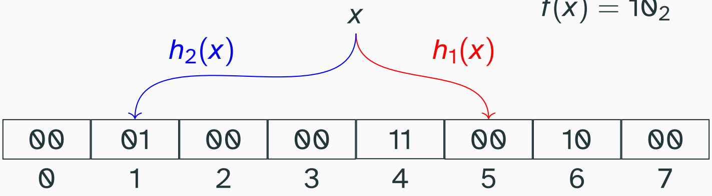
- Same for the fingerprint hash f: any $x \in U$ is mapped to a given fingerprint $f_x \in \{1, \ldots, 1/\varepsilon\}$ with probability ε.

Analysis of Cuckoo Filters

$f(x) = 10_2$



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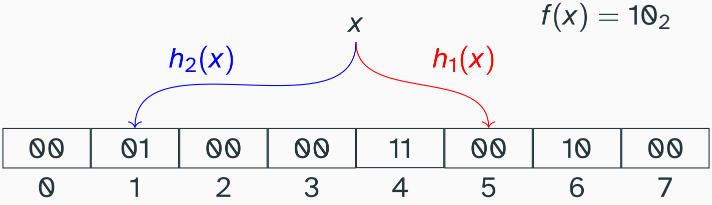

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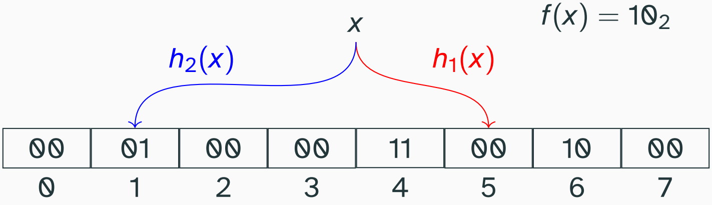

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- We will analyze *without* partial-key cuckoo hashing (we'll assume independent h_1 and h_2)
- We'll analyze with 1 slot per bin, $2n$ total slots. On Assignment 3, you'll do the *same* analysis for the actual cuckoo filter you use (with 4 slots per bin, and $1.05n$ total slots).

First Guarantee: No False Negatives

Guarantee (No False Negatives)

A filter is always correct when it returns that $q \notin S$.
Equivalently, if we query an item $q \in S$, then a filter will always correctly answer $q \in S$.

Invariant

For every $x \in S$, there exists an $i \in \{1, \ldots, k\}$ such that $f(x)$ is stored in $T[h_i(x)]$.

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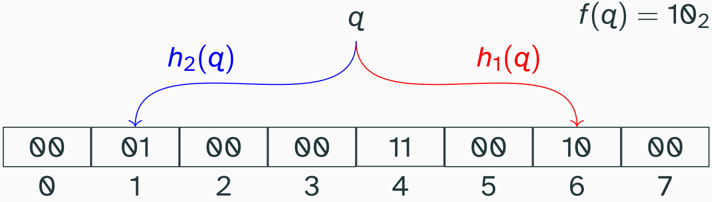
- We can see that the invariant means that there are no false negatives.

Second Guarantee: False Positive Rate



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A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

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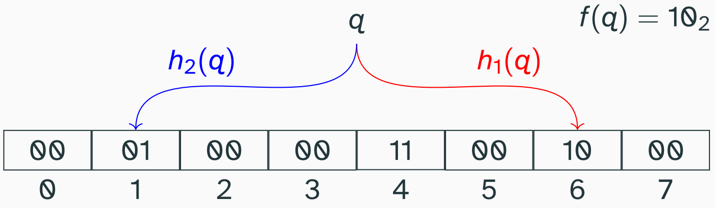

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- A query $q \notin S$ is a false positive if, for some h_i, $T[h_i(q)] = f(q)$.
- Let's examine each hash h_1 and h_2 individually.

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- $1/2$, because we are storing n elements in $2n$ slots.
- If $T[h_1(q)]$ contains a fingerprint, the probability that $f(x) = f(q)$ is ε.
- Therefore, the probability that $T[h_1(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.

Second Guarantee: False Positive Rate

- What about h_2?

Second Guarantee: False Positive Rate

- What about h_2?

- Same exact analysis: probability that $T[h_2(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.

Second: Guarantee: Putting it Together

- q is a false positive if either $T[h_1(q)]$ contains a fingerprint $f(x_1)$ such that $f(x_1) = f(q)$, or $T[h_2(q)]$ contains a fingerprint $f(x_2)$ such that $f(x_2) = f(q)$

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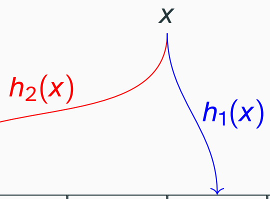
- By union bound, one or the other happens with probability at most $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

Cuckoo Filter Performance



1000	1010	0000	0000	0101	0000	0100	0000
0000	0000	0000	0000	1001	0000	0110	0000
0000	0000	0000	0000	0010	0000	0101	0000
0000	0000	0000	0000	1001	0000	1111	0000
0	1	2	3	4	5	6	7

- Query time?

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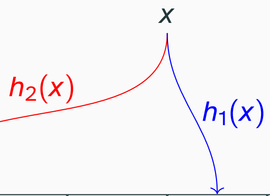

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- Query time?
    - $O(1)$: just check 8 slots for the fingerprint and you're done

Cuckoo Filter Performance

x

$h_2(x)$ $h_1(x)$

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- Insert time? (Recall that Cuckoo Hashing was $O(1)$ in expectation.)

