# Lecture 12: Locality-Sensitive Hashing and MinHash

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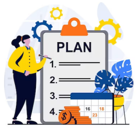
October 18, 2024

Williams College

# Admin

- I’ll make grading progress this coming week
- Homework 4 in
- Assignment 2 out
- Last assignment/homework in C!
    - We’ll use LP solvers for Homework 5 and Assignment 3
    - Then just the project

# Plan for rest of course




- Today: topic for Assignment 2
- I’m trying to get a decent recording of today’s lecture; I’ll post it if I do
- Tuesday: SIMD instructions (necessary for part of Assignment 2)
- Next Friday: code review/wrapup. We’ll talk about the project around then
- Then we’ll get started on the third part of the class with LP/ILP

# Assignment 2

- Idea: give two weeks so that you can fit around midterms/etc.
- (Basically: a lot of students had trouble finishing the second assignment—on a different topic–last time and I'm pretty sure it was time rather than difficulty. So don't put it off too much!)
- We are skipping a topic, "Robin Hood Hashing," as a result. ☹ or maybe ☺
- This assignment should be roughly similar to Homeworks 3 and 4; almost definitely a little harder than Homework 4.
- I'll release it this afternoon (need to fix some stuff)

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- Extra credit counts towards Homework 4
  - (Since assignments are weighted more I want the extra credit to be on a homework)

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  - Please don't do these things; do it yourself
  - Our implementation has some nontrivial differences from theirs (in short: we have much smaller, denser sets than most people work with). If you copy/use/etc. others' work it will be very clearly different, and also slower

# Finding Similar Items

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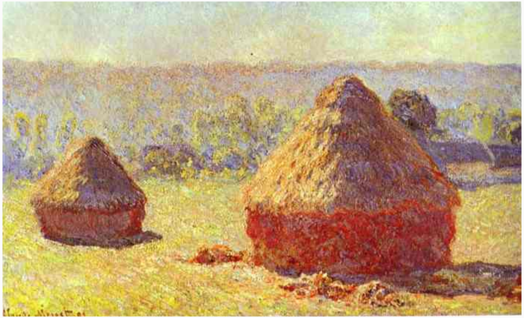
- But data is still big!
  - In particular: high-dimensional
  - Table with many columns
  - For each Netflix user, what movies have they seen
- Goal: solve a difficult, but important, problem. If you've taken an ML course it's reasonably likely that you've seen some variant of this problem.

# Finding Similar Pair



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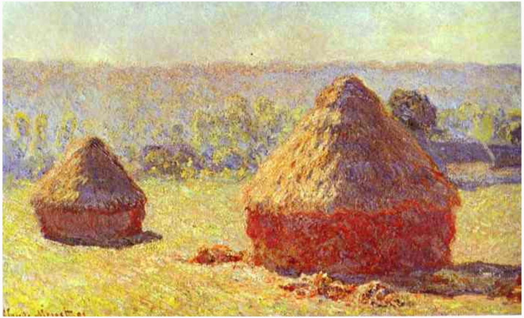

