Lecture 17: Integer Linear Programming Continued

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Williams College

Admin

- Assignment 3 out; due *Saturday* the 16th (basically: built-in 2 day extension for everyone)
- Some difficult problems; it's OK if you don't get all of them completely correct. Just write what you know.
- Questions?

Survey

- Assignment 3 is your last assignment/homework, we'll just work on the project afterwards
- We'll have some classes for going over previous solutions/doing short project presentations, but there are a few more extra slots
- Two options:
  - The extra class slots will basically be extra office hours where you can work on the project and discuss it with me
  - Normal lectures going over a few cool topics (Burrows-Wheeler Transform, Suffix Trees, Van Emde Boas trees)
- Core question is really: If we have normal lectures on cool (but not easy) topics that are not ever tested, are you interested/will you attend?

Solving ILPs and MIPs

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- *LP relaxation:* just remove the integer constraints

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- *LP relaxation:* just remove the integer constraints
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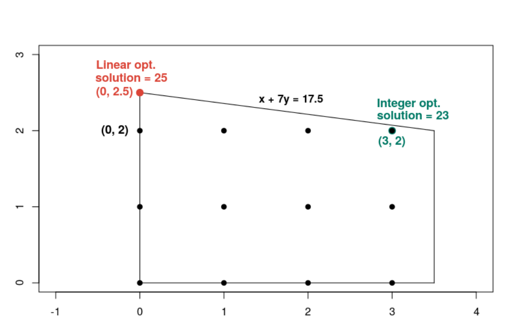
- How badly can this do?

Rounding MIPs



From Google OR Tools Documentation

- Can do *arbitrarily* badly, even for simple ILPs

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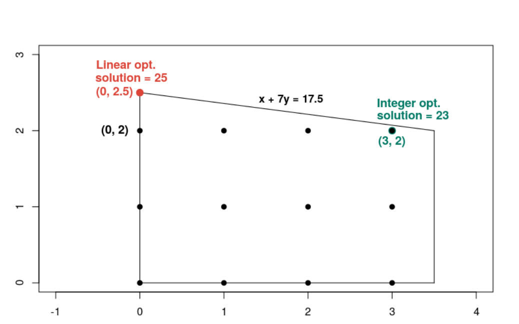

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- May work effectively if the problem has a special structure that makes rounding effective

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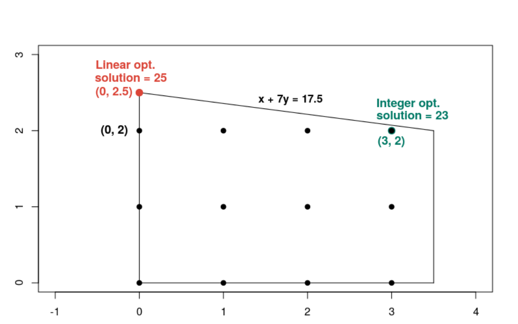


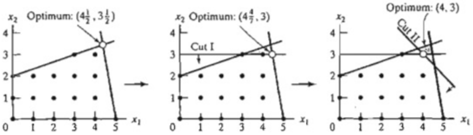
From Google OR Tools Documentation

- Can do *arbitrarily* badly, even for simple ILPs
- May work effectively if the problem has a special structure that makes rounding effective
- Example: the diet problem is probably solved fairly well by rounding (will only be off by 1 unit of each food)

