Lecture 10: Bloom Filters and Cuckoo Filters

Sam McCauley

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Williams College

Admin

- Mini-midterm 1 done!
- Assignment 3 out tonight
- Lecture notes from today's class right after
- I think I'll make a very short lab video for reference. (Totally optional; just to help you find your way around the assignment.)

Wrapping up Probability

Limits of Expectation

- Let's say I charge you \$1000 to play a game. With probability 1 in 1 million, I give you \$10 billion. Otherwise, I give you \$0.
- Would you play this game?
- Answer: probably not. You're just going to lose \$1000.
- But expectation is good! You expect to win \$9000.

Concentration bounds

- Rather than giving the *average* performance, bound the probability of bad performance.
- Let's say I flip a coin k times. On average, I see $k/2$ heads. But what is the probability I never see a heads?
- Answer: $1/2^k$
- Quicksort has expected runtime $O(n \log n)$. What is the probability that the running time is more than $O(n \log n)$?
- Answer: $O(1/n)$

With High Probability

- An event happens *with high probability* (with respect to n) if it happens with probability $1 - O(1/n)$
- So: quicksort is $O(n \log n)$ with high probability
- Cuckoo hashing can maintain its invariant with high probability
- Cuckoo hashing inserts require $O(\log n)$ swaps with high probability
- Linear probing queries require $O(\log n)$ time with high probability. (Contrast to $O(1)$ in expectation!)
- With high probability is always with respect to a variable. Assume that it's with respect to n unless stated otherwise.

WHP example

- How many coins do I need to flip before I see a heads with high probability? (With respect to some variable n)
- If I flip k times, I see a heads with probability $1 - 1/2^k$.
- So I need $1/2^k = O(1/n)$. Solving, $k = \Theta(\log n)$.

Expectation vs Concentration (WHP)

- We'll usually use "with high probability" for concentration bounds
- Expectation states how well the algorithm does on average. Could be much better or worse sometimes!
- With high probability gives a guarantee that will almost always be met: if n is large it becomes vanishingly unlikely that the bound will be violated.

Filters: Goals for Today

What we want

- Worst-case compression
- Lossy compression with algorithmic guarantees
- That is to say: we know what we're losing and what we're not

Filter

- Stores a set S of size n
- Answers queries q of the form: "is $q \in S$?"
  - Really just a very simple dictionary that only returns whether or not a key exists (no values)
- All elements $x \in S$ and all queries must be from some universe U
- (Only need U to make sure that we can hash everything.)

Filter Guarantees

Guarantee 1 (No False Negatives)

A filter is always correct when it returns that $q \notin S$.

Equivalently, if we query an item $q \in S$, then a filter will always correctly answer $q \in S$.

Guarantee 1 Sanity check

- Can you create a very simple data structure that has no false negatives?

- Easiest option: my data structure stores nothing. On every query q, my data structure responds "$q \in S$."

- Another easy option: I store the entire set S using a standard dictionary (perhaps using a hash table). On a query q, I look it up and give the correct answer.

Filter Guarantees

Guarantee 2 (Bounded False Positive Rate)

A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.
We want our filter to have a false positive rate $\varepsilon < 1$.

The filters we will talk about today will work for *any* false positive rate ε, so long as $1/\varepsilon$ is a power of 2.[1] So we can, if we want, guarantee a false positive rate of $1/2$, or of $1/1024$—whatever is best for your use case.

[1] The cuckoo filter will actually need $1 + 1/\varepsilon$ to be a power of 2.

Guarantee 2 Sanity check

- Can you create a very simple data structure that has a good false positive rate?
- I store the entire set S using a standard dictionary (perhaps using a hash table). On a query q, I look it up and give the correct answer. This satisfies Guarantee 2 with $\varepsilon = 0$.
- What about my other solution? If I always return $q \in S$, what false positive rate do I have?
- Answer: 1. (So this is not a filter!)

Tradeoff

- Obviously, smaller ε is better-it means we make fewer mistakes.
- So what's the tradeoff?
- We tradeoff space versus accuracy using ε.
  - Smaller ε means the compression is not as lossy
  - We make fewer mistakes, but we need more space
  - Larger ε means more aggressive compression
  - Space is very small, but filter is very inaccurate!
- A filter generally requires $O(n \log 1/\varepsilon)$ bits of space.

Space bounds

We talk about two filters today:

- A Bloom filter requires $1.44n \log_2(1/\varepsilon)$ bits of space.
- The cuckoo filter uses $1.05n \log_2(1 + 1/\varepsilon) + 3.15n$ bits of space.

How can we interpret this?

- Plugging in numbers: if we have a cuckoo filter with $\varepsilon = 1/63$, the filter takes less than 1 byte of space per element being stored.
- Notice that this space does *not* depend on the size of the original elements. We can store very long strings and still require only one byte per string stored.

History and Discussion

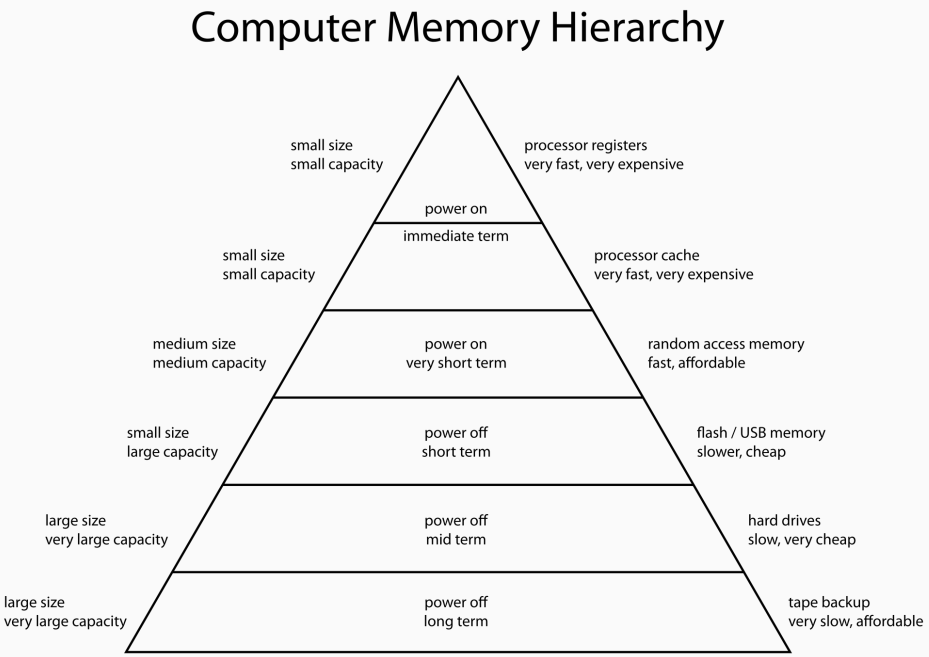
Bloom filter

- Invented by Burton H. Bloom in 1970
- Original publication only talked about good practical performance; theoretical analysis came later.

