Applied Algorithms Lec 5: Hirshberg's Algorithm

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September 20, 2024

Williams College

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- I have to try to get some work done but you can come and work and I can answer questions
- Wednesday will stay 2-5 as before

Admin: Homeworks

- Homework 1 in. How was it?
- Some really cool ideas! We'll talk about some of them next week.
- Homework 2 is out
  - It is probably the most difficult homework this semester (not because it's complicated per se—it's recursive, which makes it hard to debug, and off-by-1s are very consequential)
  - Start early (!)
  - I took out a question from last time the course was taught so it should be a touch shorter
- For what it's worth: Homeworks 3 and 4 are perhaps the easiest; so things will ease up a bit in a couple weeks

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- Please don’t take them out of the lab so other students can use them

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- More external memory at the end if we have time
- Monday: mostly focus on reviewing Homework 1 and going over some `gcc` features; perhaps another external memory model example (lighter day in terms of concepts)

External Memory Wrapup

Let’s revisit `sortedlinkedlist.c` and `unsortedlinkedlist`

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- What is the cost of our algorithm in the external memory model if the items have stride $B + 1$?
- Answer: $O(n)$
- The external memory model predicts the real-world slowdown of this process.
- (Actual performance is *worse* in this case: we get a slowdown of ≈ 30, whereas the number of nodes in a cache line is 8. I imagine that this is due to prefetching; seem to be some further optimizations internally.)

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- Running time is almost as good as `sortedlinkedlist.c`
- The linked list *stays in cache*. So it is cheap to access!

Homework 2: Hirschberg's Algorithm

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- True even though the space-efficient approach does extra computations!

Edit Distance

- Minimum number of inserts/deletes/replaces to get from one string to another
- Useful in comp bio. Classic dynamic programming solution.

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Delete C

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Replace E with A

OCURRANCE

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Recursive edit distance (building up to D.P.)

- Base case: if X has length 0, what is the edit distance between X and some string Y?

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    - Length of Y

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- If the last characters of X and Y match, what is $ED(X, Y)$?
  - If X' and Y' are X and Y respectively with the last character removed, then $ED(X, Y) = ED(X', Y')$

OCCURRA**N**

OCCURRE**N**

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  - **Replace:** $1 + ED(X', Y')$
  - **Insert:** $1 + ED(X', Y)$ (Insert the last character of X into Y. The characters of Y must match the remaining characters of X)

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- Min of three options: (X' and Y' are X and Y with one character removed)
  - **Replace:** $1 + ED(X', Y')$
  - **Insert:** $1 + ED(X', Y)$ (Insert the last character of X into Y. The characters of Y must match the remaining characters of X)
  - **Delete:** $1 + ED(X, Y')$ (delete the last character of Y; match the rest to X)

OCCURR**A**

OCCURR**E**

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- Let $m = |X|$, $n = |Y|$.
- Build an $n + 1 \times m + 1$ table
  - ($+1$s are so we can have 0-length entries)
- Fill out the table row-by-row using our recursive method (doing lookups instead of recursive calls)

Example DP execution

		O	C	C	U	R	R	E	N	C	E
	0	1	2	3	4	5	6	7	8	9	10
O	1	0	1	2	3	4	5	6	7	8	9
C	2	1	0	1	2	3	4	5	6	7	8
U	3	2	1	1	1	2	3	4	5	6	7
R	4	3	2	2	2	1	2	3	4	5	6
R	5	4	3	3	3	2	1	2	3	4	5
A	6	5	4	4	4	3	2	2	3	4	5
N	7	6	5	5	5	4	3	3	2	3	4
C	8	7	6	6	6	5	4	4	3	2	3
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- $O(mn)$ time (to fill out a table entry just need to look in three other table slots)

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- Probably not by more than log factors
- [Backurs Indyk 2014]: if you can solve edit distance in less than $O(nm)$ time, you can solve 3SAT in less than 2^n time

Edit Distance Cannot Be Computed in Strongly Subquadratic Time (unless SETH is false)

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ABSTRACT

The edit distance (a.k.a. the Levenshtein distance) between

with many applications in computational biology, natural language processing and information theory. The problem of computing the edit distance between two strings is a classical

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- Optimal # cache misses required to fill out that table

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O	1	0	1	2	3	4	5	6	7	8	9
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U	3	2	1	1	1	2	3	4	5	6	7
R	4	3	2	2	2	1	2	3	4	5	6
R	5	4	3	3	3	2	1	2	3	4	5
A	6	5	4	4	4	3	2	2	3	4	5
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- $O(\frac{n+m}{B})$: the only cache misses are from reading in the strings!
- WAY better than $O(\frac{mn}{B})$!