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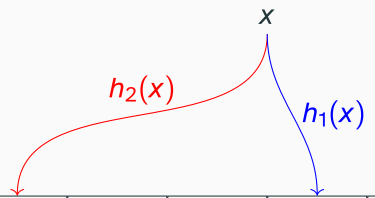

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  - Can we be more specific about this?

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- Answer: some of you might, but I'm guessing many of you would not. You're just going to lose \$1000.

- But expectation is good! You expect to win \$9000.

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- Quicksort has expected runtime $O(n \log n)$. What is the probability that the running time is more than $O(n \log n)$?
- Answer: $O(1/n)$ (this is why quicksort is not worse than merge sort even though it can be $\Theta(n^2)$: you're very unlikely to see a bad case if n is at all large)

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  - Cuckoo hashing inserts finish without looping with high probability

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  - Chaining queries require $O(\frac{\log n}{\log \log n})$ time with high probability. (Don't need to know the $\log \log$ for the midterm.)
- With high probability is always with respect to a variable. Assume that it's with respect to n unless stated otherwise.

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- This is (a simplified version of) the analysis leading to the $O(\log n)$ worst case bounds on the last slide

Expectation vs Concentration (WHP)

- We’ll usually use “with high probability” for concentration bounds

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- Expectation states how well the algorithm does on average. Could be much better or worse sometimes!
- "With high probability" gives a guarantee that will almost always be met: if n is large it becomes vanishingly unlikely that the bound will be violated.
- Largely fulfills the promise of classic worst-case algorithm analysis, but applied to randomized algorithms

Streaming

Really Large Data

- Netflix sends (so far as I can tell) about 300–500TB per minute on average to its customers
- Google's search index has been over 100,000,000 GB for most of a decade
- Brazil Internet Exchange processes 35 trillion bits every second

Really Large Data

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- If is possible to store, can be very difficult to access particular pieces

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- All you can do is sample pieces as they pass by

Streaming Model

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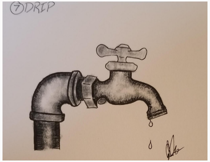
- Stream is incredibly long; you can't store all of the items
- Can't move forward or backward either; just come in one at a time

Streaming Model



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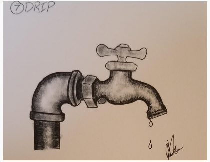

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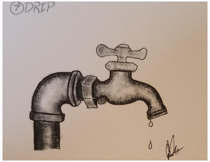

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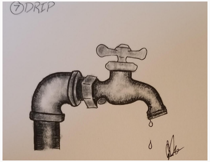

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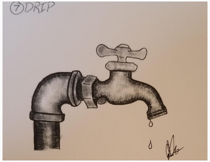

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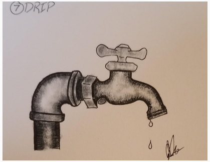

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- Today we'll look at two classic results

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- Today: two data structures
    - Count-min sketch: More aggressive than a filter. Good guarantees for counting how many times a given element occurred in a stream.
    - HyperLogLog: Only uses a few bytes. Estimates how many unique items appeared in the stream.

When to Use Streaming Algorithms?

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    - N/B cache misses

Actual Applications

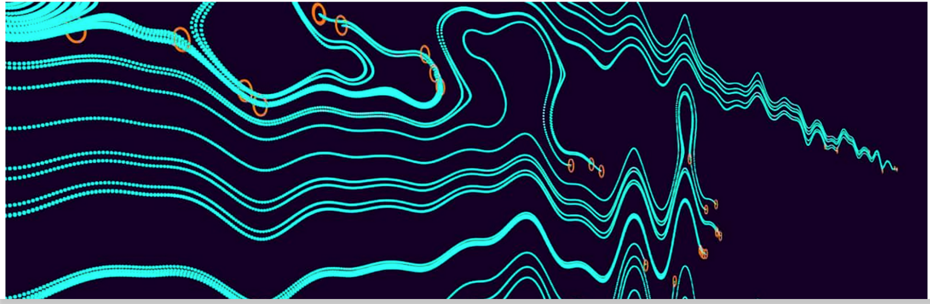
- DDOS attack: keep track of IP addresses that appear too often
- Keep track of popular passwords
- Google uses an improved HyperLogLog to speed up searches
- Reddit uses HyperLogLog to estimate views of a post
- Facebook uses HyperLogLog to estimate number of unique visitors to site.

HyperLogLog at Facebook

FACEBOOK Engineering

POSTED ON DECEMBER 13, 2018 TO DATA INFRASTRUCTURE, OPEN SOURCE

HyperLogLog in Presto: A significantly faster way to handle cardinality estimation



“Doing this with a traditional SQL query on a data set as massive as the ones we use at Facebook would take days and terabytes of memory… With HLL, we can perform the same calculation in 12 hours with less than 1 MB of memory.”