Cuckoo filter

- Invented by Fan et al. in 2014
- Provides better space usage for small ε (i.e. when the compression is not too lossy)
- Requires fewer hashes; has better cache performance.

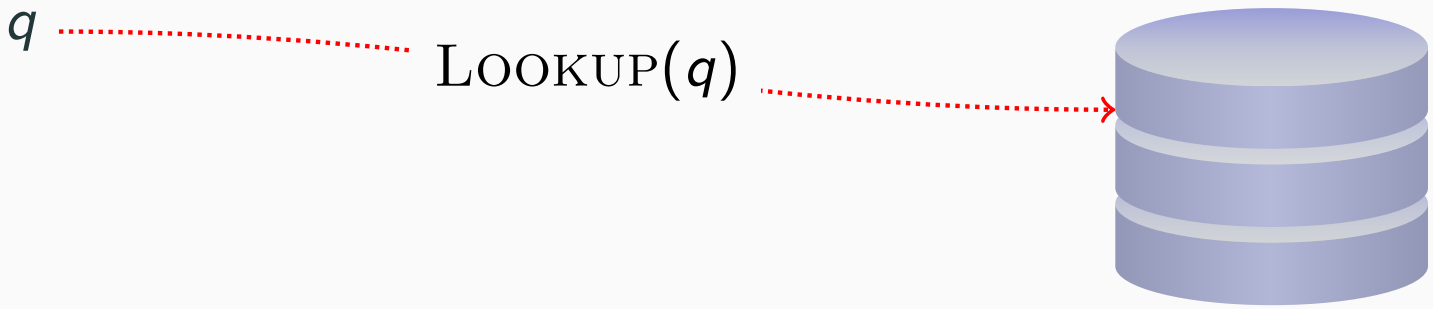
When should you use a filter?



1st example: avoiding cache misses

- Let's say we have a very large table of data
- Large enough that it doesn't fit in L3
  - Maybe it doesn't even fit in RAM
- Frequently query items not in the table

Common filter usage



Queries to the entire dataset are very expensive!

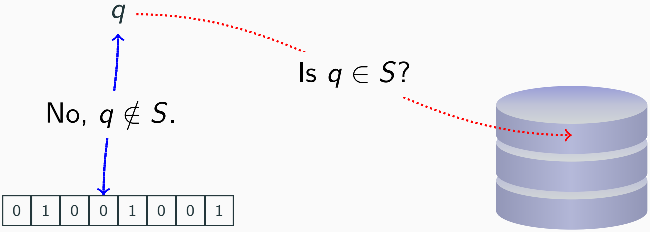
Queries are often "unnecessary"

Many workloads involve mostly "negative" queries: queries to keys not stored in the table. (query $q \notin S$)

- Classic example: dictionary of unusually-hyphenated words for a spellchecker.
- Checking if key already exists before an insert (deduplication in general)
- Check for malicious URLs
- Table with many empty entries

Classic filter usage: succinct data structure that will allow us to "filter out" negative queries.

Common filter usage



Filters are so small that they can fit in local memory.

Filters can [illegible]ship queries,
improving performance.
If $q \notin S$ (false positive), still do an unnecessary access.

Common filter usage

- With $O(n \log 1/\varepsilon)$ local memory (perhaps fitting in L3 cache), can filter out $1 - \varepsilon$ cache misses for keys $q \notin S$.

- Greatly reduces number of remote accesses, thereby reducing time.

When should you use a filter?

2nd example: Approximately storing a set

- Before, we stored the actual set S. (It was expensive to access, but we stored it.)
- But what if we don't want to?
- Example: approximate spell checker

Approximate spell checker

- Want to build a spell checker; don't have room to store dictionary
- Store the words in a filter
- Guarantee 1: if we query a correctly-spelled word, it is never marked as misspelled
- Guarantee 2: if we query a misspelled word, we only miss it (don't mark it misspelled) with probability ε
- Using only a byte or so per item, can do almost as well as storing a full dictionary!

Bloom Filters

Bloom Filter

A Bloom filter consists of:

- $k = \log_2 1/\varepsilon$ hash functions, which I will denote using $h_1, h_2, \ldots, h_k$,
- Bit array A of $m = (\log_2 e)nk \approx 1.44n \log_2(1/\varepsilon)$ bits.
- For each $i = 1, \ldots, k$, $h_i : U \to \{0, m-1\}$ (that is to say, h_i maps an element from the universe of possible elements U to a slot in the hash table).
- Assume $1/\varepsilon$ is a power of 2. Round m up to the nearest integer

Building a Bloom Filter

- Begin with $A[i] = 0$ for all i. (Basically, just calloc the bit array.)

- For each $x \in S$:
  - For each hash $h_i = h_1, h_2, \ldots, h_k$:
    - Set $A[h_i(x)] = 1$.

Building a Bloom Filter

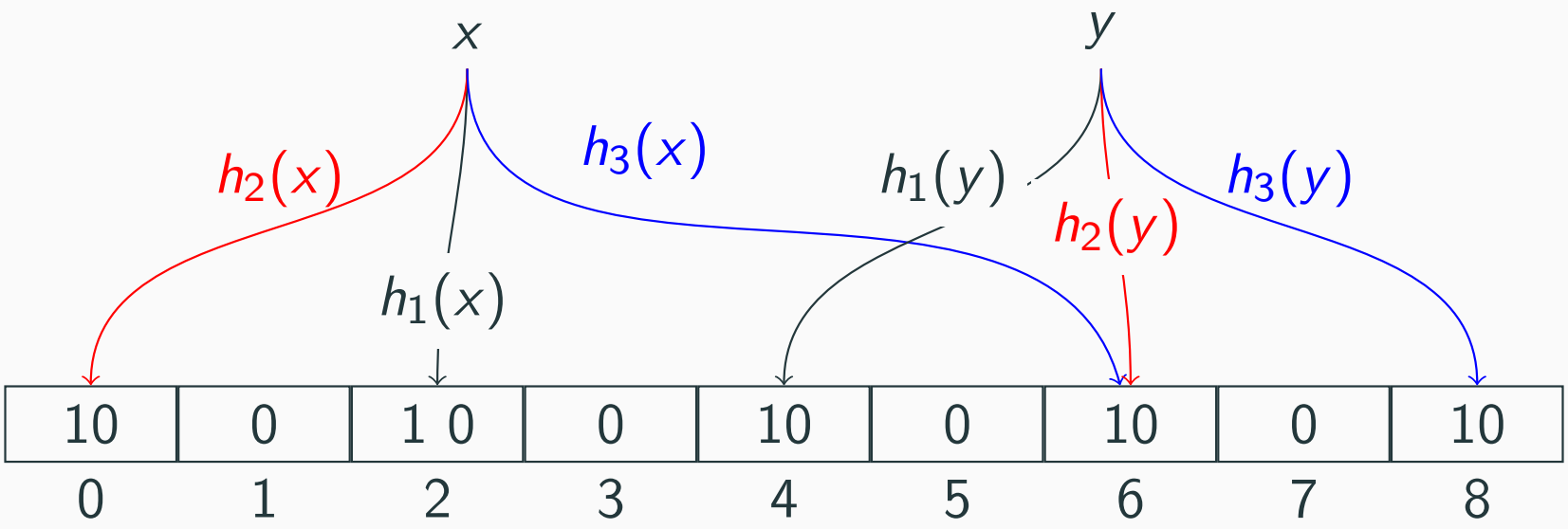


Figure 1: Inserting two elements x and y into a Bloom filter with $\varepsilon = 1/8$. We have three hash functions, and (rounding up) the array is of length $m = 9$ bits.

