Lecture 23: van Emde Boas Trees

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December 6, 2021

Williams College

Admin

- Assignment 5 back (MM2 and MM3 back soon; Assignment 8 probably not back until after reading period unfortunately)
- Today: an SA-IS example run beginning to end, then van Emde Boas trees

Next Class

- Review!
- I have some topics I want to go over:
  - One example of algorithm analysis in the external memory model
  - One probabilistic algorithm example (Random shuffle from Assignment 5)
  - One LP or ILP
- If you have something you want to see, please *email me* the topic before Thursday
- Questions during class are also OK, but it's better if I have prep
- We'll also do course evaluations at the end of class (please bring a laptop or something)

Final Exam Info

- Comprehensive (on all parts of course)
- Remote 24 hour take home exam. Can take at any point starting Dec 11 (Saturday); must finish by 8:30pm Dec 19 (Sunday)
- Start thinking of when you want to take it
- I'll be handing them out manually (current plan is email). You'll need to let me know when you want to take it. If you don't I'll send it to you at 8:30pm Dec 18
- Not much coding (just a little bit). You should have access to the lab computers. Let me know if that's a problem. (Probably not a problem if you have access to the American internet)

SA-IS Example

String: `CACATACACAGACACAC$`

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Correct Suffix Array: 17 15 13 11 5 7 1 9 3 16 14 12 6 0 8 2 10 4

Predecessor and Successor Queries

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- In CS 136 we saw how to answer this using a balanced binary search tree in $O(\log n)$ time

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  - Successor: Find the smallest $i \in S$ such that $i \geq q$
- In CS 136 we saw how to answer this using a balanced binary search tree in $O(\log n)$ time
- This is optimal if all you can do is compare items

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- Know much more about the relative values of integers or strings
- Today: let's say that the items of S are taken from a bounded set $\{0, \ldots, M-1\}$
- For example: if the items of S are 64-bit integers, then we have $M = 2^6 4$. If items of S are k-character strings, we have $M = 8^k$.

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- This assumption is often too restrictive! Often we want to perform predecessor queries on integers or strings
- Know much more about the relative values of integers or strings
- Today: let's say that the items of S are taken from a bounded set $\{0, \ldots, M-1\}$
- For example: if the items of S are 64-bit integers, then we have $M = 2^64$. If items of S are k-character strings, we have $M = 8^k$.
- In this case, we will show how to get predecessor and successor in $O(\log \log M)$ time.
  - For a w-bit integer, get $O(\log w)$ time
  - For a k-character string, get $O(\log k)$ time

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- Van Emde Boas tree!
- Clever data structure. Very good constants, but still used sometimes in practice
- We'll only look at inserts, successor. Can generalize to predecessor queries and deletes.
- Let's not worry about space today (we'll wind up with $O(M)$ space). Some techniques to achieve $O(n)$ space.
- Also, let's assume that $\log_2 \log_2 M$ is an integer (M is 2 to a power of 2; like 2^8 or $2^6 4$

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  - Time for insert?
  - $O(1)$
  - Time for successor?
  - $O(M)$
- Insert is really fast. Can we try to speed up successor?

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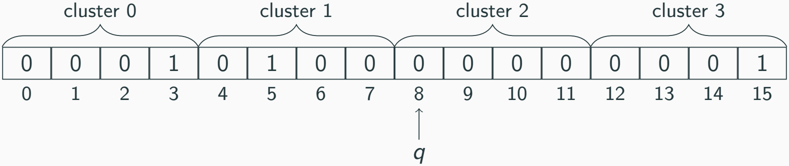
    - Otherwise, find the next nonempty cluster
    - Then, query within the correct cluster for the minimum element ($O(\sqrt{M})$ time as before)
    - How can we query for minimum using a successor query?
    - How can we find the next nonempty cluster?



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- We want to find the next nonempty cluster
- That's a successor query!
- Let's create a second, identical data structure to hold whether or not each cluster is empty

Summary array:

1	1	0	1
0	1	2	3

cluster 1				cluster 2				cluster 3				cluster 4			
0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
								q							

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$O(1)$ insert, $O(\sqrt{M})$ successor query:

Successor:

- Figure out which cluster q is in (can calculate: $\lfloor q/\sqrt{M} \rfloor$)

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- If it's not found:
  - Find the next nonempty cluster by looking in the summary array ($O(\sqrt{M})$ time)
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- Insert is still really fast, we want to improve successor.
- Where can we improve?
- All our time is spent doing array scans for successor queries within a cluster...
- But we know how to do better-than-linear successor queries! Let's recurse.