Second Method: Cutting ILPs

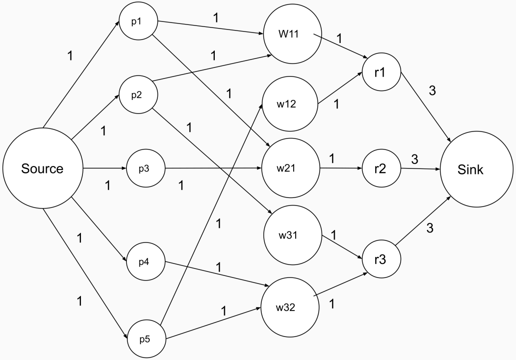
FIGURE 9.10

Illustration of the use of cuts in ILP



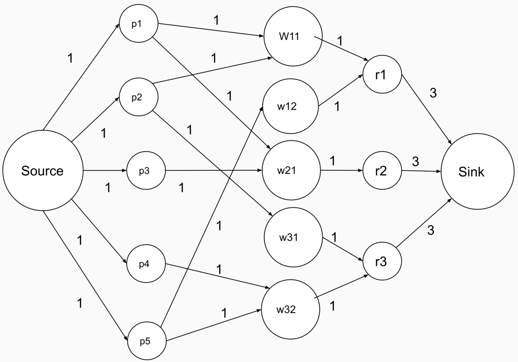
- We *won't cover* in this class
- Cut the LP without removing integer solutions
- After enough cuts, can round and get a good solution!
- Not always possible, but surprisingly effective methods in practice for some types of problem
- Many MIP solvers find these cuts for you

Third Method: Prove the LP has integral soln



- Broad class of LPs are guaranteed to give optimal solutions
- We *won't cover* in this class
- Example for linear algebra people: if your constraint matrix is totally unimodular then there exists an optimal integer solution

Third Method: Prove the LP has integral soln



- Broad class of LPs are guaranteed to give optimal solutions
- We *won't cover* in this class
- Example for flow-reduction-lovers: if you write a flow problems as an LP where all constraints are integers, there exists an optimal integer solution

Main MIP Solving Method: Branch and Bound

Branch and Bound

- Two towers: meet-in-the-middle was faster since we could “rule out” some of the search space

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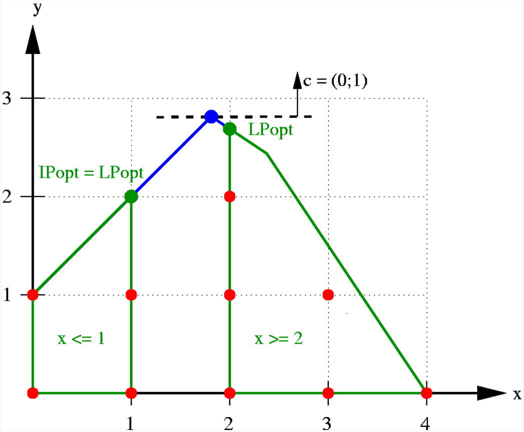
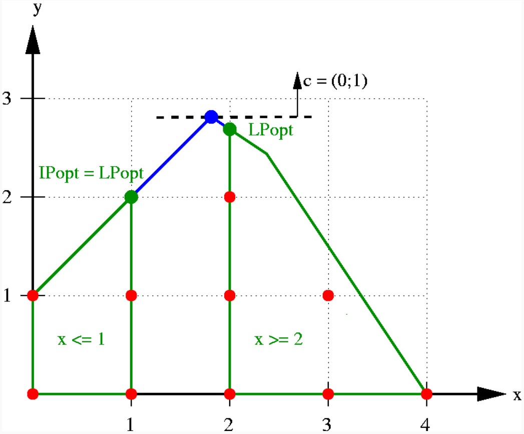
- Maintain worst-case guarantees
- Branch and bound: a less-problem-specific way to do the same thing
- This is a large *class* of algorithms; I’m giving a high level description of the idea
- (There is a question about this on Assignment 3.)

Branching



- First, we divide the problem into several subproblems

Branching



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- Visualization is useful: just partition the feasible region into several pieces

