Lecture 9: Probability and Hashing

Sam McCauley

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Williams College

Admin

- Apply to be a TA in the Spring (you should have gotten an email)
- Assignment 1 deadline Saturday
- Any lingering Assignment 1 questions?
- Homework 2 back
- Homework 3 out tonight
- Maybe short day today? We'll wrap up probability analysis and some other lingering topics

Useful formulas for probability ($e = 2.71\ldots$)

Two useful approximations for simplifying exponents (presented as inequalities, but really quite tight even for moderate n):

$$(1 + 1/n)^n \leq e \qquad\qquad (1 - 1/n)^n \leq 1/e$$

Example: $(1.1)^{10} = 2.593\ldots$

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With probability we often use choose (a.k.a. binomial) notation, but it's unweidly. These inequalities can help approximate it:

$$\left(\frac{x}{y}\right)^y \leq \binom{x}{y} \leq \left(\frac{ex}{y}\right)^y$$

Example: $\binom{n}{10} = \Theta(n^{10})$

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- Since each card is a club with probability (about) $1/4$, and we draw 4 cards, it seems like S should generally be around 1. Can we formalize this intuition?

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- This algorithm takes $O(n \log n)$ time in expectation.

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- It is a weighted average of the outcomes

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- $\mathrm{E}[X] = \frac{20 \cdot 21}{2 \cdot 20} = 10.5$
- So you’ll win \$.50 on average; you should probably play the game

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- Example: let X_1 denote the number of heads on my first coin flip, and X_2 denote the number of heads on my second coin flip. These are independent.
- But: let X_H denote the number of heads I flip over k coin flips, and X_T denote the number of tails. These are not independent.

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- True even if the X_i are not independent!!!

Using Linearity of Expectation

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- Let's figure this out on the board using linearity of expectation
- $X =$ number of heads I see in 100 flips.
- $$X_i = \begin{cases} 1 & \text{if the } i\text{th flip is heads} \\ 0 & \text{otherwise} \end{cases}$$
- So then $X = X_1 + X_2 + \ldots X_{100}$
- We can see that $\mathrm{E}[X_i] = 1/2$.
- $\mathrm{E}[X] = 50$ by linearity of expectation

Linearity of Expectation Example 2

- Let’s say we want to Bubble Sort an array *A*, where *A* is randomly permuted

```
bubbleSort(A : list of sortable items)
    n = length(A)
    do
        swapped = false
        for i = 1 to n-1 do
            if A[i-1] > A[i] then
                swap(A[i-1], A[i])
                swapped = true
            end if
        end for
    while not swapped
```

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- Rephrasing: how many inversions are there in a randomly-permuted array?

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    - Reason: know $a < b < c$, but $A[a] > A[b]$ and $A[b] > A[c]$, so $A[a] > A[c]$.

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- $\mathrm{E}[X] = n(n-1)/4$

Cuckoo Hashing

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- Hashing: way to implement a dictionary with constant-time insert, delete, lookup
- Hashing is randomized, so performance can be bad sometimes
- Cuckoo *hashing*: $O(1)$ worst-case lookup. (Inserts are usually constant-time, but can be expensive sometimes.)

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- Cuckoo hashing is widely used and highly relevant to the course anyway

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- How fast can we do inserts and queries? How much space do we need?
    - Can get $O(1)$ expected time for both operations using $O(n)$ space.

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- Assume that M is much bigger than the number of items in our dataset. (Like $M = 2^{64}$)

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- What's the problem with this approach?

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- Disadvantages?
  - Cache inefficient

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- $\mathrm{E}[X_i^j] = 1/cn$
- $\mathrm{E}[X^j] = \sum_{i=1}^{n} \mathrm{E}[X_i^j] = 1/c$
- So the expected length of the chain is $O(1 + 1/c) = O(1)$

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  - Not that efficient; performance is terrible if table fills up

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    - Linear probing: more cache-efficient, but both inserts and queries are only $O(1)$ on average

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c			x		u
0	1	2	3	4	5

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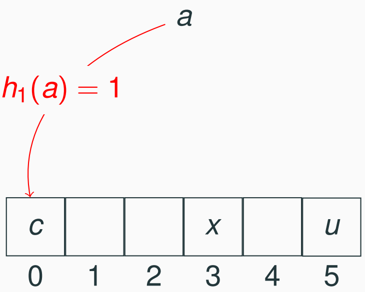