Invariant

- What invariant does this data structure satisfy?

Invariant 1

A Bloom filter storing a set S using hashes $h_1, \ldots h_k$ satisfies $A[h_i(x)] = 1$ for all $x \in S$ and all $i \in \{1, \ldots, k\}$.

Querying a Bloom filter

On a query q:

- For each hash $h_i = h_1, h_2, \ldots h_k$:
  - If $A[h_i(q)] = 0$, return "$q \notin S$."

- Otherwise, $A[h_i(q)] = 1$ for all h_i; return "$q \in S$."

Query example

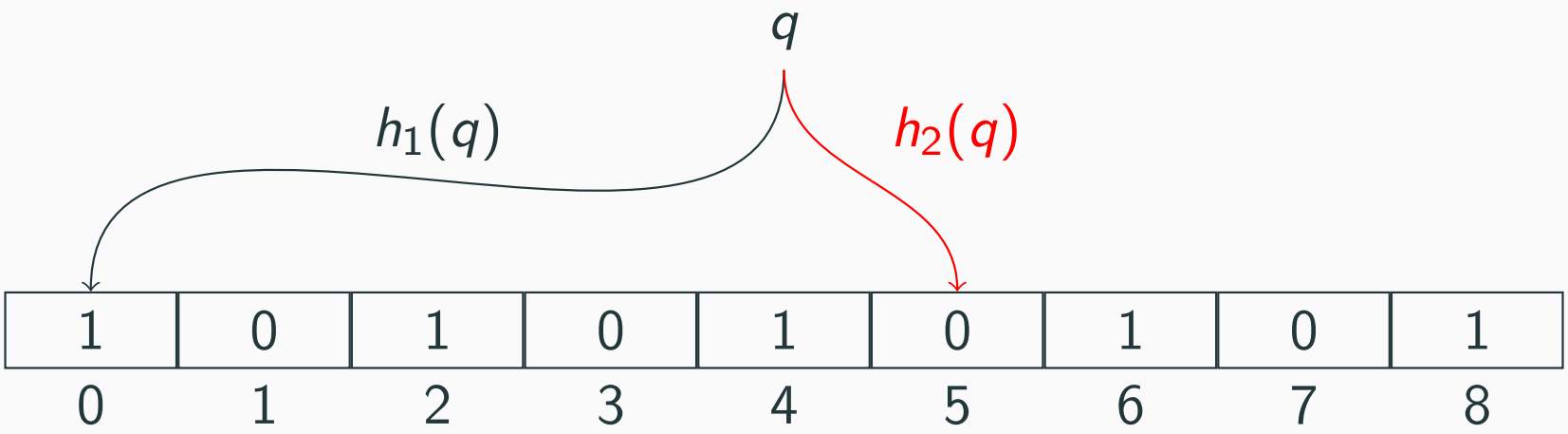


Figure 2: An example query to an element not in the set; $k = 3$.

Query example 2

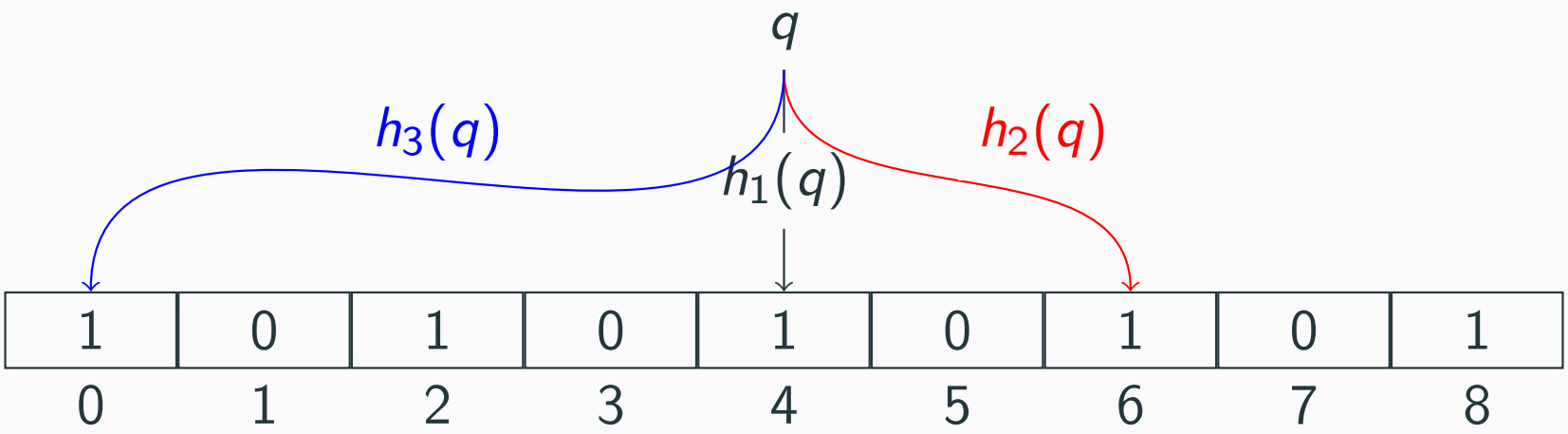
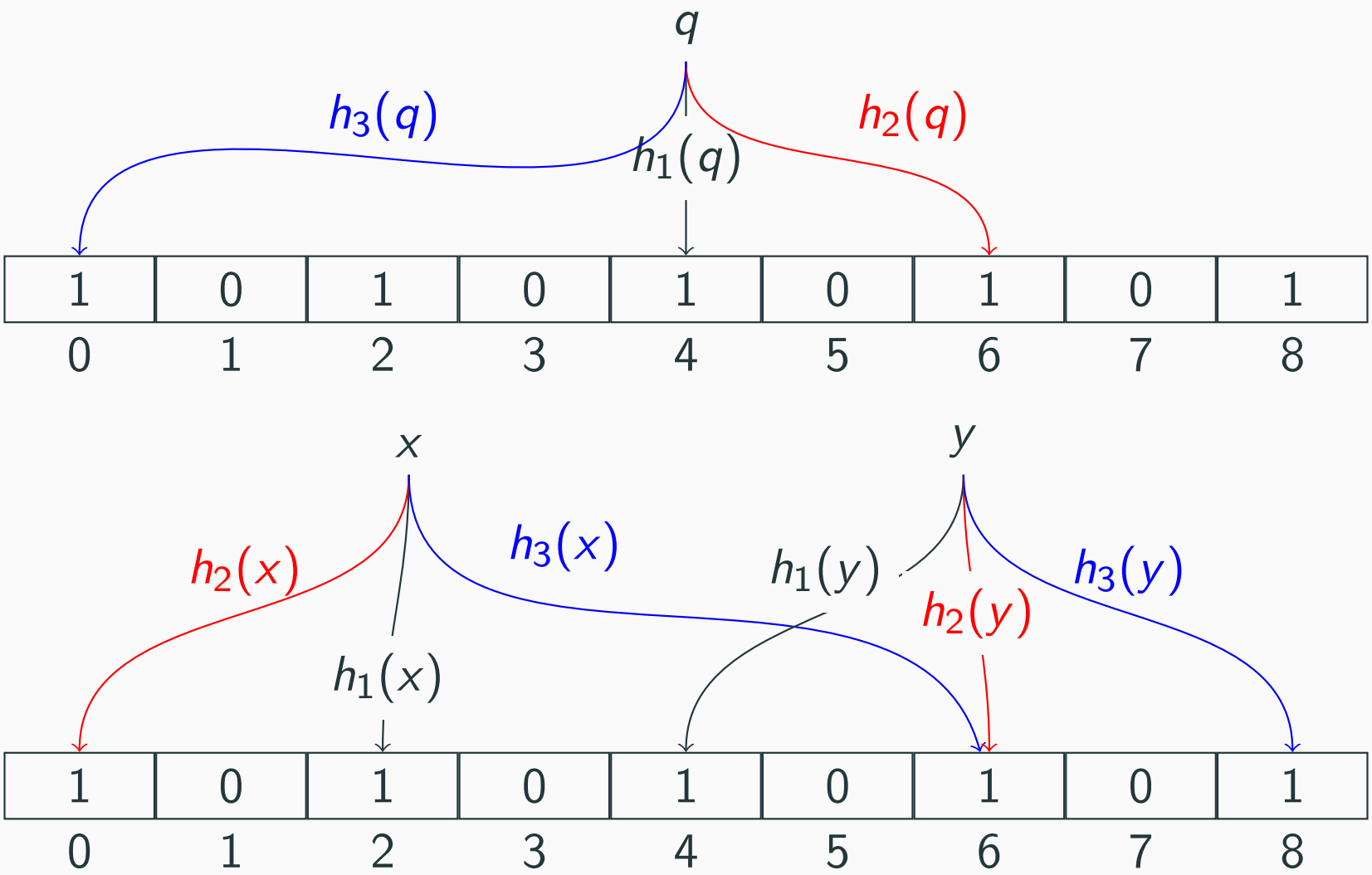