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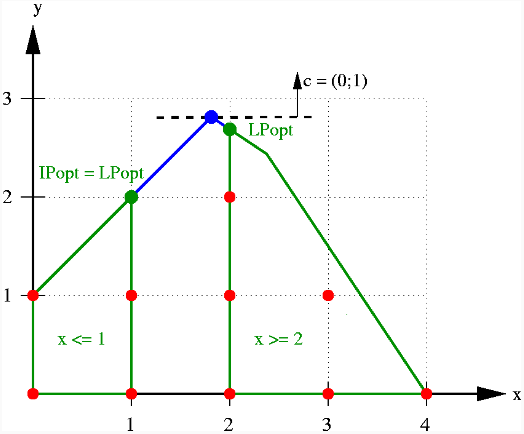
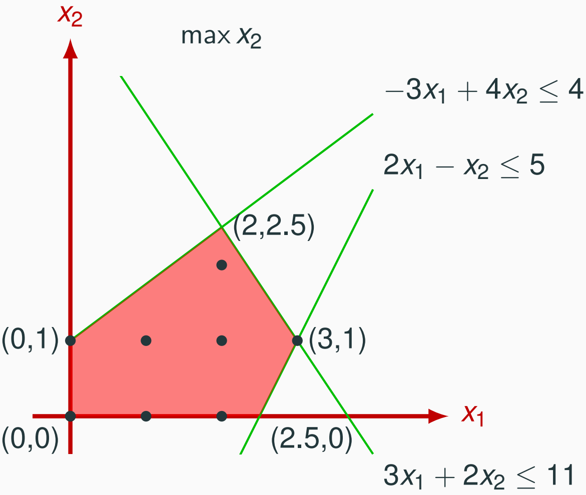


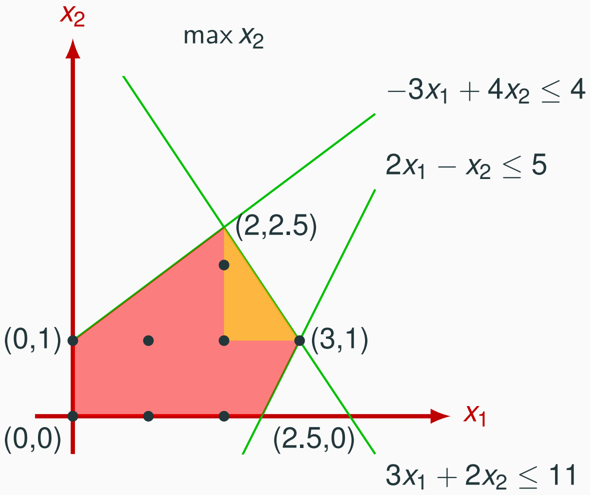
- First, we divide the problem into several subproblems
- Visualization is useful: just partition the feasible region into several pieces
- So far, still need to search through all of them (same as brute force)

Branching and Bounding



- Partition region

Branching and Bounding



- Partition region
- Find best solution in orange piece

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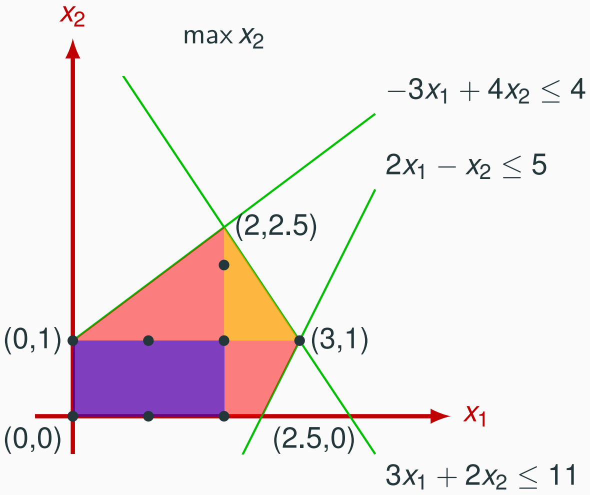
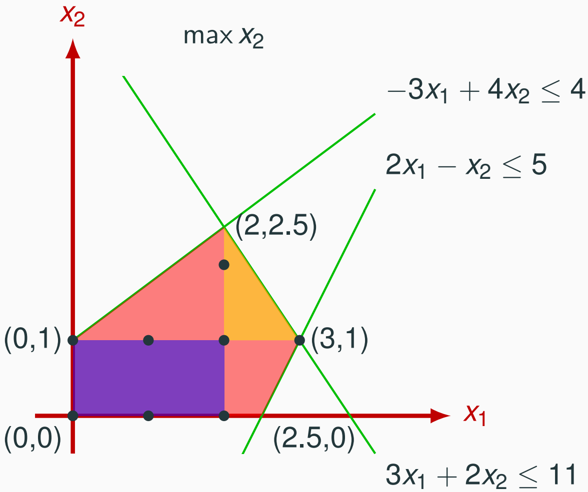


- Partition region
- Find best solution in orange piece
- When can we avoid searching in purple?