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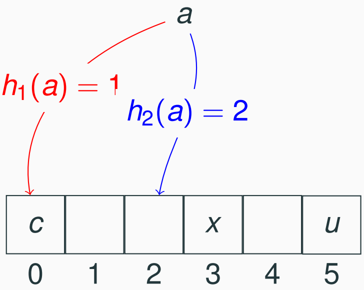

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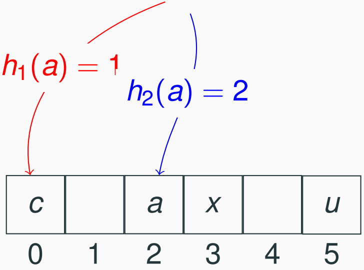

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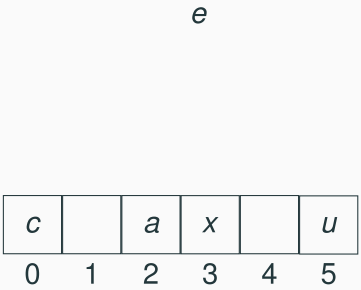

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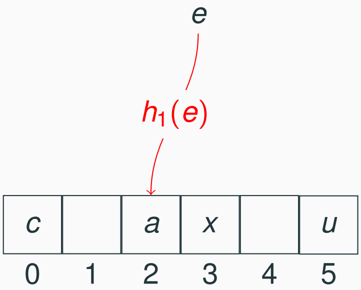

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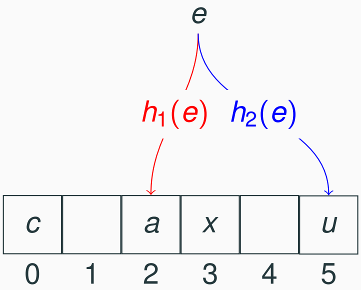

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Cuckooing!

Cuckoos kick other birds' eggs out of the nest, replacing them with their own.

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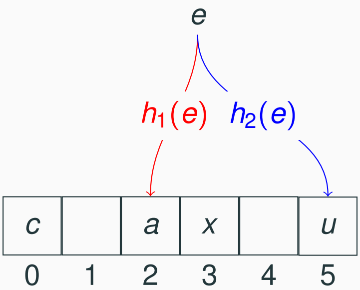

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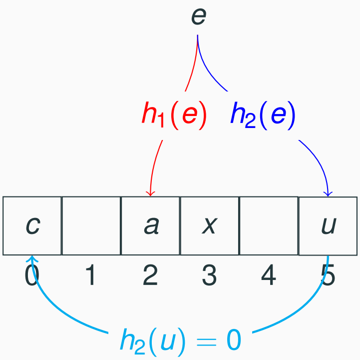

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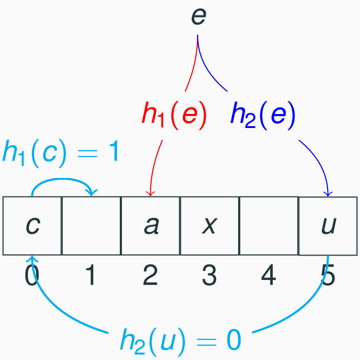

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- Answer: since we are storing a dictionary, we have access to the item x *itself*
    - In the filter we only stored $f(x)$ rather than x
- So we can just rehash it!

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    - So this happens with probability $\Theta(n^3 m^2 / m^6) = \Theta(1/n)$.
    - Probability of *any* impossible configuration is $O(1/n)$ (outside the scope of the course)

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- Idea: cuckoo hashing does great on queries (though with potentially worse cache efficiency than linear probing), but pays for it with expensive inserts

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- Answer: maybe, but probably not. You’re just going to lose $1000.
- But expectation is good! You expect to win $9000.

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- Answer: $O(1/n)$ (this is why quicksort is not worse than merge sort: you'll never see the worst case in your life if n is at all large)

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- Linear probing queries require $O(\log n)$ time with high probability. (Contrast to $O(1)$ in expectation!)
- With high probability is always with respect to a variable. Assume that it's with respect to n unless stated otherwise.

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- So I need $1/2^k = O(1/n)$. Solving, $k = \Theta(\log n)$.

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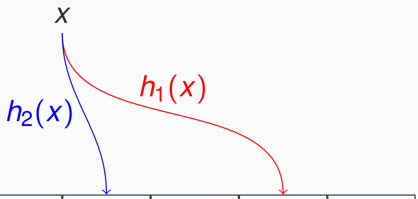
- With high probability gives a guarantee that will almost always be met: if n is large it becomes vanishingly unlikely that the bound will be violated.