- Given a set of objects
- Find the most similar pair of objects in the set

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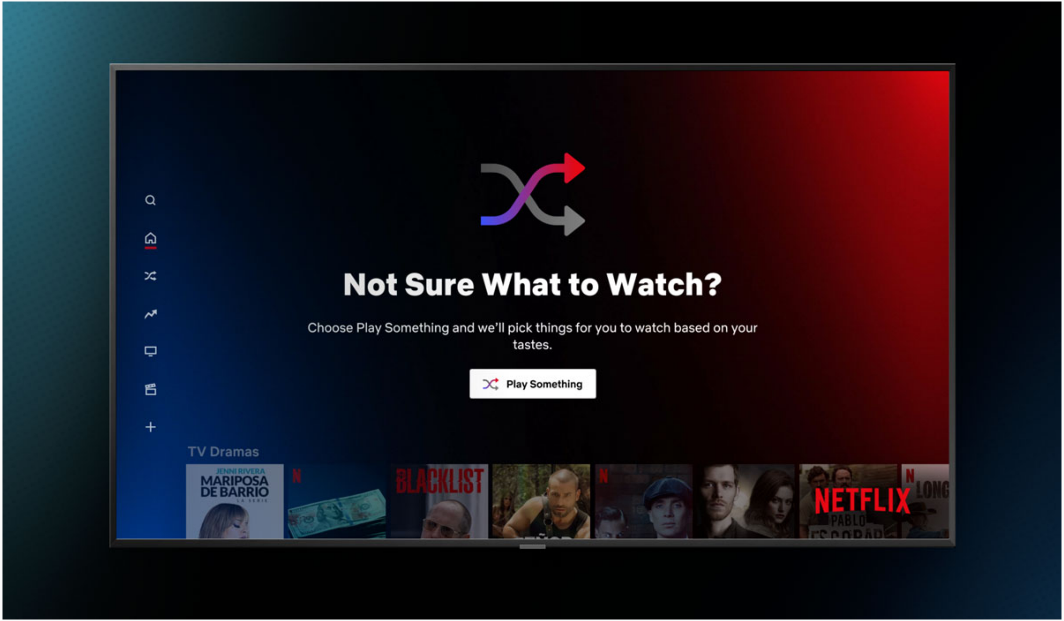
- Similar music: Spotify suggests music by finding similar users, and selecting what they listen to
- Machine learning in general (training, evaluation, actual algorithms, etc.)
- Data deduplication, etc.
- “Give me a similar pair in this dataset” is a common query!

# Similarity Search Example



# Strategies for Similarity Search

# First attempt: 1-dimensional data

| |
|---|
| 92 |
| 44 |
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- “Similarity” is the absolute value of the difference between them

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- Given a list of numbers
- "Similarity" is the absolute value of the difference between them
- How can we find the closest numbers (i.e. ones with smallest difference)?

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- How efficiently can we do this?

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- $O(n \log n)$ time

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Aside: can we do better? Yes, there's a clever $O(n)$ algorithm based on sampling.

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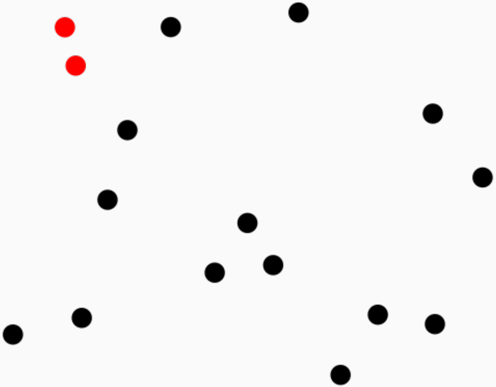

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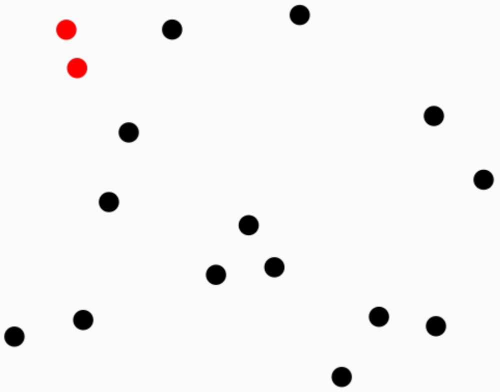

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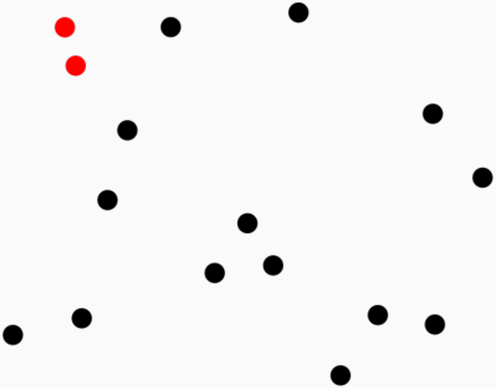

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- Divide and conquer, $O(n \log n)$ time.
- (Again, possible in $O(n)$ with sampling.)