Figure 3: An example false positive query.

Query example 2



Discussion

- Can we insert into a Bloom filter?
  - Yes, but performance degrades as it fills up. We are OK so long as no more than n items are inserted.

- Can we delete?
  - No. If we flip a bit from 1 to 0, it may cause a false negative, violating Guarantee 1.

Bloom filter analysis

- Assume our hashes h_i are perfectly uniform random: any $x \in U$ is mapped to any hash slot $s \in \{0, \ldots, m-1\}$ with probability $1/m$; independently of any other hash.

- Let's strategize: what about the Bloom filter can we use to prove that Guarantee 1 and Guarantee 2 hold?

Guarantee 1

Guarantee (No False Negatives)

If we query an item $q \in S$, then a filter will always answer $q \in S$.

- By the Bloom filter Invariant, if $q \in S$, then $A[h_i(q)] = 1$ for all $i \in \{1, \ldots k\}$.

- This means that the query algorithm always returns "$q \in S$."

Guarantee 2 (False positive rate)

Guarantee (Bounded False Positive Rate)

A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

High-level argument:

- Assume: each entry of A is 1 with probability $1/2$
- Only get a false positive if every bit is a 1
- Are these events independent?
  - No! But it seems like the independence isn't too big of a deal...let's assume they're independent for now.
- Occurs with probability $(1/2)^k = (1/2)^{\log_2(1/\varepsilon)}$
- $(1/2)^{\log_2(1/\varepsilon)} = \varepsilon$.

Cuckoo Filter

Assignment 3

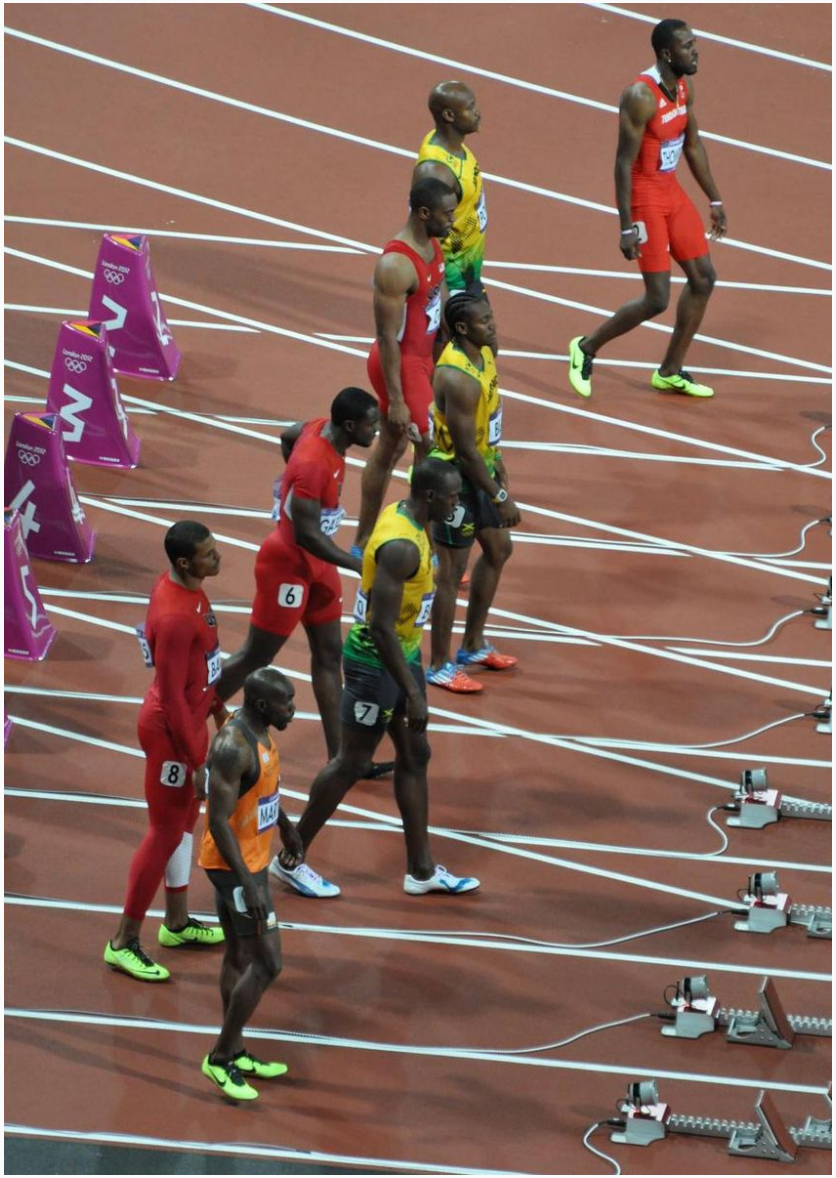
- In short: you'll implement a cuckoo filter to speed up a sequence of dictionary queries
- You're looking for "bilingual palindromes": strings whose reverse is a word in another language
- Most words are not bilingual palindromes, so a filter can significantly speed up queries