Count-Min Sketch

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Goal:

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Goal:

- Maintain a data structure on a *stream* of items
    - See the items one at a time; you have no control over how they are given to you
    - Want to be *extremely* space efficient
- At any time, estimate how frequently a given item appeared

Example

You see the following items one by one:

adhesive

Example

You see the following items one by one:

flawless

Example

You see the following items one by one:

closed

Example

You see the following items one by one:

adhesive

Example

You see the following items one by one:

describe

Example

You see the following items one by one:

closed

Example

You see the following items one by one:

sea

Example

You see the following items one by one:

illustrious

Example

You see the following items one by one:

describe

Example

You see the following items one by one:

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Example

- Now, answer questions of the form: how many times did some item x_i occur in the stream?

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- Now, answer questions of the form: how many times did some item x_i occur in the stream?

- Example: how many times did adhesive appear? How about closed?
  - (2 times and 3 times respectively)

Formally

- See a stream of elements $x_1, \ldots x_N$, each from a universe U[1]

[1] Like in the last lecture, this is just a requirement to make sure that we can hash them.

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    - Don't depend on N, or $|U|$: so you can upper bound this as $O(\log N)$ space
    - This is asymptotically the same space it takes to store *the answer itself*: a number from 0 to N

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- Keep a hash table with all elements
- Increment a counter each time you see an element
- $O(N)$ space, $O(1)$ time per query
- Pretty efficient! But we want way way less space.

Sketching: A first attempt

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- If an item appears k times in the stream, we record it $k/100$ times in expectation.

Sketching: A first attempt

- If an item appears k times in the stream, we see it $k/100$ times in expectation.
- So, if we wrote an item down w times, we can estimate that it probably occurred $100w$ times in the stream.

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Sketching: A first attempt

What are some downsides to this approach?

- It's pretty loose. If our counter is just one off, that changes our guess by $+100$
- Could have a fairly frequent item that we never write down.
- Can't guarantee much about our estimate

Second attempt: hash counts

- Maintain a hash table A with $1/\varepsilon$ entries, each of at least $1 + \lfloor \log_2 N \rfloor$ bits
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Counters of length $1 + \lfloor \log N \rfloor$ so don't overflow

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Since we always increase this counter when we see $x_i = q$

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But, also increase it when $h(x_i) = h(q)$, but $x_i \neq q$

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- What guarantees does this give?
    - Always *overestimates* the number of occurrences
    - How much does it overestimate by?
    - Each of N items hashes to same slot with probability ε, so $N\varepsilon$ in expectation

Second attempt: hash counts (Analysis)

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Expectation is not that great!

- Let's say we have only two items; A appears 100 times and B appears 900
- What are the possibilities for what happens when we query A?
- With probability $1 - \varepsilon$ we get 100; with probability ε we get 1000

What do we really want?

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- How can you increase the reliability of a random process?

- For example, let's say we're rolling a die. We want to be sure we see a 6 at least once. How can we do that?

- Of course: roll the die many times!

Repetitions

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We'll come back to δ later.

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The e is important for the analysis.

Repetitions

- Rather than having one hash table A, let's have a two-dimensional hash table T
- T has $\lceil \ln(1/\delta) \rceil$ rows
- Each row consists of $\lceil e/\varepsilon \rceil$ slots
- Different hash function for each row

Example Insert

x

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7

Example Insert

x

$h_1(x)$

1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	1	2	3	4	5	6	7

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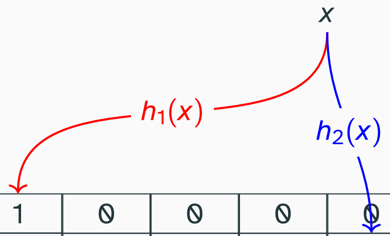

1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0
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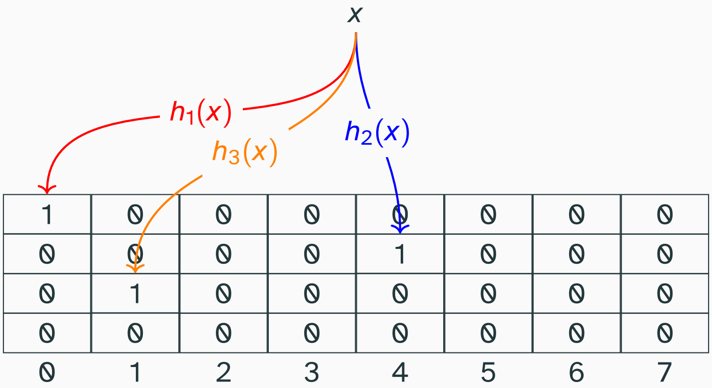

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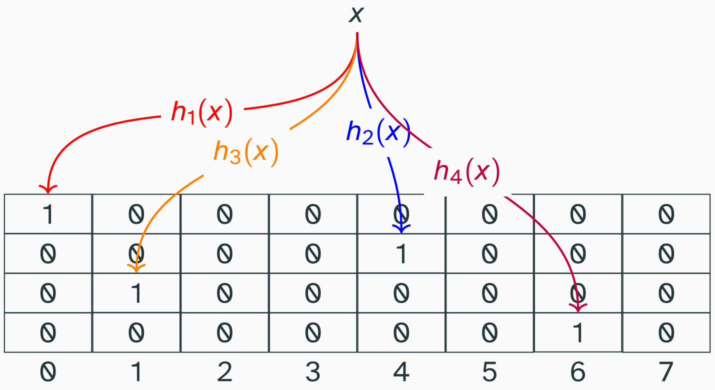

Example Insert

y

1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0
0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0
0	1	2	3	4	5	6	7