Quick Cuckoo Filters Review

Cuckoo Filter

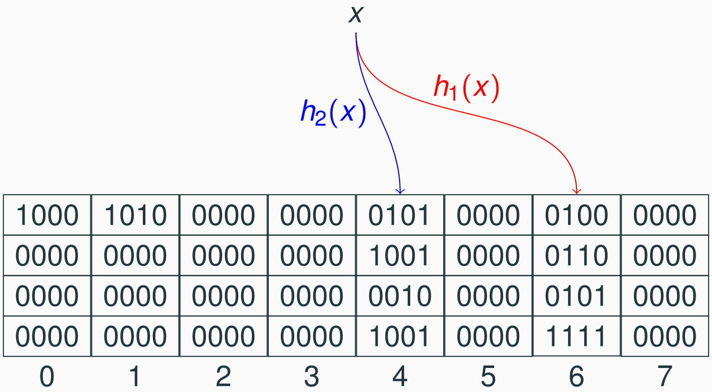


1000	1010	0000	0000	0101	0000	0100	0000
0000	0000	0000	0000	1001	0000	0110	0000
0000	0000	0000	0000	0010	0000	0101	0000
0000	0000	0000	0000	1001	0000	1111	0000
0	1	2	3	4	5	6	7

A cuckoo filter with fingerprints of length 4, $k = 2$, and 4 slots per bin.

Reminder: how do we calculate $h_2(x)$?

Cuckoo Filter



1000	1010	0000	0000	0101	0000	0100	0000
0000	0000	0000	0000	1001	0000	0110	0000
0000	0000	0000	0000	0010	0000	0101	0000
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Reminder: what happens if all slots store a value > 0?

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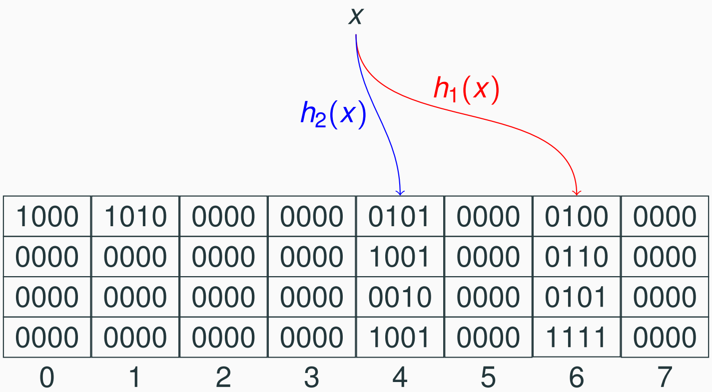

1000	1010	0000	0000	0101	0000	0100	0000
0000	0000	0000	0000	1001	0000	0110	0000
0000	0000	0000	0000	0010	0000	0101	0000
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A cuckoo filter with fingerprints of length 4, $k = 2$, and 4 slots per bin.

Please note: when choosing what slot to cuckoo out of, you cannot always use the same slot! Easy solution: increment every time.

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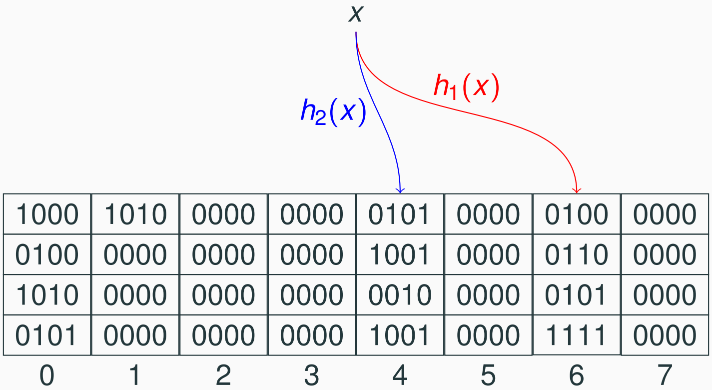

0	1	2	3	4	5	6	7
1000	1010	0000	0000	0101	0000	0100	0000
0100	0000	0000	0000	1001	0000	0110	0000
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Remember to rotate what slot in a bit you cuckoo out of.

