Vindictive Bidding in Keyword Auctions

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ABSTRACT

We study *vindictive bidding*, a strategic bidding behavior in keyword auctions where a bidder forces his competitor to pay more without affecting his own payment. We show that most Nash equilibria (NE) are vulnerable to vindictive bidding and are thus unstable. Pure strategy Nash equilibria (PSNE) may not exist when there are at least three players who are all vindictive with each other. And there always exists a PSNE if there is only one pair of vindictive players. Given the set of vindictive bidding pairs, we show how to compute a pure strategy Nash equilibrium (if one exists). As an ongoing work, we also propose several interesting open problems related to vindictive bidding.

1. INTRODUCTION

The sponsored search keyword auction is one of the most important economic mechanisms used on the Internet since it effectively monetizes search activities for internet companies like Yahoo! and Google. Their effectiveness has helped make the web a remarkably efficient advertising medium, and the associated economic benefits have stimulated innovation in search as well as further investment in the Internet at large by companies who benefit from the increased efficiencies.

As sponsored search auctions continue to mature and become even more mainstream and competitive, there has been much interest in the research community on keyword bidding strategies and basic mechanism design questions. Most of this work, e.g. [6, 10, 3, 9], has emphasized the aspect of the keyword auctions that differentiates them from the most frequent types of auctions studied previously. This aspect is their positional nature, in which bidders are not simply competitively bidding on one object, but on the position of their advertisement on the search result page.

Most of the previous work has focused on the scenario where bidding agents are completely self-interested in the sense of most classical auction theory, i.e., they maximize their own utility without considering the utility of other agents. In keyword auctions, a keyword normally corresponds to a specific product or service. When a specific product or service is considered, usually there may be a handful of top players who control the majority of the market share. In such competitive markets, it can often be in a business's interest to try to squeeze other players out of the market, thus reducing market competition. In starkly competitive markets, companies have to take into consideration their competitors. For the corresponding keyword auction, it is even more evident. Normally each bidder in a specific keyword auction market has a budget (either daily budget, monthly, quarterly or annual budget). There are a few big players, and if a bidder can drive one player out of the market, then that bidder can benefit from reduced competition to lower their advertising costs and increase their profit.

In this paper we study *vindictive bidding*, a strategic bidding behavior in keyword auctions, where a bidder forces his competitor to pay more without affecting his own payment. This is perfectly legal in current keyword auction systems, and we show some empirical evidence that supports the fact that it happens quite frequently. However, vindictive bidding creates quite a few complicated issues. If there are too many vindictive players involved, then the market may become unstable: it may be that no pure strategy Nash equilibria exist. By explicitly adding the vindictive component to bidder behavior, we show that the huge space of pure strategy Nash equilibria considered in prior work is reduced or sometimes eliminated. So there is a tradeoff between vindictiveness and stability. And it is an intriguing decision whether a bidder should be either vindictive to or cooperative with another bidder.

Our results and the empirical evidence also suggest that many keyword markets may not be in competitive equilibrium, and there is more learning to be done on the part of market participants. The empirical evidence also suggests some clear opportunities for benefit from mixed strategies. Although we do not currently have time-course bidding data, our analysis suggests there may be additional interesting unobserved (by us) dynamics going on.

1.1 Organization

The rest of the paper is organized as follows. In Section 2, we discuss related work. In Section 3, we show that traditional Nash equilibria become no longer stable with vindictive bidders, and give an example of bidding dynamics with two vindictive players. We also show *statistically* that vindictive bidding pairs (two consecutive bids with price difference 1 cent) is very common in practice. In Section 4, we show that most Nash equilibria (NE) are vulnerable to

Second Workshop on Sponsored Search Auctions, June 11, 2006, Ann Arbor, Michigan.

vindictive bidding and thus unstable. Pure strategy Nash equilibria (PSNE) may not exist when there are at least three players who are all vindictive with each other. And there always exists a PSNE if there is only one pair of vindictive players. In Section 5, we show how to compute pure strategy Nash equilibria given a set of vindictive bidding pairs, and also show that efficient algorithms for computing pure strategy Nash equilibria exist for the special case of *Symmetric Nash equilibria*. We propose several interesting open problems in Section 6 and conclude in Section 7.

2. RELATED WORK

A number of recent papers have analyzed keyword auctions. Edelman et al.[6] gave a nice description of the evolution of sponsored search auctions. Varian [10] and Edelman et al. [6] analyzed the one round keyword auction deployed by Yahoo! and showed that it is not incentive compatible, and also characterized a subclass of Nash equilibria, termed *symmetric Nash equilibrium* by Varian and *locally envy-free equilibria* by Edelman et al. Their work is extended by Aggarwal, Goel and Motwani [3] and Lahaie [9], who analyzed a *rank by weighted bid* winner selection scheme. This includes both the *rank-by-bid* scheme currently deployed by Yahoo! and *rank-by-revenue* scheme currently deployed by Google.