We stopped here in class!

- Idea of the rest: we need to recurse rather than doing these expensive searches

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- Idea of the rest: we need to recurse rather than doing these expensive searches
- Key obstacle to overcome: need to only do *one* recursive call on each; that will get us recurrence $T(M) = T(\sqrt{M}) + O(1)$ which solves to $O(\log \log M)$.

Rest of Slides (We didn't get to in class)

Recursing: van Emde Boas Tree (almost)

If $M = 1$, just store the array.

Otherwise:

- Store a summary vEB tree of size $\sqrt{|M|}$ to keep track of which clusters are full

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- For each cluster, store a vEB tree of size $\sqrt{M}$

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- (Keep an array with a pointer to each of these vEB trees)

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- For each cluster, store a vEB tree of size $\sqrt{M}$
- (Keep an array with a pointer to each of these vEB trees)
- Let's draw a picture of it on the board

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- Recurrence:
- $T(M) = 2T(\sqrt{M}) + O(1)$
- Solves to $O(\log M)$ insert time (too slow!)

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- Let's draw what this might look like on the board.
- Recurrence:
- $T(M) = 3T(\sqrt{M}) + O(1)$
- Solves to $O((\log M)^{\log_2 3}) = O(\log^{1.585} M)$ insert time (way too slow!)

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- Can we get rid of some of them? Let's focus on successor

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    - Then query that cluster for the minimum element
- Finding the minimum element doesn't require a whole successor call! Let's just store the minimum element in each cluster. Then finding the minimum element is $O(1)$.

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- Recurrence for both: $T(M) = 2T(\sqrt{M}) + O(1)$; solves to $T(M) = \log M$.

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- How can we make this just one call?
- *Hint:* Can we store something to help us determine if q has a successor in its cluster without a recursive query?

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- Otherwise, use a successor query on the summary vEB tree to find the next nonempty cluster
- Return the minimum element in that cluster
- Example on board: store $3, 5, 15$ from universe $\{0, \ldots 15\}$; query for element 8.

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- In that case: just store q as min!
- Change to the algorithm: don't store minimum element recursively!
- Only need to recurse on summary data structure

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- No, but it's easy to fix: just check if $q <$ the minimum element. If so, the minimum element is the successor.
- Done!

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- Otherwise, have a "summary" vEB tree of size $\sqrt{M}$; and, divide M into $\sqrt{M}$ parts, with one vEB tree for each
- Plus the minimum and maximum elements in our structure, if they exist

van Emde Boas Tree Summary: Insert

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To insert an item x:

- Find x's cluster c. If c has no minimum, set the minimum of c to be x, and insert c into the summary data structure.
- Otherwise:
  - Check if x is less than the minimum m.
  - If so, set x to be the minimum, and insert m into x's cluster.
  - Do the same for the maximum.
  - Otherwise, insert x into its cluster.

van Emde Boas Tree Summary: Successor

To find the successor of an item x:

van Emde Boas Tree Summary: Successor

To find the successor of an item x:

- If x is less than the current minimum element m, return m.

van Emde Boas Tree Summary: Successor

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- Otherwise, query the summary vEB tree for the successor of c; call it c'. Return the minimum element of c'.

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- Insert does $O(1)$ work and makes one recursive call of size $\sqrt{M}$; also $O(\log\log M)$ time

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- Possible to get $O(n)$ space deterministically using another, more complicated data structure (y-fast tries)

Predecessor/Successor data structures

For a set S from $\{0, \ldots, M-1\}$:

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- van Emde Boas trees: $O(\log \log M)$
- Takeaway: unless M is very large or n is very small, vEB trees are quite a lot faster
- But, they're probably a bit more complicated

That's all for today!

Remember to:

- Think about when you might want to take the exam

That's all for today!

Remember to:

- Think about when you might want to take the exam
- Let me know (email or slack) if there are any topics you'd like me to cover on Thursday