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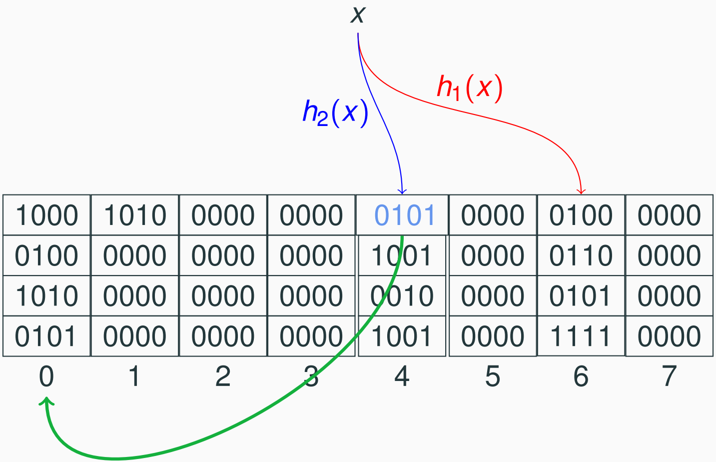

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1000	1010	0000	0000	0101	0000	0100	0000
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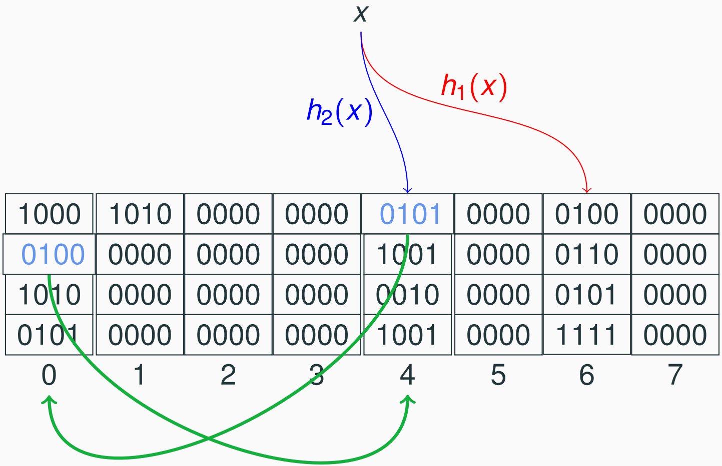

1000	1010	0000	0000	0101	0000	0100	0000
0100	0000	0000	0000	1001	0000	0110	0000
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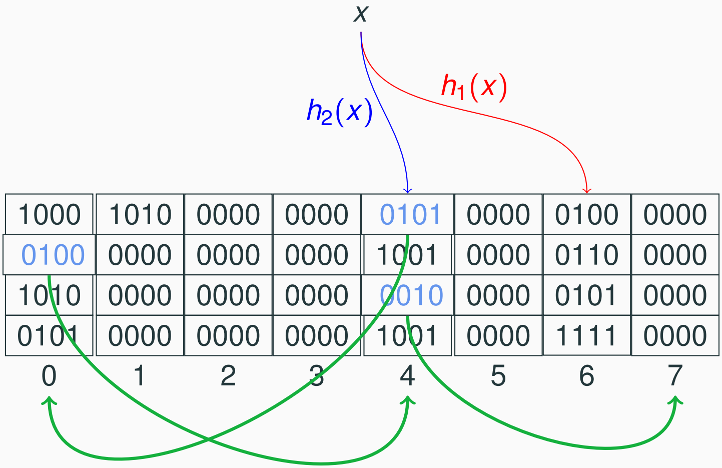

0	1	2	3	4	5	6	7
1000	1010	0000	0000	0101	0000	0100	0000
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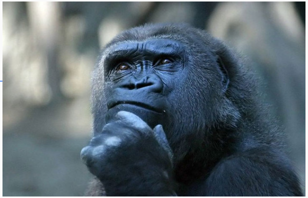
Choosing Hash Functions in Practice

Choosing a hash function



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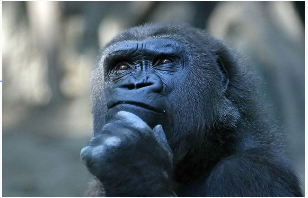

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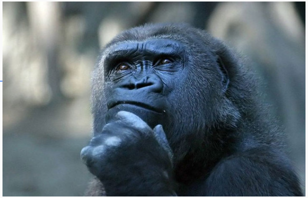

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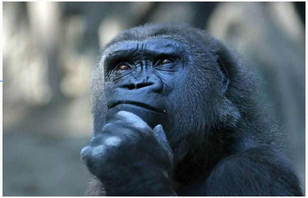

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- Random enough that different elements usually don’t hash together.

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- $h(x) = x \wedge (x >> 32)$ (the $\wedge$ means XOR)
- Is this going to work well for a filter or dictionary?
- No: if $x < 2^{32}$ then $h(x) = x$!

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  - Big problem for cuckoo hashing/cuckoo filters/etc.

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- We'll look at this in detail the lecture after next.

Have a great weekend!