Branching and Bounding



- Upper bound best solution in purple

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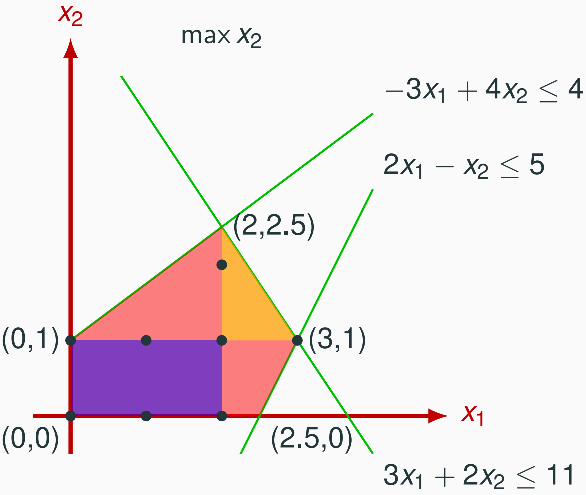
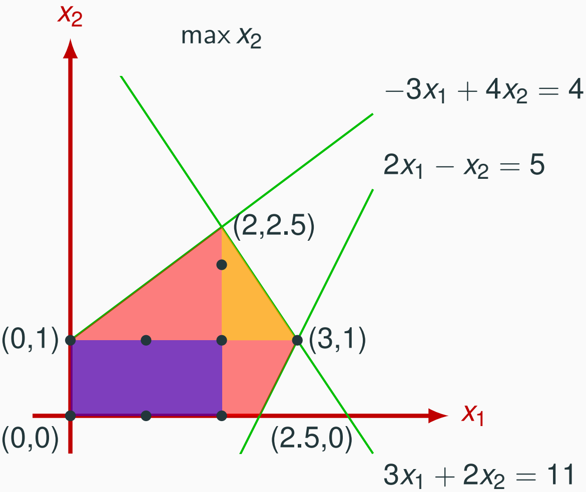


- Upper bound best solution in purple
- If best possible soln in purple is worse than best soln in orange, can *safely skip* it

Branching and Bounding



Safe to skip: *always* still gives an optimal solution.

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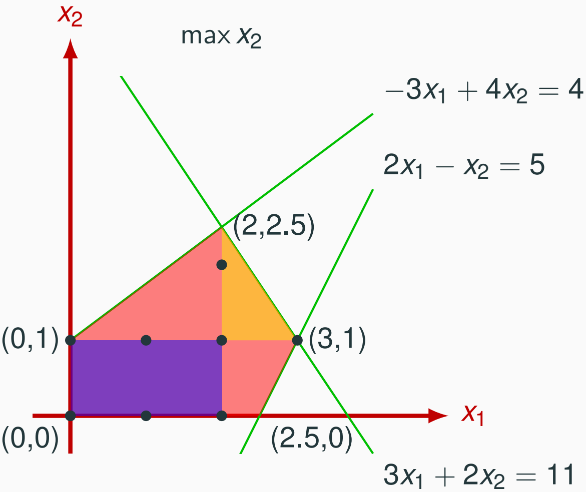

Safe to skip: *always* still gives an optimal solution.

But, can't skip anything in worst case.

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  - Relax to an LP! Might not give a good upper bound, but will give *an* upper bound (Recall: LPs are relatively fast to solve)

What do we need?