In terms of bidding strategies for keyword auctions, Edelman and Ostrovsky [5] discussed the "sawtooth" bidding patterns observed from Overture bidding data, when the Overture mechanism was a first-price auction. They argued that this kind of bidding pattern was caused by the "winner's curse" of first price auctions. Later both Yahoo! and Google switched to variants of second price auctions. Thus this kind of bidding patterns is no longer relevant. Several companies market bid management software for pay-per-click auctions, such as Efficient Frontier, Performics, Silicon Space, GoToast, etc. GoToast offers rule-based systems in which you can set minimum prices, maximum prices, desired positions, and so on. Among many rules provided by GoToast [1], *Gap Jammer* is one rule corresponding to vindictive bidding. Kitts and LeBlanc [8] presented a trading agent for keyword auctions which explores various bidding strategies to optimize the advertising effectiveness crossing multiple keyword auction markets.

The other work most relevant to us is the spiteful bidding work by Brandt, Sandholm and Shoham [4]. They studied the problem where a bidder is interested in not only maximizing its own profit, but also minimizing its competitors' profits. In contrast to the keyword position auctions, they consider a traditional auction where only one single object is auctioned and the payoff of player i equals to $u_i - \alpha_i \sum_j u_j$ where $\alpha_i \in [0, 1]$ is a constant describing how spiteful the i-th player is. For us, vindictive bidding is spiteful, but it is only weakly so. In our model, we assume *weak vindictiveness* in which each player behaves vindictively only when it is not affecting its own payment, at least for just a single round. I.e., in our case the $\alpha_i = 0$, but a bidder still prefers a competitor to pay more, other things being equal.

3. BIDDING DYNAMICS AND VINDICTIVE BIDDING PAIRS

Throughout the paper, we consider a single keyword auction where there are n bidders, k positions and bids are ranked in strictly decreasing order: $b_1 > b_2 > \ldots > b_n$. Assume that $n > k$ and the reservation price is 0. [1] Let c_j denote the clickthrough rate for ads displayed on the j-th position, for $j = 1, \ldots, k$. [2] Assume that we use the generalized second price auction where the i-th bidder wins the i-th position and pays b_{i+1}, for $i = 1, 2, \ldots, k$. Bidder i has its own private and independent valuation per click (VPC) v_i, for $i = 1, \ldots, n$. Bidders are rational players, and the payoff of bidder i is

$$U_i = c_i(v_i - b_{i+1}), \quad 1 \le i \le k.$$

With vindictive bidders, traditional Nash equilibria become unstable as the vindictive bidder has the incentive to increase its bid to undermine the one immediately above it. In turn, for the player being hurt, his best response is no longer staying in his original position, but narrowly undercut the vindictive bidder. This leads to multiple rounds of undercut alternatively by these two vindictive players, and eventually they reach a price level when the one on top has no incentive to undercut the other player and they reach an equilibrium.

An example best illustrates this bidding dynamics. Suppose that there are two bidders with $v_1 = \$1.0$, $v_2 = \$0.5$, $c_1 = 1$, $c_2 = 0.5$, and $b_1 = \$1.0$, $b_2 = \$0.5$, $b_3 = \$0.1$. This is a PSNE if bidders are self-interested since both bidders have no incentive to change their positions. However, if the bidders are vindictive to each other, this is no longer stable. Bidder 2 has an incentive to raise his bid to \$0.99, thus forcing bidder 1 to pay \$0.99 per-user-click and the utility of bidder 1 drops from \$0.5 to \$0.01. Bidder 1 has an incentive to undercut bidder 2 by bidding \$0.98. Bidder 2 suffers afterwards and will undercut bidder 1 again. These two players alternatively undercut each other and eventually they reach the state $b_1 = \$0.56, b_2 = \$0.55$, where they stabilize and have no incentive to undercut each other anymore. Figure 1 shows the bidding dynamics for this example.

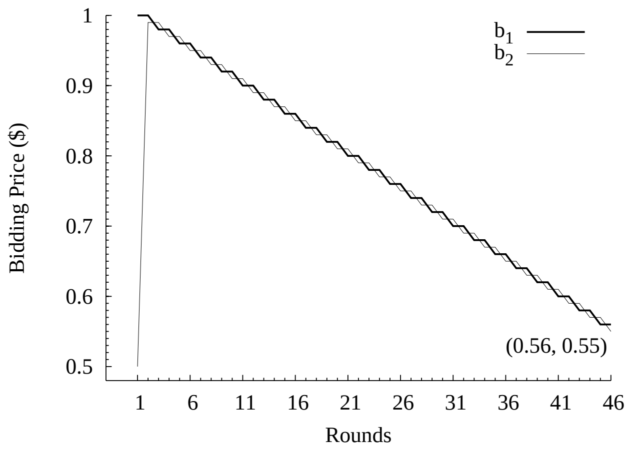

Figure 1: An example of bidding dynamics with two vindictive bidders.