Example Insert

y

$h_1(y)$

1	$\emptyset$	$\emptyset$	1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1	$\emptyset$
$\emptyset$	1	2	3	4	5	6	7

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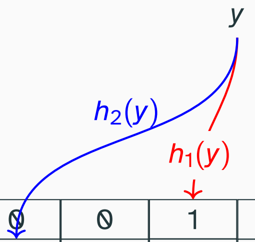

1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	0
0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0
0	1	2	3	4	5	6	7

Example Insert

y

$h_2(y)$ $h_1(y)$ $h_3(y)$

1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	0
0	2	0	0	0	0	0	0
0	0	0	0	0	0	1	0
0	1	2	3	4	5	6	7

Example Insert

y

$h_2(y)$ $h_3(y)$ $h_1(y)$ $h_4(y)$

1	0	0	1	0	0	0	0
0	1	0	0	1	0	0	0
0	2	0	0	0	0	0	0
0	0	0	0	0	0	1	1
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Inserts

To insert x_i:

- For $j = 0 \ldots \lceil \ln(1/\delta) \rceil - 1$:

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- For $j = 0 \ldots \lceil \ln(1/\delta) \rceil - 1$:
  - Increment $T[j][h_j(x_i)]$

We now have $\lceil \ln(1/\delta) \rceil$ independent counters for each item. How can we query?

Example Query

q

28	10	78	9	26	69	39	28
85	40	52	70	11	84	65	99
56	82	34	75	99	35	14	55
10	20	17	80	92	89	71	13
0	1	2	3	4	5	6	7

Example Query

q

$h_1(q)$

28	10	78	9	26	69	39	28
85	40	52	70	11	84	65	99
56	82	34	75	99	35	14	55
10	20	17	80	92	89	71	13
0	1	2	3	4	5	6	7

Example Query

q

$h_2(q)$ $h_1(q)$

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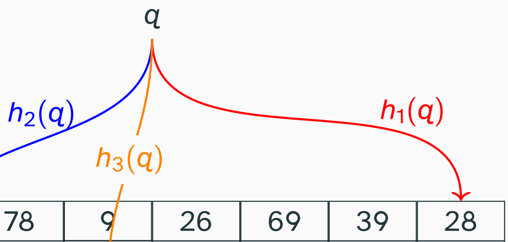
0	1	2	3	4	5	6	7

Example Query



28	10	78	9	26	69	39	28
85	40	52	70	11	84	65	99
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$h_2(q)$ $h_3(q)$ $h_4(q)$ $h_1(q)$

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We have 4 numbers for q: 28, 40, 75, 71. In pairs: what do we know about each of these numbers? How can we combine them into a single answer to the query?

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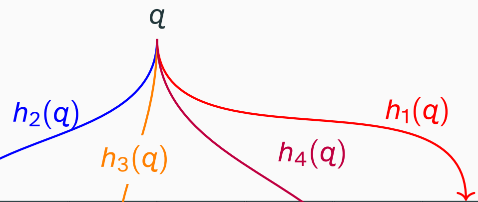

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We have 4 numbers for q: 28, 40, 75, 71. In pairs: what do we know about each of these numbers? How can we combine them into a single answer to the query?

Answer: each is an overestimate; take the *min*. It must be the closest to the true answer!

Queries

Each entry is an *overestimate*.

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- Find $\min_j T[j][h_j(x_i)]$.

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- Table T with $\lceil \ln(1/\delta) \rceil$ rows, each with $\lceil e/\varepsilon \rceil$ columns. Cells of size $1 + \lfloor \log N \rfloor$

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- To query q: return $\min_{j \in \{0, \ldots, \lceil \ln(1/\delta) \rceil - 1\}} T[j][h_j(q)]$

Count-Min Sketch Guarantee: Lower bound

- On query q, let's say the filter returns that there were o_q occurrences
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- First: always have $\widehat{o_q} \leq o_q$.

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- Can we get *concentration bounds?*

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- Let's prove this with $C = 2$ on the board.

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- In a *given row*, we are at most εN over with probability $1/e$

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$$\left(\frac{1}{e}\right)^{\#\text{ rows}} = \left(\frac{1}{e}\right)^{\lceil \ln 1/\delta \rceil} \leq \delta$$

Count-Min Sketch Bounds

- $\left\lceil \frac{e}{\varepsilon} \right\rceil \left\lceil \ln \frac{1}{\delta} \right\rceil (1 + \lfloor \log_2 N \rfloor)$ bits of space

- For any query q, if the filter returns o_q and the actual number of occurrences is $\widehat{o_q}$, then with probability $1 - \delta$:

$$\widehat{o_q} \leq o_q \leq \widehat{o_q} + \varepsilon N.$$

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- 32-bit counters (a little wasteful!)
- 7.3MB of data summarized in 4.8KB
- Really accurate still: in 1.2 million word stream, can estimate num occurrences of each word within ± 1500
- Often more accurate! Also: feel free to try 1000 or 10000 entries per row; it gets quite accurate

Hyper Log Log Counting

Setting up

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- Count-min sketch takes up a lot of space!
- OK not really. But, it stores a lot of information about the stream
- Common question: how many *unique* elements are there in the stream?