- Way to get a good solution in orange region: recurse!
- Base case: can just do a simple greedy method if the region is small enough.
- Way to upper bound best solution in purple region??
    - Relax to an LP! Might not give a good upper bound, but will give *an* upper bound (Recall: LPs are relatively fast to solve)
    - Duality can help (we won't talk about in this class)

Branch and Bound Intuition

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- Let us rule out big parts of the search space
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- Many practical problems have large parts that are easy to skip. (If we’re stacking groceries on pallets, no need to spend time looking at solutions with bread on the bottom.)
- The more we branch (find good solutions), the more we can bound (rule out parts of the search space whose solutions are suboptimal)

Branch and Bound in Practice

- Advanced methods to figure out how to split into pieces; how much to search each piece before doing more bound calculations

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Branch and Bound in Practice

- Advanced methods to figure out how to split into pieces; how much to search each piece before doing more bound calculations
- The better your choices, the more you can rule out
- Other methods (greedy, LP cuts, duality, heuristic search, etc.) can be integrated into this method

Branch and Bound in Practice

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- Dedicated solvers for TSP, Knapsack, that make branching decisions and use bounding methods particularly effective for that problem
- This is how you get the optimal, giant TSP tours
- Also some general-purpose solvers

Branch and Bound Summary

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- May not find it *quickly* on tricky problems
- Can be very fast even on reasonably hard, reasonably large instances.

Solvers

These solvers have both LP and MIP solvers (using different algorithms):

- GLPK (simplex, branch and bound). Open source. Standalone program is fairly easy to use; can also access from C.
- CPLEX - IBM software for MIPs. Old but reliable. Proprietary. Effective, but can be difficult to work with
- COIN-OR - open source solver
- Google OR tools - wrapper for COIN-OR. Has a really nice TSP and Knapsack solvers. More user friendly

More ILP and MIP Examples

Scheduling

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Scheduling

- (Aside: scheduling is a major application of ILPs. Lots of different techniques; this is just one example.)
- Assign n unit-cost jobs to machines.
- Each job j_i has a type t_i. Two jobs of the same type cannot be assigned to the same machine.
- How can we schedule the jobs with the minimum number of machines?

Scheduling Jobs with Types

- n jobs, job i has type t_i
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- At most n. So have n^2 variables: $s_{i,m} \in \{0, 1\}$, for $1 \leq i \leq n$ and $1 \leq m \leq n$.

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- For all $1 \leq i \leq n$, $\sum_{m=1}^{n} s_{i,m} = 1$

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- Constraints?
- Two jobs of the same type can't be assigned to the same machine
- Rephrased: for every machine m, no two jobs of the same type can be assigned to m

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- (Up to n^3 constraints. Also: constraints depend on the input.)

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- Objective?

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- Let c_m be the cost of machine m. Want $c_m = 1$ if there is a job assigned to machine i, $c_m = 0$ otherwise.

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- Constraint for c_m?
- For all jobs i and all machines m, $c_m \geq s_{i,m}$

Size and Computation Time for an ILP/MIP

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- In other words: the time to calculate all the *constants*!
- We also want this to be *polynomial* in the size of the original problem input
- I will not ask you to calculate these values. I am going over this because any ILP/MIP you give should have polynomial size and polynomial computation time.

Scheduling Jobs with Types

Objective: $\min \sum_{m=1}^{n} c_m$

Constraints:

For all $1 \leq m \leq n$ and $1 \leq i \leq n$, $c_m \geq s_{i,m}$

For all $1 \leq m \leq n$, for all jobs i_1 and i_2 with the same type $t_{i_1} = t_{i_2}$, $s_{i_1,m} + s_{i_2,m} \leq 1$

For all $1 \leq i \leq n$, $\sum_{m=1}^{n} s_{i,m} = 1$

$s_{i,m} \in \{0, 1\}$ for all $1 \leq i \leq n$, $1 \leq m \leq n$.

What is the size of this ILP?

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$s_{i,m} \in \{0, 1\}$ for all $1 \leq i \leq n$, $1 \leq m \leq n$.

What is the size of this ILP?

$n + n^2 = O(n^2)$ variables, at most $n^2 + n^3 + n = O(n^3)$ constraints. Multiplying, total size is $O(n^5)$

So the size is polynomial in n.

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Computation time?