Cuckoo Filter

A cuckoo filter consists of:

- k hash functions denoted by $h_1, h_2, \ldots, h_k$ (k is a constant)
  - We'll only use one of these hash functions (h_1) in our implementation!
- a fingerprint hash function f that takes an item from the universe and outputs a number from 1 to $1/\varepsilon$ (we'll call this number the *fingerprint* of the item)
- a cuckooing hash function h that takes in a fingerprint and outputs a number from 1 to m, and
- a hash table T of m slots, where each slot has room for $\log_2(1/\varepsilon)$ bits.

Some initial parameters



- $k = 2$ hash functions
- $m = 2n$ slots
- These parameters are easy to analyze, but space inefficient. We'll fix it later.

- Also assume that $1/\varepsilon + 1$ is a power of 2, and m is a power of 2.

Initializing a Cuckoo Filter

- Make sure all slots of T are empty

- Today: we'll set all slots to 0. A slot in T is nonempty if and only if it stores a number larger than 0.

Inserting into a Cuckoo Filter

- If there is an h_i such that $T[h_i(x)]$ is nonempty, then store $f(x)$ in $T[h_i(x)]$.
- Otherwise, we cuckoo:
  - Choose some $i \in \{1, \ldots, k\}$
  - Let's say that x_1 is the element stored in $T[h_i(x)]$.
  - Then we store $f(x)$ in $T[h_i(x)]$ and "cuckoo" x_1 to another slot
- If we cuckoo more than $\log n$ elements, we rebuild the filter.

Implementing Insertions

There's a problem with what I said!

- We don't have access to x_1. So how can we calculate $h_2(x_1)$?
- If $k = 2$, we can use *partial-key cuckoo hashing*
- Set $h_2(x) = h_1(x) \hat{} h(f(x))$. (XOR)
- Note that then
  $h_2(x) \hat{} h(f(x)) = h_1(x) \hat{} h(f(x)) \hat{} h(f(x)) = h_1(x)$.

Cuckooing

So to cuckoo a fingerprint ϕ stored in a slot s to its other location:

- Calculate $h(\phi)$
- Its other slot is $s \hat{} h(\phi)$.
- If that other slot is empty we can store ϕ in it (woo)! Otherwise, take the fingerprint stored there and cuckoo it to its other slot.

Cuckoo Filter Insert Example 1

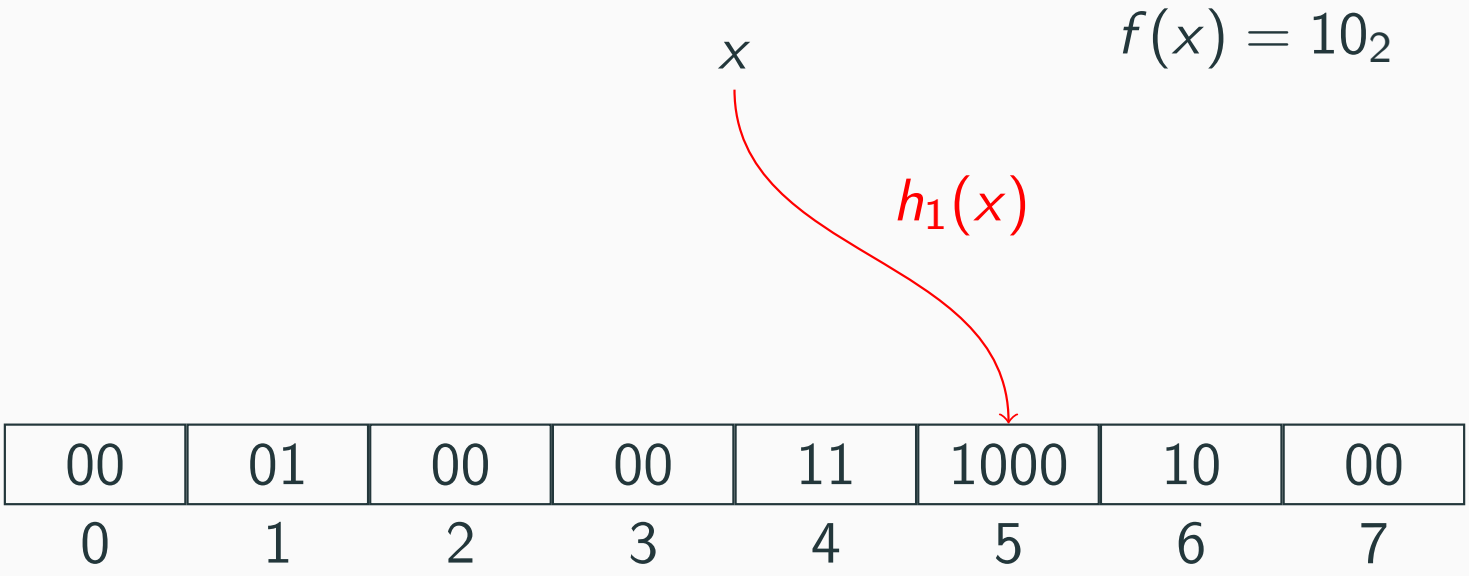


Figure 4: A cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Cuckoo Filter Insert Example 2