The problem we're trying to solve

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- (Compare to CMS: stores approximately how many there are of *each* element)

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The problem we're trying to solve

- Stream of N elements
- Approximate number of unique elements
- To do this exactly: need dictionary of all elements we've already seen.
- How can you count unique elements approximately? Challenge: don't want to double-count when we see an element twice.

Cool way to solve this

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- Idea: let's look at a rare event in these hashes. The more often it happens, the more *distinct hashes* (and thus distinct items) we must be seeing!

Cool way to solve this

- Let's hash each item as it comes in

- Then instead of a list of items, we get a list of random hashes

- Idea: let's look at a rare event in these hashes. The more often it happens, the more *distinct hashes* (and thus distinct items) we must be seeing!

- In particular: how many 0s does each hash end with?

Hashes ending in 0s

- What is the probability that a hash ends in ten `0`’s?

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- If we have $2^{10} = 1024$ distinct elements, it's pretty likely that the hash of one will end with 10 0's!
- Note "distinct!" All of this comes back to estimating how many *unique* elements there are. Unique elements give a new hash, and a new opportunity for many zeroes. Non-unique elements don't give a new hash.

Example

You see the following hashes one by one:

0010000101010001

Example

You see the following hashes one by one:

0010110010111101

Example

You see the following hashes one by one:

0001000011101111

Example

You see the following hashes one by one:

0000001011000011

Example

You see the following hashes one by one:

0110010010011100

Example

You see the following hashes one by one:

1000101011100001

Example

You see the following hashes one by one:

0110100100111101

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You see the following hashes one by one:

0011101001100010

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You see the following hashes one by one:

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Example

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How many unique items were there?

Example 2

You see the following hashes one by one:

0010001010101001

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How many unique items were there? Was it more or less than the last one?

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- Answer: 1st had 14 items, 2nd had 3
- Notice that only one hash in the second example ended with 0
  - Extremely unlikely if there were 14 different elements!
- One of the items in the first example ended with 4 0's
  - Unlikely if there were 3 elements!

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- So it seems that, there are probably something like 2^k items
- But: if we're just off by 1 or 2 zeroes, that affects our answer by a lot! (We don't get good *concentration bounds*)

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- How do we improve the consistency of a random process?

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- Answer: It's complicated. (And the rationale is outside the scope of the course.)

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- **Make sure to add 1 to your count of the number of zeroes**

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- Return bm^2/Z.

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index $= 110$ Remaining: $010001000111110111111101010$

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The remaining hash ends with 1 zero, so we want to store 2. The counter stores less than 2, so we store it.

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index $= 110$ Remaining: $01111000110010000111101001 0$

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000	001	010	011	100	101	110	111

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The remaining hash ends with 1 zero, so we want to store 2. The counter stores 2, so we keep it as-is.

Example (with $m = 8$; in practice m is higher)

x_3

Example (with $m = 8$; in practice m is higher)

$$x_3$$

$$h(x_3) = 110011011101100000011010000001$$

Example (with $m = 8$; in practice m is higher)

$$x_3$$

$$h(x_3) = 110011011101100000011010000001$$

index = 001 Remaining: 110011011101100000011010000

0	0	0	0	0	0	2	0
000	001	010	011	100	101	110	111

Example (with $m = 8$; in practice m is higher)

$$x_3$$

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index $= 001$ Remaining: $110011011101100000011010000$

000	001	010	011	100	101	110	111
0	0	0	0	0	0	2	0

Example (with $m = 8$; in practice m is higher)

x_3

$h(x_3) = 1100110111011000000011010000001$

index $= 001$ Remaining: $1100110111011000000011010000$

0	0	0	0	0	0	2	0
000	001	010	011	100	101	110	111

The remaining hash ends with 4 zeroes, so we want to store 5. The counter stores 0, so we store 5 in the slot.

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index $= 001$ Remaining: 100010011101101110110110111

000	001	010	011	100	101	110	111
0	5	0	0	0	0	2	0

The remaining hash ends with 0 zeroes, so we want to store 1. The counter stores 5, so we keep the slot as-is.

Example (with $m = 8$; in practice m is higher)

x_4

$h(x_4) = 100010011101101110110110111001$

index $= 001$ Remaining: $100010011101101110110110111$

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index $= 110$ Remaining: $011110001100100001111010010110$

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index $= 110$ Remaining: $011110001100100001111010010110$

0	5	0	0	0	0	2	0
000	001	010	011	100	101	110	111

The remaining hash ends with 1 zero, so we want to store 2. The counter stores 2, so we keep it as-is.

At the end of the day

Have an array of counters:

0	5	0	0	0	0	2	0
000	001	010	011	100	101	110	111

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000	001	010	011	100	101	110	111

- Sum up $(1/2)^{M[j]}$ across all $j = 0$ to $m - 1$; store in Z
- Return bm^2/Z. Here $m = 8$. We would have to look up the value of b for 8. (No one does HyperLogLog with 8)

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Discussion

- How big do our counters need to be?
- Need to be long enough to count the longest string of 0s in any hash
- Size $> \log\log(\text{number of distinct elements})$ (hence the *loglog* in the name)
- 8-bit counters are good enough, so long as the number of elements in your stream is less than the number of particles in the universe
- Note: one thing to be careful of is hash length. But 64 bit hashes should be good enough for any reasonable application (and 32 bits is usually fine)