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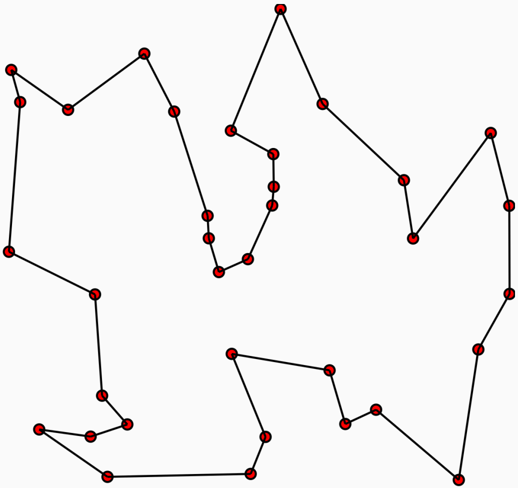
$s_{i,m} \in \{0, 1\}$ for all $1 \leq i \leq n$, $1 \leq m \leq n$.

Computation time?

Polynomial. (All the constants can be calculated in $O(1)$ time.)

More specifically, this can be caluclated in $O(n^5)$ time.

Travelling Salesman



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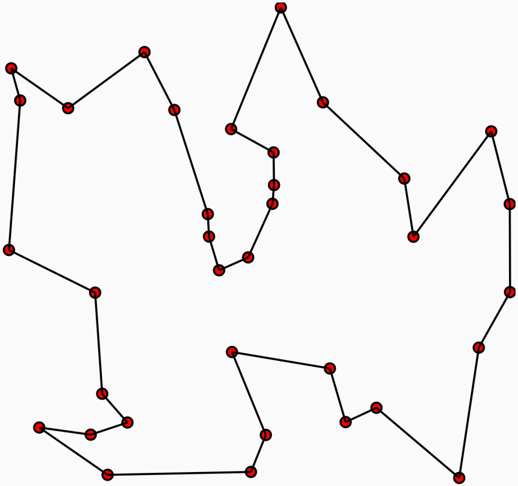

- Find minimum-length cycle through vertices such that each is visited exactly once
- Given: set of n points, for each pair of points i and j the cost $c_{i,j}$ to get from i to j. Have $c_{j,i} = c_{i,j}$

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It does work! But the number of variables may be *exponential* in the number of vertices n, and calculating all the d_is also takes (in sum) *exponential* time.

Travelling Salesman: Polynomial Size Solution

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Travelling Salesman

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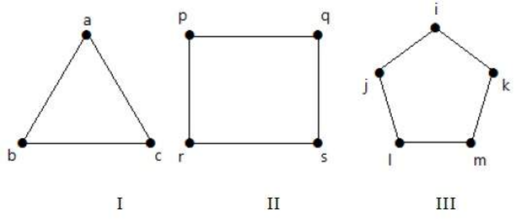
- Is this sufficient?

Travelling Salesman



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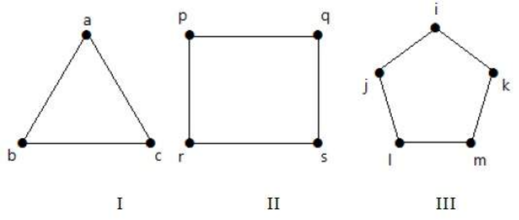

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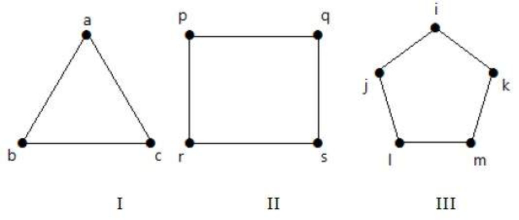

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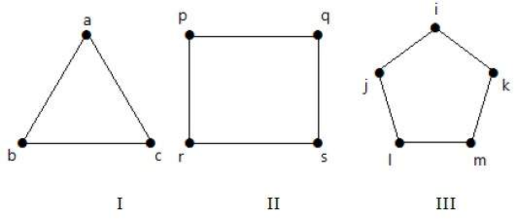