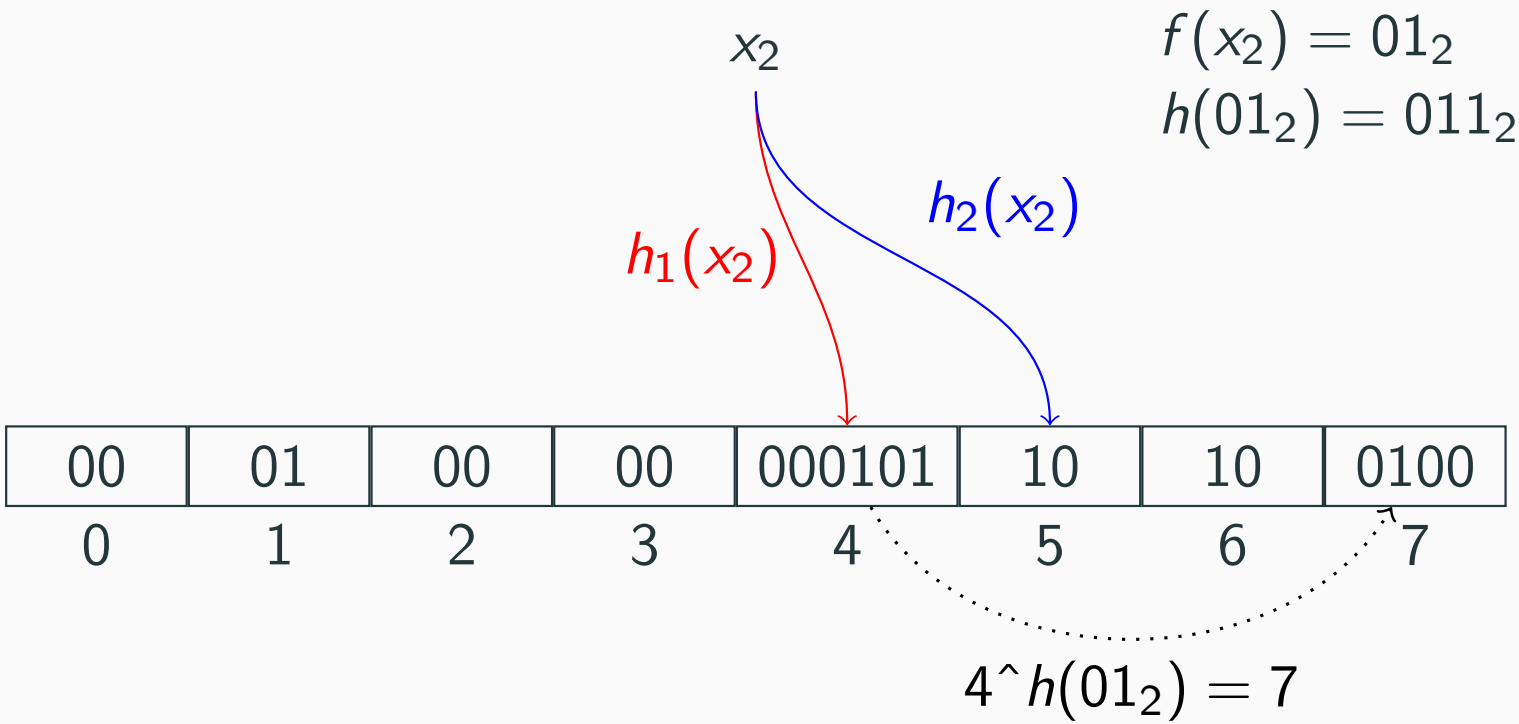


Figure 5: A cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Cuckoo Filter Invariant

Invariant 2

For every $x \in S$, there exists an $i \in \{1, \ldots, k\}$ such that $f(x)$ is stored in $T[h_i(x)]$.

Querying a Cuckoo Filter

To query an element q

- For each hash $h_i = h_1, h_2, \ldots h_k$:
  - If $T[h_i(q)] = f(q)$, return "$q \in S$."
- Return "$q \notin S$."

Querying a Cuckoo Filter: Example

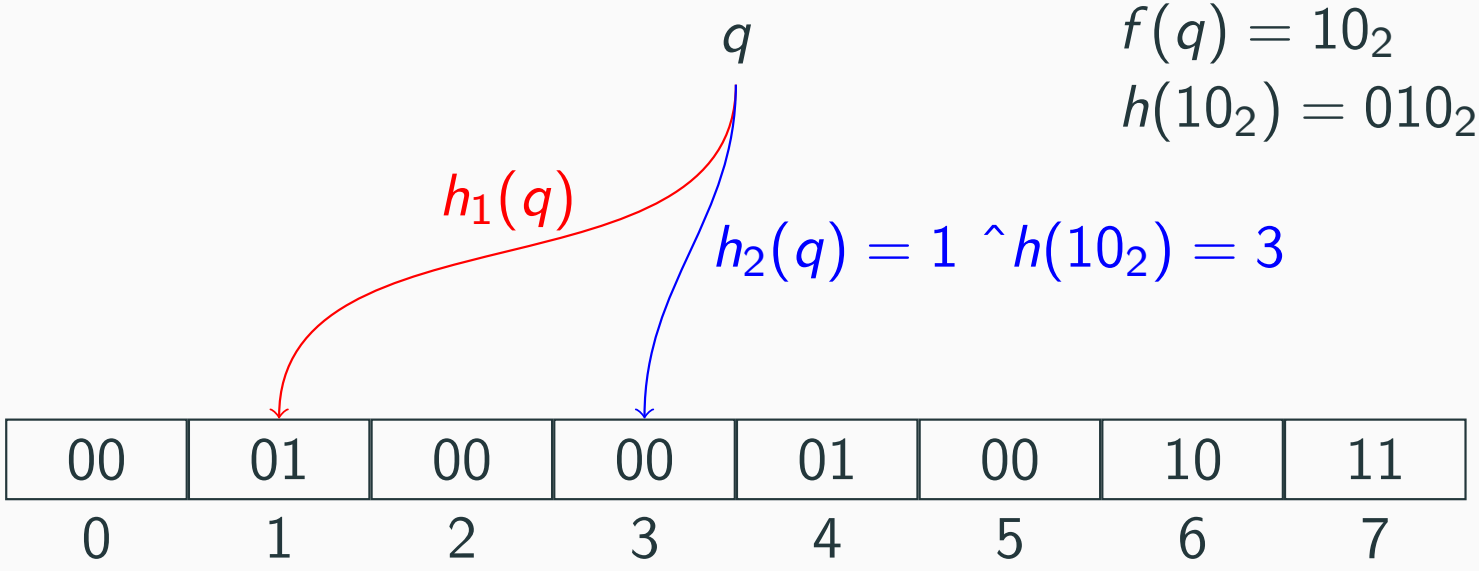


Figure 6: Querying a cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Querying a Cuckoo Filter: Example 2