Takeaway: Improved Space Can Imply Improved Cache Efficiency

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- Actually not so bad: follow the path back!

Recovering the edits

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R	4	3	2	2	2	1	2	3	4	5	6
R	5	4	3	3	3	2	1	2	3	4	5
A	6	5	4	4	4	3	2	2	3	4	5
N	7	6	5	5	5	4	3	3	2	3	4
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- How can we tell where each entry came from?

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O	1	0	1	2	3	4	5	6	7	8	9
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U	3	2	1	1	1	2	3	4	5	6	7
R	4	3	2	2	2	1	2	3	4	5	6
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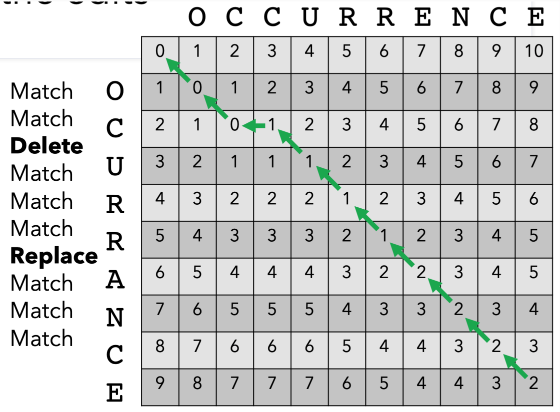
- Redo same min computation from the normal dynamic program. (Break ties arbitrarily—for now.)

Recovering the edits



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- Once you have the path back, can essentially read back the edits: a diagonal is a match or replace; right is a delete; down is an insert. (This is if we're putting the target string vertically—if Y is being edited to become X, then X is vertical.)

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  - This problem was originally looked at in 1975 with the goal of limiting space to *fit the problem* on computers at that time

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  - This problem was originally looked at in 1975 with the goal of limiting space to *fit the problem* on computers at that time
  - Now it's still used, but the goal is to fit the problem in cache

Intro to Hirshberg's Original Paper

Introduction

The problem of finding a longest common subsequence of two strings has been solved in quadratic time and space [1, 3]. For strings of length 1,000 (assuming coefficients of 1 microsecond and 1 byte) the solution would require 10^6 microseconds (one second) and 10^6 bytes (1000K bytes). The former is easily accommodated, the latter is not so easily obtainable. If the strings were of length 10,000, the problem might not be solvable in main memory for lack of space.

Answer: Hirschberg's algorithm!

- Recursive approach that extends the dynamic program to make it space-efficient

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- Recursive approach that extends the dynamic program to make it space-efficient
- Can find in textbook (woo); I also posted the original paper (a tad old but still a reasonable resource).

(Slightly odd) Thought question

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- Specifically: the edit in the middle row
- In other words: what square in the middle row is on my solution path?

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	0	1	2	3	4	5	6	7	8	9	10
O	1	0	1	2	3	4	5	6	7	8	9
C	2	1	0	1	2	3	4	5	6	7	8
U	3	2	1	1	1	2	3	4	5	6	7
R	4	3	2	2	2	1	2	3	4	5	6
R	5	4	3	3	3	2	1	2	3	4	5
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Structural Lemma

Lemma 1

Let's say that X and Y have edit distance k. Divide X into two halves X_1 and X_2. Then there is some way to partition Y into two parts Y_1 and Y_2 such that $ED(X_1, Y_1) + ED(X_2, Y_2) = k$.

For example:

ADVICE and VINCENT have edit distance 5.

What parts of VINCENT match up with ADV? ICE?