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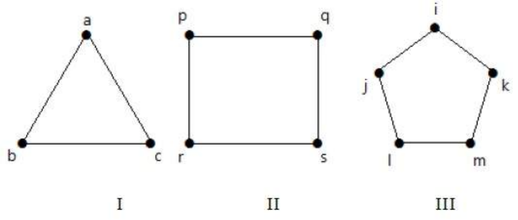

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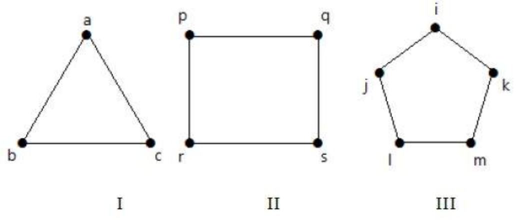

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- $1 \leq u_i \leq n - 1$ for $2 \leq i \leq n$

Travelling Salesman LP

minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} e_{i,j} c_{i,j}$

For all i, $\sum_{j \neq i} e_{i,j} = 1$ and $\sum_{\ell \neq i} e_{\ell,i} = 1$

For all $2 \leq i \neq j \leq n$, $u_i - u_j + n e_{i,j} \leq n - 1$

- Let's prove that this is correct!

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- Then we'll talk a little bit about intuition.

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- Assume by contradiction that C' doesn't contain city 1. Then we'll *sum all* the constraints for edges in C'
- All the u_i and u_j cancel, and we get $n \leq n - 1$. Since this is impossible, one of the original constraints must not have been satisfied.

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- Yes, just need to look up the costs $c_{i,j}$.

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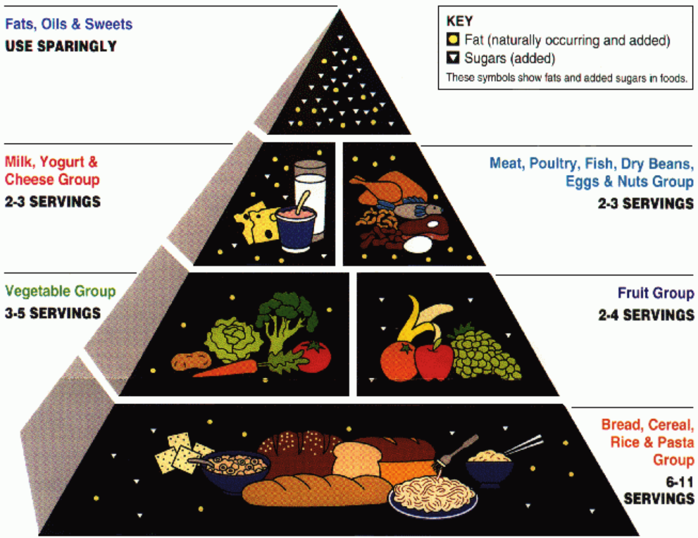
- Simple example: optimal eating while being able to choose your diet

Food Pyramid



Choice of diet

- You need to satisfy one of the three following diet goals:
  - 46 grams of protein and 130 grams of carbs every day; or
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What is the cheapest way you can hit one of these diet goals?

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- $25.8p + 2.5r + 13.5c + 46(1 - x_1) \geq 46$

Choice of diet LP

- Diet options:
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  - 20 g protein; 200 g carbs; or
  - 100 g protein; 30 g carbs
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- 100g Rice: 2.5g protein, 28.7g carbs, $.79
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$\min 1.61p + .79r + .7c$

- $25.8p + 2.5r + 13.5c + 46(1 - x_1) \geq 46;$
- $16.1p + 28.7r + 130(1 - x_1) \geq 130$
- $25.8p + 2.5r + 13.5c + 20(1 - x_2) \geq 20;$
- $16.1p + 28.7r + 200(1 - x_2) \geq 200$
- $25.8p + 2.5r + 13.5c + 100(1 - x_3) \geq 100;$
- $16.1p + 28.7r + 30(1 - x_2) \geq 30$
- $x_1 + x_2 + x_3 = 1$
- $p, r, c \geq 0; p, r \in \mathbb{Z}; x_i \in \{0, 1\}$

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- What happens with rounding when you use this technique?

That’s all for today

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