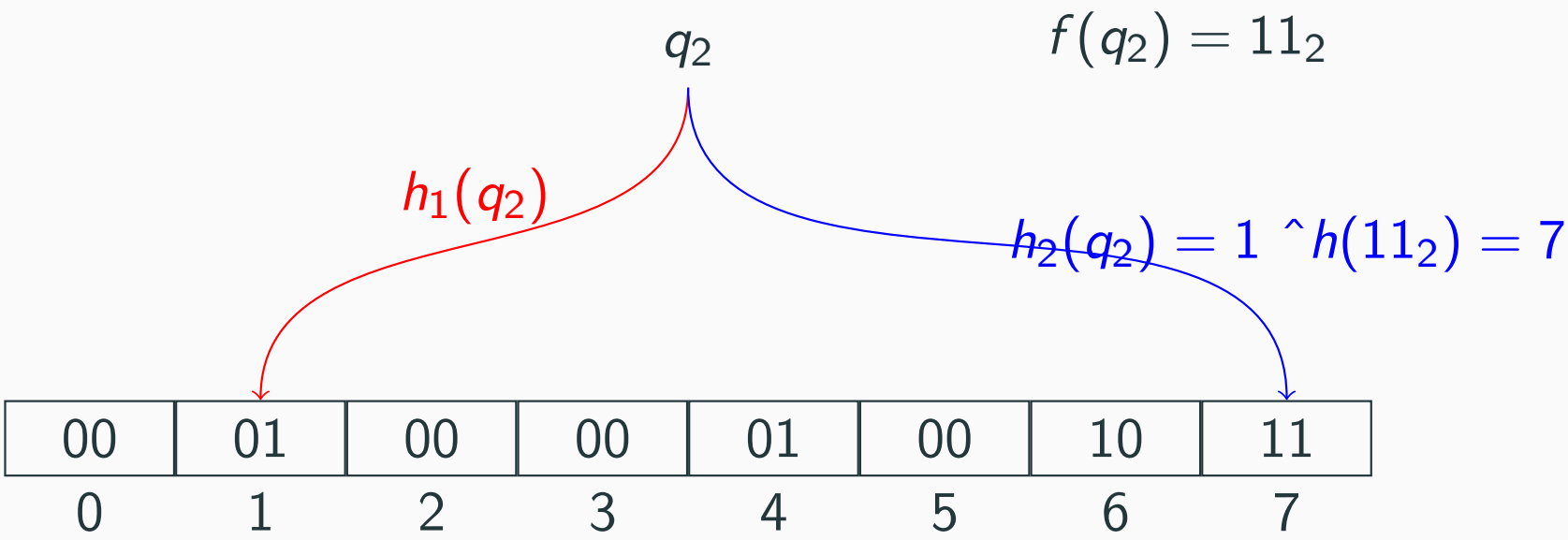


Figure 7: Querying a cuckoo filter with $\varepsilon = 1/3$ and $k = 2$.

Discussion

- Can a cuckoo filter handle inserts?

- How about deletes?

Union Bound

- Simple but useful tool in randomized algorithms
- Always works, even for events that are not independent

Theorem 1

Let X and Y be random events. Then

$$\Pr(X \text{ or } Y) \leq \Pr(X) + \Pr(Y).$$

More generally, if $X_1, X_2, \ldots, X_k$ are any random events, then

$$\Pr(X_1 \text{ or } X_2 \text{ or } \ldots \text{ or } X_k) \leq \sum_{i=1}^{k} X_k.$$

Union Bound Example

- Let's say I have 10 students in a course, and I randomly assign each student an ID between 1 and 100 (these IDs do not need to be unique).

- Can you upper bound the probability that some student has ID 1?

Exact Analysis of Student ID Problem

- The probability that at least one student has ID 1 is

$$1 - \Pr(\text{no student has ID 1}).$$

- The probability that a single student has an ID other than 1 is $99/100$.
- Thus, the probability that all 10 students have an ID other than 1 is $(99/100)^{10}$.
- Thus, the probability that at least one student has ID 1 is $1 - (99/100)^{10} \approx 9.56\%$.

Exact Analysis of Student ID Problem

- The probability that at least one student has ID 1 is

This is messy! And it would be even worse if the IDs were not independent!

The union bound lets us avoid this work.

- The prob[illegible]er than 1 is $99/100$.
- Thus, the probability that all 10 students have an ID other than 1 is $(99/100)^{10}$.
- Thus, the probability that at least one student has ID 1 is $1 - (99/100)^{10} \approx 9.56\%$.

Union Bound Analysis of Student Problem

- The probability that a given student has ID 1 is $1/100$.
- From Union bound: The probability that *any* student has ID 1 is the sum, over all 10 students, of $1/100$.
- This gives us a value of $10/100 = 10\%$.

Analysis

Some assumptions:

- all hash functions h_i are uniformly random: any $x \in U$ is mapped to any hash slot $s \in \{0, \ldots, m-1\}$ with probability $1/m$.

- Same for the fingerprint hash f: any $x \in U$ is mapped to a given fingerprint $f_x \in \{1, \ldots, 1/\varepsilon\}$ with probability ε.

First Guarantee: No False Negatives

Guarantee (No False Negatives)

A filter is always correct when it returns that $q \notin S$. Equivalently, if we query an item $q \in S$, then a filter will always correctly answer $q \in S$.

Invariant

For every $x \in S$, there exists an $i \in \{1, \ldots, k\}$ such that $f(x)$ is stored in $T[h_i(x)]$.

- We can see that the invariant means that there are no false negatives.

Second Guarantee: False Positive Rate

Guarantee 3 (False Positive Rate)

A filter has a false positive rate ε if, for any query $q \notin S$, the filter (incorrectly) returns "$q \in S$" with probability ε.

- A query $q \notin S$ is a false positive if, for some h_i, $T[h_i(q)] = f(q)$.

- Let's examine each hash h_1 and h_2 individually.

Second Guarantee: False Positive Rate

- Let's start with h_1. What is the probability $T[h_1(q)]$ contains a fingerprint?
- $1/2$, because we are storing n elements in $2n$ slots.
- If $T[h_1(q)]$ contains a fingerprint, the probability that $f(x) = f(q)$ is ε.
- Therefore, the probability that $T[h_1(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.

Second Guarantee: False Positive Rate

- What about h_2?

- Same exact analysis: probability that $T[h_2(q)]$ contains a fingerprint $f(x) = f(q)$ is $\varepsilon/2$.

Second: Guarantee: Putting it Together

- q is a false positive if either $T[h_1(q)]$ contains a fingerprint $f(x_1)$ such that $f(x_1) = f(q)$, or $T[h_2(q)]$ contains a fingerprint $f(x_2)$ such that $f(x_2) = f(q)$

- Each happens with probability at most $\varepsilon/2$

- By union bound, one or the other happens with probability at most $\varepsilon/2 + \varepsilon/2 = \varepsilon$.

Improving the Cuckoo Filter

- Currently, have $m = 2n$ slots, so the space is $2n \log_2(1/\varepsilon)$.
- Here is one way to improve that:
- Store room for *four* fingerprints in each hash slot, and make the fingerprints hash to $\{1, \ldots, 8/\varepsilon\}$. Assume that $8/\varepsilon + 1$ is a multiple of 2.
- Then can set $m = 1.05n/4$, giving total space usage $1.05n \log_2(8/\varepsilon + 1) \approx 1.05n \log_2(1/\varepsilon) + 3.15n$.