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ED(ADV, V) = 2

ED(ICE, INCENT) = 3

Structural Lemma

Lemma 2

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Proof idea: there is some optimal sequence of edits applied to Y that obtain X. Let's apply those edits left to right. As we apply those edits, more and more of Y will match X (let's do an example with ADVICE and VINCENT on the board).

At some point, the beginning of Y will match the first half of X (that is to say: will match X_1). We can take that as Y_1, and the remainder of Y as Y_2.

Structural Lemma

Lemma 3

Let's say that X and Y have edit distance k. Divide X into two halves X_1 and X_2. Then there is some way to partition Y into two parts Y_1 and Y_2 such that $ED(X_1, Y_1) + ED(X_2, Y_2) = k$.

Note: I am not showing you this lemma just to be formal. This is a useful reference for when you're coding so that you know *exactly* how subproblems fit together. Perhaps most importantly: Y_1 and Y_2 do not overlap; nor do X_1 and X_2.

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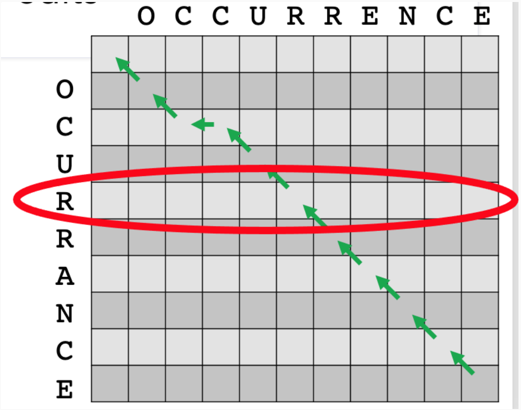
- Idea: for *every possible* Y_1, Y_2, calculate $ED(X_1, Y_1) + ED(X_2, Y_2)$ (slow for now! But bear with me)
- By the above lemma, there is at least one of these with sum exactly $ED(X, Y)$. These correspond to optimal paths through the matrix!

Using the Structual Lemma



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- Kind of like $T(X) = T(X/2) + O(X)$

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- Let's calculate them separately: let's calculate $ED(X_1, Y_1)$ for all Y_1, and $ED(X_2, Y_2)$ for all Y_2.

Calculating $ED(X_1, Y_1)$ for all Y_1

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The values we want *are* the entries in row $m/2$ of the DP table! So we already know how to calculate these in $O(nm)$ time and $O(n)$ space

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- Problem: this doesn't quite correspond to a table row

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Let X^R be the reverse of X, and let Y^R be the reverse of Y. Then $ED(X, Y) = ED(X^R, Y^R)$.

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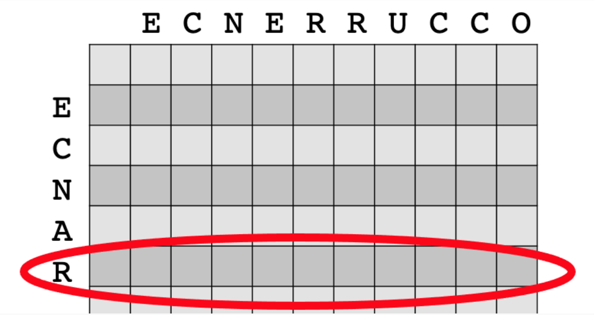
- We want to calculate, for all $i = 0, \dots, n$, the edit distance between the first i characters of Y^R and the first $m - m/2$ characters of X^R
- We know how to do this from last slide! It's just the middle row of the DP table between the reversed strings

Calculating the edit distances of the last characters



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Let X_1 be the first half of X, and X_2 be the second half of X. Let Y_i be the first i characters of Y, and Y_i' be the last $n-i$ characters of Y.

Here's how to calculate $ED(X_1, Y_i)$ and $ED(X_2, Y_i')$ for all i, in $O(nm)$ total time and $O(n)$ space:

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- Reverse X_2 to get X_2^R. Reverse Y to get Y^R.
- Perform the space-efficient dynamic program between X_2^R and Y^R (i.e. fill out the middle row of the reversed)
- Entry $(m - m/2, n - i)$ holds $ED(X_2, Y_i')$ by definition (and since edit distance is retained through reversal).