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**Josefine S. (Protected by ...**

\+ Follow

**(k) The 38 movies I saw in 2010 :)**

Only counting movies I saw for the first time. Faves: 4, atm. Best: "Harry Potter and the Deathly Hallows: Part I". :D
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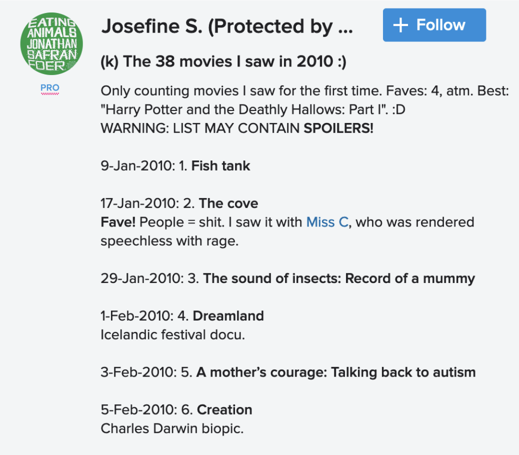


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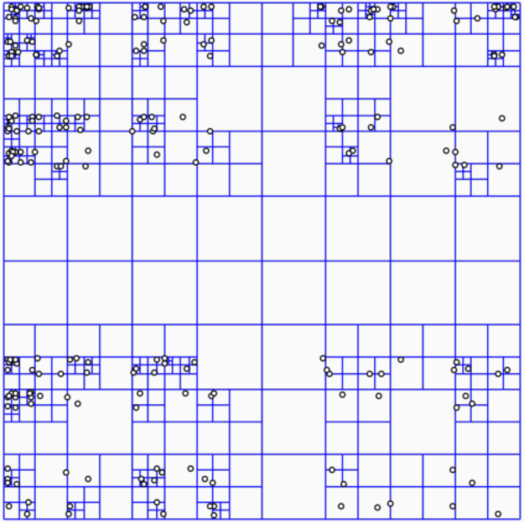
- “The tags generated for this image are: [. . .]”

# What About Higher Dimensions?

- We want VERY high dimensions (tens, hundreds, or even millions)
- Songs listened to, movies watched, image tags, etc.
- Words that appear in a book, k-grams that appear in a DNA sequence
- Classic options for finding similar items in high dimensions: quad trees, kd trees

# How Efficient are High-dimensional Algorithms?



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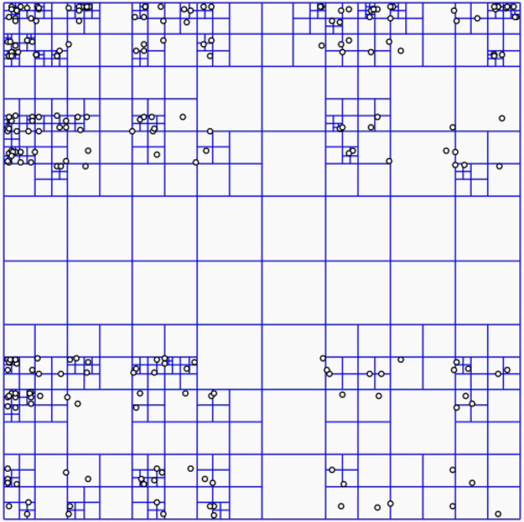

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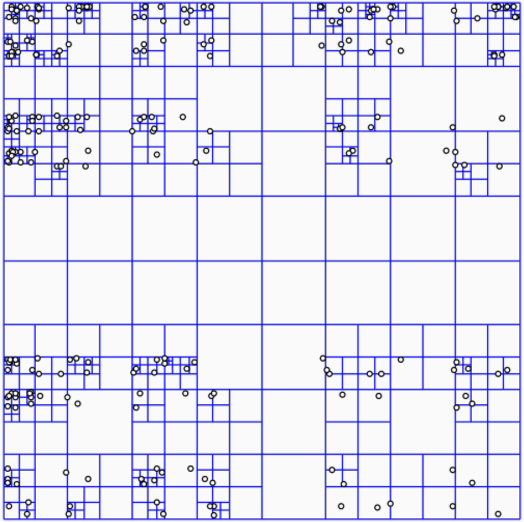