Example

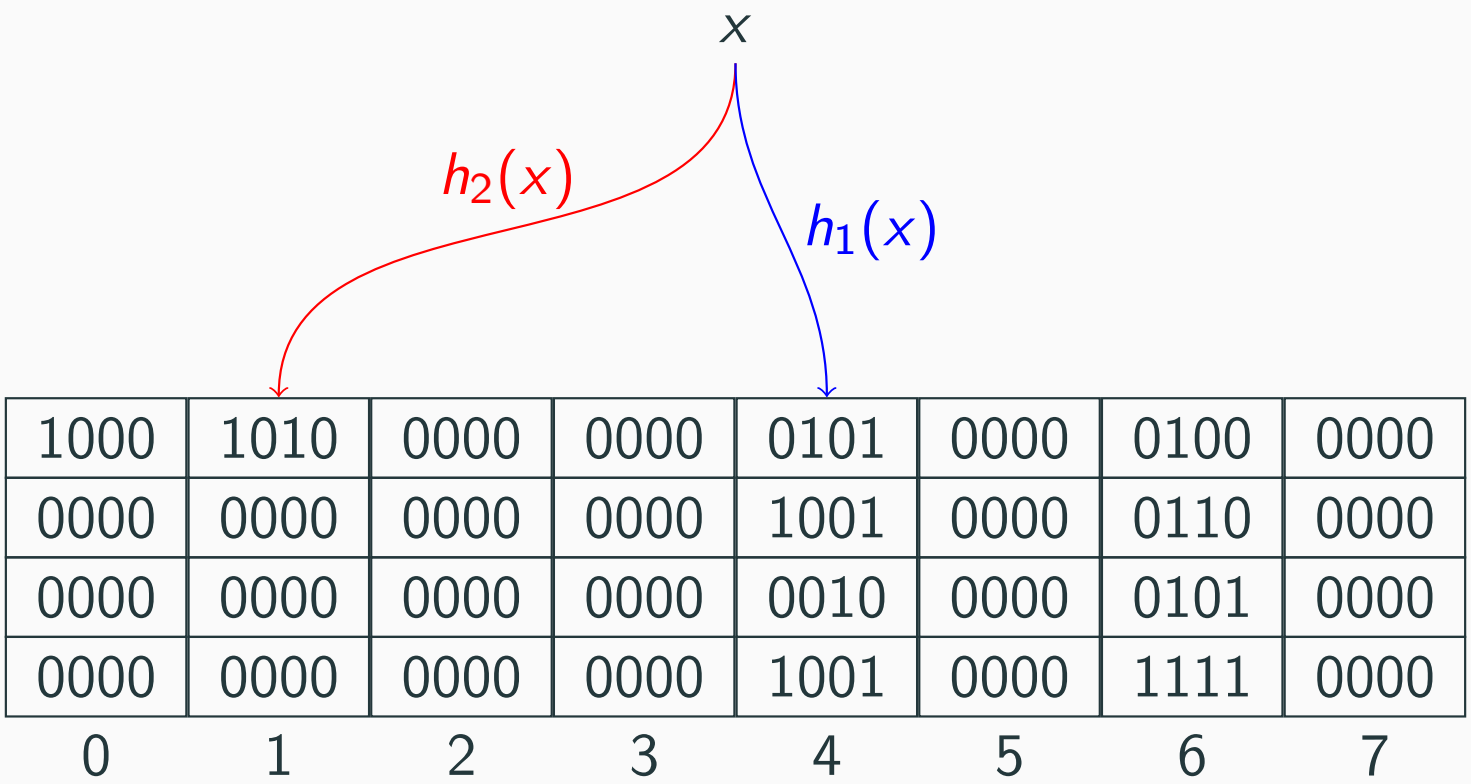


Figure 8: A cuckoo filter with fingerprints of length 4, $k = 2$, and 4 slots per bin.

Example 2

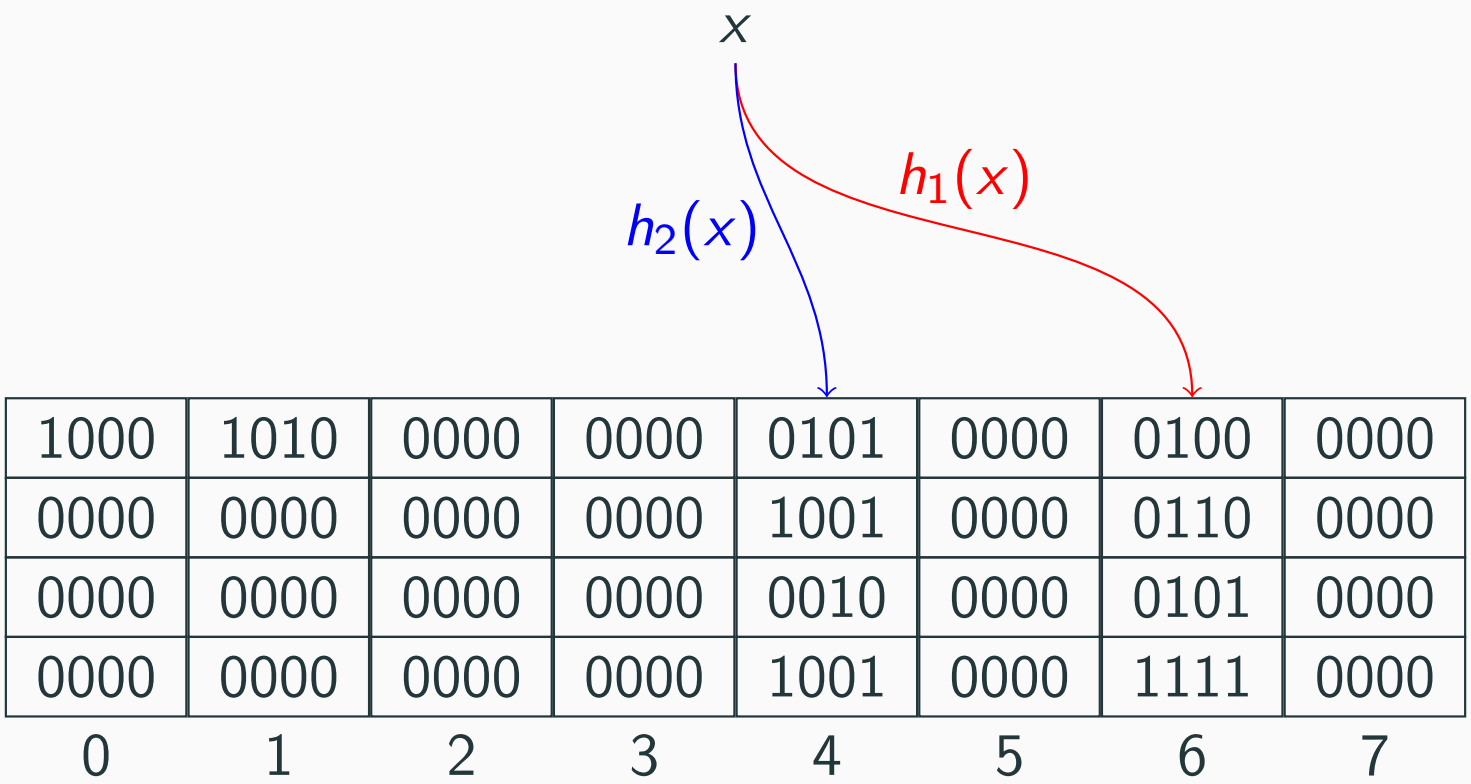


Figure 9: A cuckoo filter with fingerprints of length 4, $k = 2$, and 4 slots per bin.

Comparing the Two Filters

Bloom filters:

- Easy to implement
- Fairly efficient for large ε

Cuckoo filters:

- Much more space efficient
- Only require 2 hash functions (may improve practical performance)
- Good cache efficiency: only need to access the hash table 2 times, rather than $\log_2(1/\varepsilon)$.