HLL in the Assignment

- We'll use $m = 32$ counters
- Bias constant is .697

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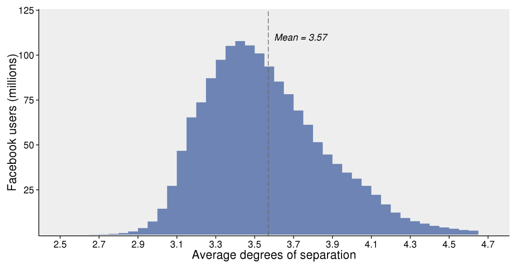
- Google developed HyperLogLog++ to help deal with these problems
- Other known improvements as well

One More Cool Thing



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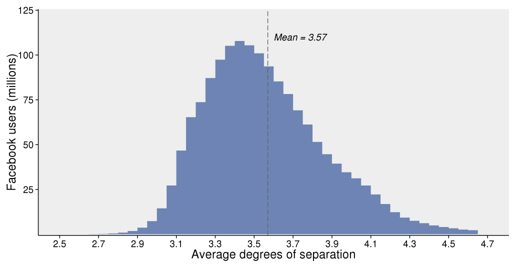

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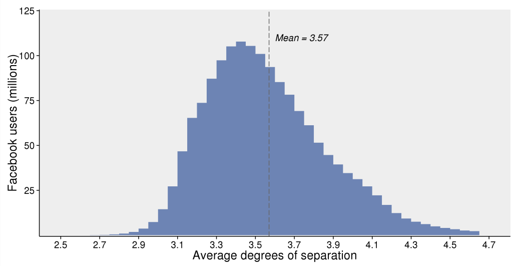

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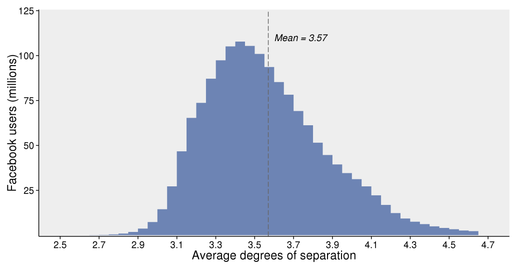

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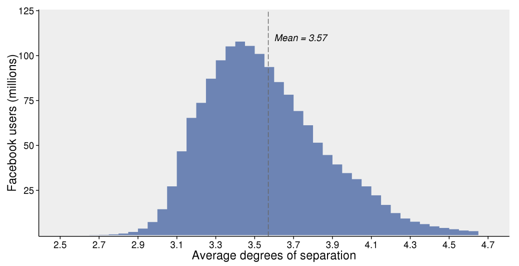

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- Usually takes $O(n^2)$ time!
- Theirs is essentially linear time, gives extremely accurate results

Hash Functions in Practice

What do we want out of a hash function?

Of course, we want consistency (each time we hash an item we get the same result back). What else might we want?

- Fast
- Low space requirements (i.e. may need to store a seed; don't want that to be too big)
- Good collision avoidance
- Bear in mind: different hashes work on different types of elements. We'll focus on integers and strings (especially strings)

Ideal Hash Functions (Independent, Uniform Hashing)

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Ideal Hash Functions (Independent, Uniform Hashing)

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- But: require extremely large space usage unless universe of possible elements is extremely small
- You did use one of these…
  - For h on Assignment 3! Those values were all chosen independently, completely at random

Hashing in Java

- Anyone know how Java hashes a 64 bit Long?
- `return x ^ (x >> 32);`
- Advantages of this?

Is this good for:

- In cuckoo filter: h_1, h, f?
  - h_1 and f: *might* work if elements are fairly well-spread (we take mod); usually won't
  - h: probably won't work (output too small)
- CMS? HLL?
  - CMS might be OK; prob not (same as above)
  - HLL likely useless unless elements *very* uniformly spread

Multiply-Shift Hashing

```
uint64_t hash3(uint64_t value){
    return (uint64_t)(value * 0x765a3cc864bd9779) >> (64 -
        SHIFT);
}
```

- Seed is a large prime number to multiply by; can also add a large random prime
- Advantages?
    - Fast! (And easy.)

Multiply-Shift Hashing

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```

- How good is it?
  - Pretty good! For any x, y, $\Pr[h(x) = h(y)] = 1/n$.
  - But unfortunately behavior doesn't extend to larger numbers of elements.
- Let's say we use this for a hash table with chaining (n items, n chains). What is the expected number of elements we find during a query q?
- $X_i = 1$ if $h(x_i) = h(q)$. Then $E[X_i] = 1/n$. By linearity of expectation, total number of items is $\sum_{i=1}^{n} 1/n = 1$.
- How big do you think the largest bucket is?
  - Best known bound: $O(n^{1/3})$ [Knudsen 2017] (very bad!!)