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- Worse than trying all pairs if $> \log n$ dimensions

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- Well-known phenomenon
- Applies to similarity search, machine learning, combinatorics
  - Approximation techniques, like those we learn about today, are underused to a slightly shocking extent—even in ML people sometimes keep dimensionality low to avoid this issue, affecting the quality of results

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  - Use hashing! ...A special kind of hashing
- For many of these problems, random inputs are worst-case inputs
  - Worst case behavior actually occurs for many common use cases; guarantees (even approximate) can be very valuable

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- This is key to hashing: no matter how clustered my data begins, I wind up with a nicely-distributed hash table
- Locality-sensitive hashing tries to act like a normal hash for items that are dissimilar, but *wants* collisions for similar elements
- Similar items are likely to wind up in the same bucket. But dissimilar items are not

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- High level idea: close items are likely to collide. Far items are unlikely to collide.
- Generally want $p_2$ to be about $1/n$; then we get a normal hash table for far (i.e. similarity $\leq cr$) elements.

# Why Locality-Sensitive Hashing Helps

| | | | | |
|---|---|---|---|---|
| | (101, 37, 65)<br>(103, 37, 64) | (91, 84, 3) | | (100, 18, 79) |
| 0 | 1 | 2 | 3 | 4 |

Ideally, close items hash to the same bucket.

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We'll put numbers on this later

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- If we have $p_2 = 1/n$, then $p_1$ is usually very small.
- How can we increase this probability?
- Repetitions! Maintain many hash tables, each with a different locality-sensitive hash function, and try all of them.

# LSH with Repetitions

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
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  - Movies watched by a user
  - Human-generated tags given to an image
  - Words that appear in a document
- Need a way to measure *set similarity*

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| User 1 | User 2 |
|---|---|
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| Ariana Grande | Khalid |
| Khalid | Drake |
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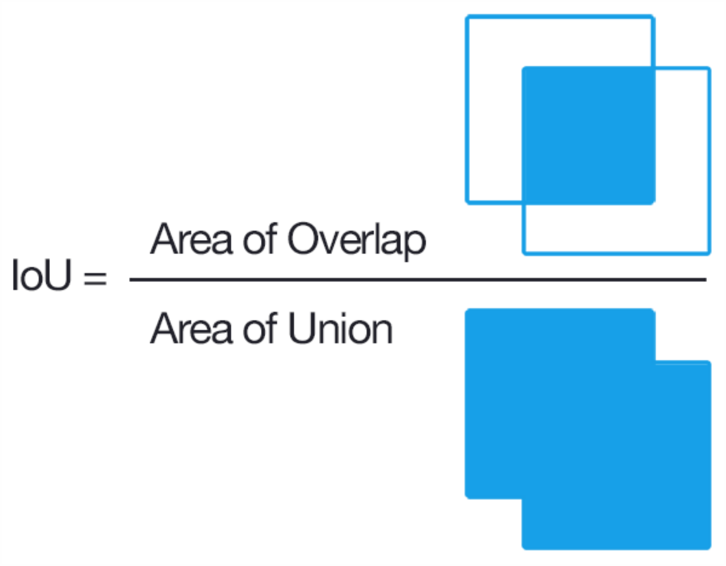
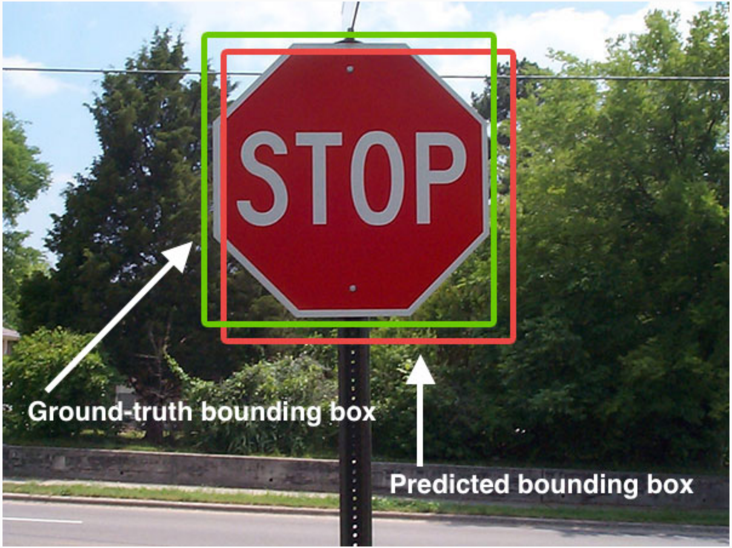
- Intuitively: what fraction of these sets overlaps?