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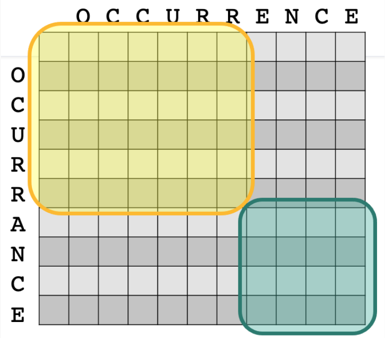
- If there's a tie, *any* of them will give an optimal solution.

Now: recurse!

- For the i we calculated as the crossing point: find the optimal sequence of edits between X_1 and Y_i. Then, find the optimal sequence of edits between X_2 and Y_i'.



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- Just concatenate the two recursive solutions.

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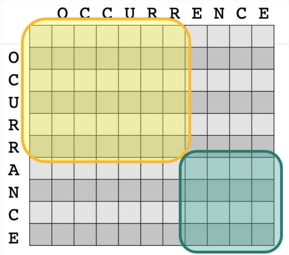
- Basic idea: the total cost of all recursive calls at a given level is the size of the table remaining; this decreases by a factor of 2 each time.



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- The space-inefficient approach may incur many cache misses to fill up the table.
- We'll have strings of length $\approx 30{,}000$. So yes, this will be the difference between fitting in (and not fitting in) L3 cache.

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- Let's look over the homework quickly

Matrix Multiplication in External Memory

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Example:

$$\begin{bmatrix} 1 & 2 \\ 8 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 17 \\ 18 & 17 \end{bmatrix}$$

Compute Product Directly

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  - Answer: $O(n^3)$ cache misses. Every operation requires a cache miss for B.

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- Another idea: swap the loops! How many cache misses is this?

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- Question: Is this worth doing?

Yep!

I am given two functions for finding the product of two matrices:

```
void MultiplyMatrices_1(int **a, int **b, int **c, int n){
    for (int i = 0; i < n; i++)
        for (int j = 0; j < n; j++)
            for (int k = 0; k < n; k++)
                c[i][j] = c[i][j] + a[i][k]*b[k][j];
 }

void MultiplyMatrices_2(int **a, int **b, int **c, int n){
    for (int i = 0; i < n; i++)
        for (int k = 0; k < n; k++)
            for (int j = 0; j < n; j++)
                c[i][j] = c[i][j] + a[i][k]*b[k][j];
}
```

I ran and profiled two executables using `gprof`, each with identical code except for this function. The second of these is significantly (about 5 times) faster for matrices of size 2048 x 2048. Any ideas as to why?

c algorithm matrix matrix-multiplication gprof

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- Note: can't do this with linear scan. $O(n/B)$ is optimal. But we did do this with `smallunsortedlinkedlist.c`

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  - Efficiently combine them for a cache-efficient solution

Blocked Matrix Multiplication

- Split A, B, and C into blocks of size $M/3$
  - $\sqrt{M/3} \times \sqrt{M/3}$ matrices
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- Multiply blocks one at a time

Decomposing matrices into blocks

Classic result: if we treat the blocks as single elements of the matrices, and multiply (and add) them as normal, we obtain the same result as we would have in normal matrix multiplication.

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- This idea is used in recursive matrix multiplication
- And Strassen's algorithm for matrix multiplication

Decomposing matrices into blocks

Example: Recall how to multiply $2x2$ matrices:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \end{bmatrix}$$