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- Hyperloglog?
  - Would have to try but I would very much suspect it would not work well at all

Murmurhash

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    - Rotate is like shift, but bits that “fall off” are replaced on other side
    - Can be implemented with two shifts and an OR

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- Uses repeated MUltiply and Rotate operations
  - Rotate is like shift, but bits that “fall off” are replaced on other side
  - Can be implemented with two shifts and an OR
- Code isn’t exactly short; 50 operations to hash a number

Murmurhash Code

```
for(i = -nblocks; i; i++)
{
  uint32_t k1 = getblock(blocks,i*4+0);
  uint32_t k2 = getblock(blocks,i*4+1);
  uint32_t k3 = getblock(blocks,i*4+2);
  uint32_t k4 = getblock(blocks,i*4+3);

  k1 *= c1; k1  = ROTL32(k1,15); k1 *= c2; h1 ^= k1;

  h1 = ROTL32(h1,19); h1 += h2; h1 = h1*5+0x561ccd1b;

  k2 *= c2; k2  = ROTL32(k2,16); k2 *= c3; h2 ^= k2;

  h2 = ROTL32(h2,17); h2 += h3; h2 = h2*5+0x0bcaa747;

  k3 *= c3; k3  = ROTL32(k3,17); k3 *= c4; h3 ^= k3;

  h3 = ROTL32(h3,15); h3 += h4; h3 = h3*5+0x96cd1c35;

  k4 *= c4; k4  = ROTL32(k4,18); k4 *= c1; h4 ^= k4;

  h4 = ROTL32(h4,13); h4 += h1; h4 = h4*5+0x32ac3b17;
}
```

Murmurhash Code

```
switch(len & 15)
{
case 15: k4 ^= tail[14] << 16;
case 14: k4 ^= tail[13] << 8;
case 13: k4 ^= tail[12] << 0;
         k4 *= c4; k4  = ROTL32(k4,18); k4 *= c1; h4 ^= k4;

case 12: k3 ^= tail[11] << 24;
case 11: k3 ^= tail[10] << 16;
case 10: k3 ^= tail[ 9] << 8;
case  9: k3 ^= tail[ 8] << 0;
         k3 *= c3; k3  = ROTL32(k3,17); k3 *= c4; h3 ^= k3;
-- 12 lines: case  8: k2 ^= tail[ 7] << 24;------------------------------
};
--  4 lines: ----------------------------------------------------------------
h1 ^= len; h2 ^= len; h3 ^= len; h4 ^= len;

h1 += h2; h1 += h3; h1 += h4;
h2 += h1; h3 += h1; h4 += h1;

h1 = fmix32(h1);
h2 = fmix32(h2);
h3 = fmix32(h3);
h4 = fmix32(h4);

h1 += h2; h1 += h3; h1 += h4;
h2 += h1; h3 += h1; h4 += h1;
```

(The light grey lines skip pieces of code.)

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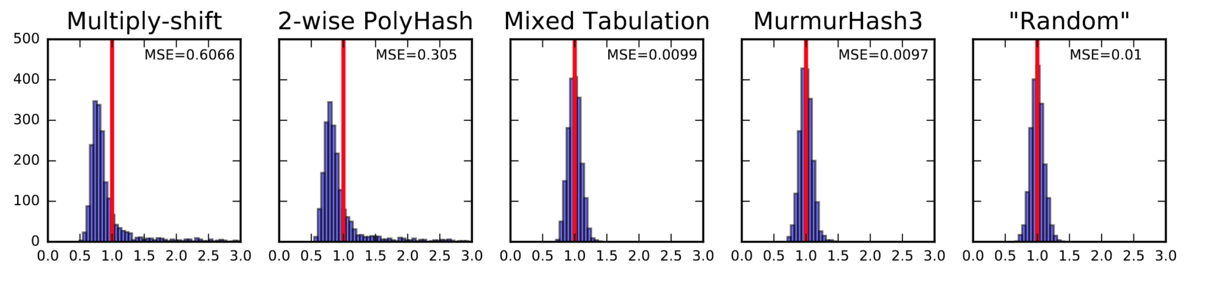
- Someday may discover: might not work well in some circumstances
- This is what happened to Murmurhash2:
    - “Will this flaw cause your program to fail? Probably not - what this means in real-world terms is that if your keys contain repeated 4-byte values AND they differ only in those repeated values AND the repetitions fall on a 4-byte boundary, then your keys will collide with a probability of about 1 in $2^{27.4}$ instead of 2^{32}. Due to the birthday paradox, you should have a better than 50% chance of finding a collision in a group of 13115 bad keys instead of 65536.”
    - `https://sites.google.com/site/murmurhash/murmurhash2flaw`

Murmurhash3 Performance



Average of square of bucket sizes. Data is an intentionally bad (albeit reasonable) case

From “Practical Hash Functions for Similarity Estimation and Dimensionality Reduction” by Dahlgaard, Knudsen, Thorup NeurIPS 2017