# Jaccard Similarity Intuition 1

# Jaccard Similarity Intuition 2



# Image Search Example



# Jaccard Example 1 (Created in 2020)

| User 1 | User 2 |
|---|---|
| Post Malone | Ariana Grande |
| Ariana Grande | Khalid |
| Khalid | Drake |
| Drake | Travis Scott |
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- Similarity: $|A \cap B|/|A \cup B|$.

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- $|A \cap B| = 4$

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- Similarity: $|A \cap B|/|A \cup B|$.
- $|A \cap B| = 4$
- $|A \cup B| = 5$

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- Similarity: $|A \cap B|/|A \cup B|$.
- $|A \cap B| = 4$
- $|A \cup B| = 5$
- Jaccard Similarity: $4/5 = .8$

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- Similarity: $|A \cap B| / |A \cup B|$.
- $|A \cap B| = 1$

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- Similarity: $|A \cap B|/|A \cup B|$.
- $|A \cap B| = 1$
- $|A \cup B| = 5$
- Jaccard Similarity: $1/5 = .2$

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- $|A \cap B| = 3$
- $|A \cup B| = 7$
- Jaccard Similarity: $3/7 = 0.428$

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- Jaccard similarity ignores items not in either set. So we learn nothing if neither of us like an artist. (Is this good?)
- Still works if one list is much longer than the other. (Generally, they'll have small similarity)

# Locality-Sensitive Hash for Jaccard Similarity

- Want: items with high Jaccard Similarity are likely to hash together

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- (AltaVista was probably the most popular search engine before Google, they wanted to detect similar web pages to eliminate them from search results)
- Now used for similarity search, database joins, clustering—LOTS of things.

# AltaVista in 2001

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- Effective if universe is fairly small; use a list for larger universe

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  - This is exactly A | B in C-style notation
- We want the size of these sets—need to count the number of 1s in A & B, or A | B.

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- To hash a set $A$: find the first item in $A$ in the order given by the permutation. That item is the hash value!

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- First set is 101000 (same as $\{$red, blue$\}$). blue is in the set, so the hash value is blue.
- Second set is 110010 (we could also write $\{$red, yellow, purple$\}$). blue is not in the set; nor is orange; nor is green. purple is, so purple is the hash value

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For the sake of space let's do 8 bits

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- Then the hash of $x$ is 5.

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- Let's do some analysis to look at this issue in more detail

# Analysis of Basic MinHash

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- Therefore: probability of hashing together is $|A \cap B| / |A \cup B|$.

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- If two items have similarity at least $r$, they collide with probability at least $p_1 = r$
- If two items have similarity at most $cr$, they collide with probability at most $p_2 = cr$

# Analysis: Phrased as bit vectors

- What is the probability that $h(A) = h(B)$?
- Let's look at the permutation that defines $h$. We can ignore any index that is 0 in both $A$ and $B$.
- Look at the first index in the permutation that is 1 in $A$ or $B$
  - If this index is in *both* $A$ and $B$, then $h(A) = h(B)$
  - If this index is in only one of $A$ or $B$, then $h(A) \neq h(B)$
- Any index that is 1 in $A|B$ is equally likely to be first. If the index is in $A\&B$, they hash together; otherwise they do not
- Therefore: probability of hashing together is (number of 1s in $A\&B$)/(number of 1s in $A|B$).

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- Probability that red or green is first out of $\{$red, blue, green, orange, purple$\}$ is $2/5$.