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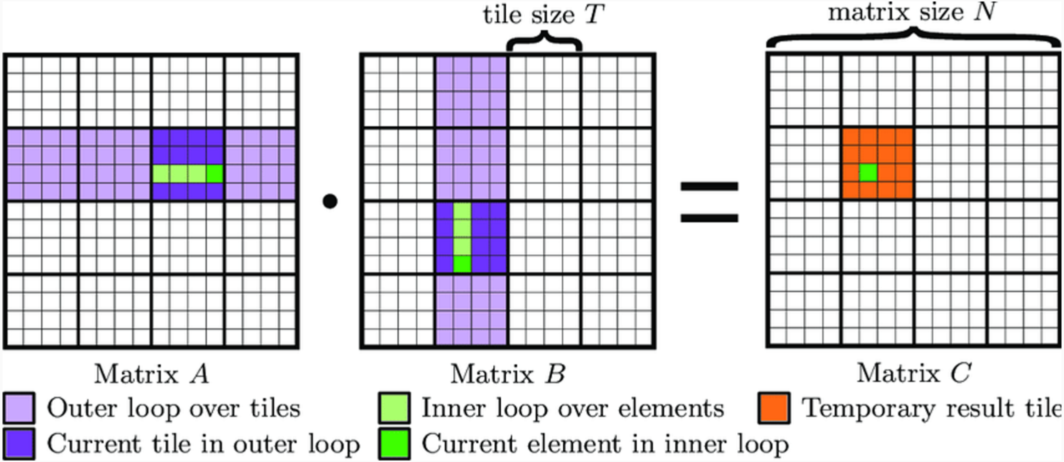
- Decompose matrix into blocks of length T (recall that $T^2 \leq M/3$)
- Do a normal $n/T \times n/T$ matrix multiplication



Blocked Matrix Multiplication Pseudocode

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MatrixMultiply(A, B, C, n, T):
    for i = 1 to n/T:
      for j = 1 to n/T:
        for k = 1 to n/T:
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          B’ = TxT matrix with upper left corner B[Tk][Tj]
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Let’s analyze the cost of this algorithm in the EM model together on the board!

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- Therefore, our total running time is the number of loop iterations times the cost of a loop. This is $O((n/T)^3 \cdot T^2/B) = O((n/\sqrt{M})^3 \cdot M/B) = O(n^3/B\sqrt{M})$.

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- Is blocking actually worthwhile?
  - Yes; it is used all the time to speed up programs with poor cache performance.
  - (Not a panacea; some programs (like linear scan, binary search) can't be blocked.)

Sorting in External Memory

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- Can we do better?

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- Each time we bring a cache line into cache, how many more things can we compare it to?

Merge sort reminder

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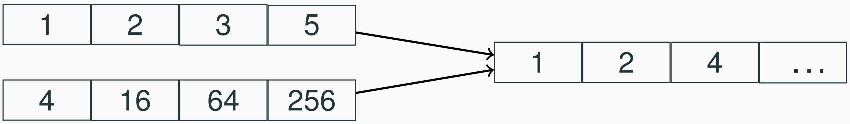

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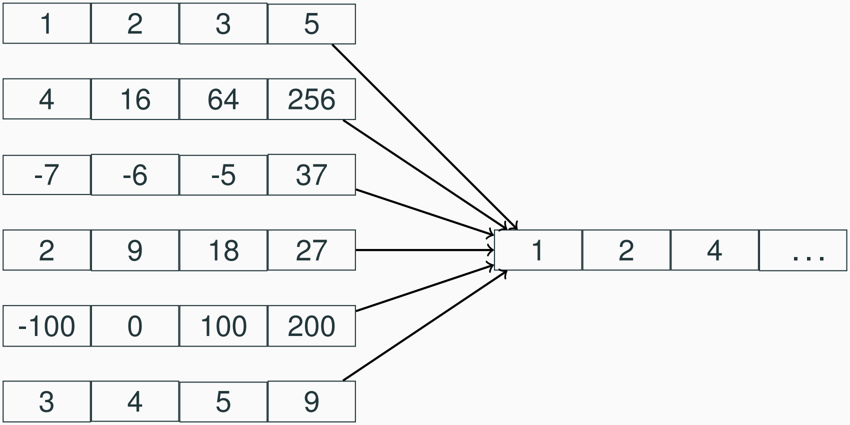
Diagram of M/B-way merge sort



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- Now also used in Java, Rust
- Keeps cache in mind, but focuses more on taking advantage of easy patterns in data

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- This is wasteful, as we empty out cache between each subarray
- Timsort starts with "run generation": a greedy version of this that uses the same cache for as long as possible. Always outputs sorted runs of length at least M; can be MUCH longer

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- Insertion sort on any very small arrays that are encountered (size < 64)

External Memory Sorting

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- Timsort is very popular in practice; uses a simpler blocking approach to stay cache-friendly.