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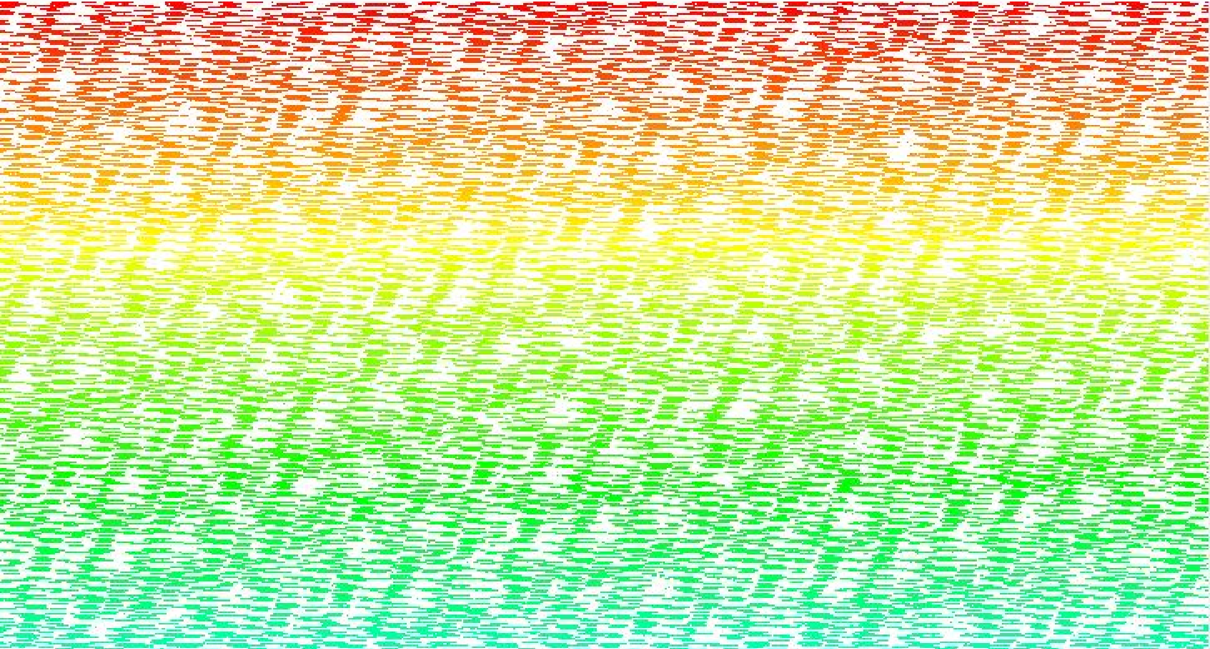
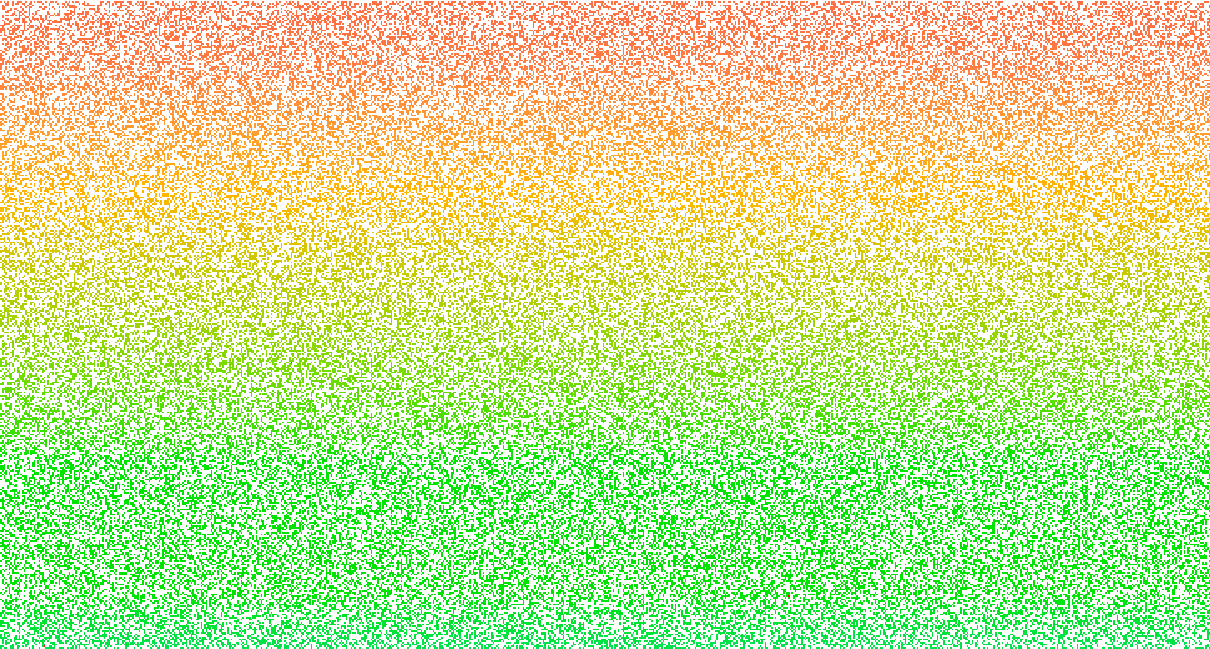
- Compare SDBM (another popular hash) with Murmurhash2; fill in pixel if corresponding table entry is hashed to

SDBM (lots of chunks of full cells!)



Murmurhash2 (visually: random)



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- Is there anything special about this specific sequence, or will any such set work pretty well?
- Answer: others might not work. Example: “SuperFastHash” also uses multiplies and rotates

Hash comparison

```
Hash             Lowercase        Random UUID  Numbers
=============    =============    ===========  =============
Murmur                145 ns      259 ns            92 ns
                        6 collis    5 collis         0 collis
SDBM                  148 ns      484 ns            90 ns
                        4 collis    6 collis         0 collis
SuperFastHash         164 ns      344 ns           118 ns
                       85 collis    4 collis     18742 collis
```

SuperFastHash has bad performance on lowercase English words, and horrendous performance on numbers-as-strings.

(Also from `https://softwareengineering.stackexchange.com/questions/49550/which-hashing-algorithm-is-best-for-uniqueness-and-speed`)

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- Broken: MD5, SHA-1, many others

(Source: `https://shattered.io`)