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- If red or green appears before blue, orange, and purple then they hash together
- If blue or orange or purple appear before red and green, then they don't hash together
- Probability that red or green is first out of $\{\text{red, blue, green, orange, purple}\}$ is $2/5$.
- Therefore, $A$ and $B$ hash together with probability $2/5$.

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# Making Sure We Find the Close Pair

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    - (We'll talk about how to do this in a moment—it's somewhat similar to Assignment 1)
- Let's say our close pair has similarity .5. How many times do we need to repeat?
- Each repetition has the close pair in the same bucket with probability .5. So need 2 repetitions in expectation.

# An Aside on Expectation

**Lemma 1**

*If a random process succeeds with probability $p$, then in expectation it takes $1/p$ iterations of the process before success.*

Examples:

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Proof: the expectation is

$$\sum_{i=1}^{\infty} ip(1-p)^{i-1} = p\sum_{i=1}^{\infty} i(1-p)^{i-1} = p\frac{1}{(1-(1-p))^2} = \frac{1}{p}.$$

# Concatenations and Repetitions

# Problems with this Approach

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- Answer: concatenate multiple hashes together.

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  - They are independent, so we can multiply to obtain probability $(|A \cap B|/|A \cup B|)^k$ of $A$ and $B$ colliding.

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- Let's hash $A$.

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    - Third hash: red is in $B$.
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- What kind of values work for $k$ and $R$?

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- We can then solve $(n-1)(1/3)^k = 1$ to get $k = \log_3(n-1)$.

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- So we expect about $R = n^{.26}$ repetitions. That's a lot!
- But it's not far from the state of the art.
- And way better than brute force!

# Finding $R$ and $k$ in general

Let's say we have $n$ points where the close pairs have similarity $j_1$, and all other pairs have similarity at most $j_2$

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  - Doable at home: show that this is the optimal value for $k$ using the below analysis.
- Then, number of $R$ we need in expectation is:

$$\left(\frac{1}{j_1}\right)^k = \left(\frac{1}{j_1}\right)^{\log_{1/j_2} n} = n^{\log_{(1/j_2)}(1/j_1)}.$$

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- For each bucket, compare all pair of items in the bucket to see if they are close. If a close pair is found, return return it
- (Our analysis shows that we'll need to hash all $n$ items $n^{\log_{1/j_2}(1/j_1)}$ times in expectation)

# Practical MinHash Considerations

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- Then $A$ hashes to redorangeblue

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  - But this works fine in practice (and is used all the time)
- We will do this on the Assignment; in fact I recommend using $\hat{k} = k$. That means that each repetition has only one permutation.
- I think it makes life very significantly easier. In the real world you want a smaller value of $\hat{k}$

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- What can we do if we happen to get a big bucket?

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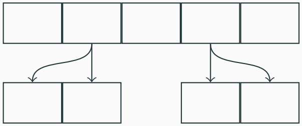

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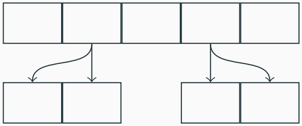

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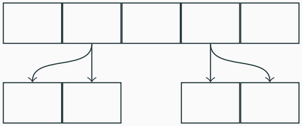

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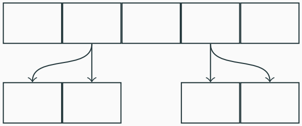

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- (Not required; just one optimization suggestion.)

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- (Not realistic case, but hard case!)

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- How can we do this?
- Most efficient way I know is not clever. Just go through each index, and check to see if that bit is set (say by calculating `x & (1 << index)` —but remember that these are 128 bits)

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- Option 2: Treat as bits. 0 to 127 can be stored in 7 bits. Store the hash as a sequence of $k$ 8-bit chunks.

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- Why? Having a large number of buckets and/or repetitions leads to bad constants
- Smaller $k$ means fewer buckets, fewer repetitions (but bigger buckets and more comparisons)
- Start with $k \approx \log_3 n$, but experiment with slightly smaller values.

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- The discussion of repetitions in the lecture is for two reasons: 1. analysis, 2. give intuition for the tradeoff by varying $k$

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- How to keep track of buckets?
- Similar to Assignment 1

# Assignment 1 Short Discussion