[1]The reservation price is typically a nonzero constant, such as \$0.10 in Yahoo!Marketplace. However the nonzero case can be easily reduced to the zero case.

[2]For simplicity, here we assume implicitly that the clickthrough rate is only dependent on the position.

Given two bidders taking positions i and $i+1$ in a keyword auction, if $b_{i+1} = b_i - 0.01$, we call the pair (b_i, b_{i+1}) a *vindictive bidding pair.* Vindictive bidding pairs do seem to exist in real auctions. If vindictiveness were not a real issue, we would not expect to see many bidding pairs that differ by 0.01. To explore this issue, we gathered bidding data on a group of 55 keywords related to the keyword "inkjet". This group represents a very competitive market with several strong competitors and, and the average second price over the 55 terms is \$1.67. In Figure 2, we show a histogram of the bid differences among the top 7 bids over all 55 keywords. The data is from Overture's publicly available tool [2]. As can be seen, a bid difference of \$0.01 is very prevalent. Bid differences of \$0.00 (ties are broken by order of arrival on Overture) and \$0.01 account for approximately 40% of the pairs.

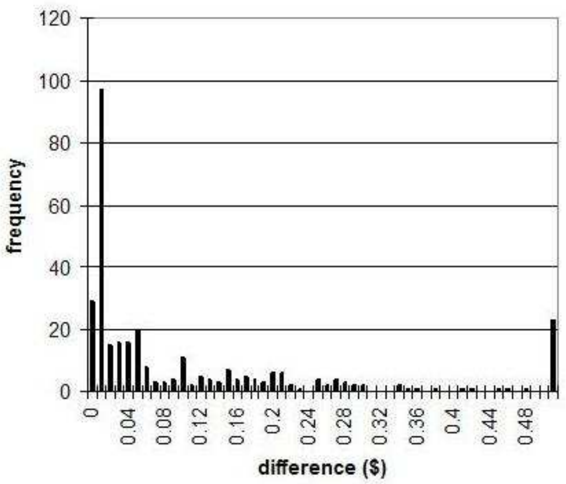

Figure 2: A histogram of bid differences (between consecutive bidders up to the seventh bidder) over a group of 55 keywords related to the keyword "inkjet". The final bin contains counts for differences $> \$0.5$

4. PSNE WITH VINDICTIVE BIDDING

In this section we study the (non) existence of pure strategy Nash equilibria with vindictive bidders. Consider a scenario where there are two big players $1, 2$ with their VPC v_1, v_2 both larger than VPC of all other players. Both players have their daily (or monthly, hourly, etc.) budgets. If one player runs out of money, it will leave the market and the other player can enjoy the 1st position with payment b_3 per click. So it has an incentive to hurt the other player by forcing him to pay more. For player 2, it pays b_3 to obtain the 2nd position for any $b_2 \in [b_1 - 0.01, b_3 + 0.01]$. Thus vindictive bidding will entice him to raise his bid such that $b_2 = b_1 - 0.01$.

Given two vindictive players with VPC v_1, v_2 with $v_1 > b_3$ and $v_2 > b_3$. Assume that the set of bids $b_1 > b_2 > b_3 > \ldots > b_n$ is a PSNE where $b_2 = b_1 - 0.01$. Let c_1, c_2 be clickthrough rates for the 1st and 2nd positions with $c_2/c_1 = c_2' \in (0, 1)$. By the definition of NE, player 1 could not benefit by bidding $b_2 - 0.01$, thus

$$\begin{aligned} U_1 = c_1(v_1 - b_2) &\geq c_2(v_1 - b_3) \\ \Rightarrow \quad (1 - c_2')v_1 + c_2'b_3 &\geq b_2 = b_1 - 0.01. \end{aligned}$$

By the definition of NE, player 2 could not benefit by bidding $b_1 + 0.01$, i.e.,

$$\begin{aligned} U_2 = c_2(v_2 - b_3) &\geq c_1(v_2 - b_1) \\ \Rightarrow \quad b_1 &\geq (1 - c_2')v_2 + c_2'b_3. \end{aligned}$$

In summary,

$$0.01 + (1 - c_2')v_1 + c_2'b_3 \geq b_1 \geq (1 - c_2')v_2 + c_2'b_3.$$

Because $0.01 + (1 - c_2')(v_1 - v_2) \geq 0$, thus $v_1 \geq v_2$. So that all PSNE solutions can be described as

$$b_1 \in [(1-c_2')v_2+c_2'b_3, (1-c_2')v_1+c_2'b_3+0.01], \quad b_2 = b_1-0.01.$$

The same analysis also applies to the case when the i-th bidder and the $(i+1)$-th bidder are vindictive to each other. We summarize the results into the following theorem:

THEOREM 1. *In a keyword auction Nash equilibrium, if the i-th bidder and the $(i+1)$-th bidder are vindictive to each other, then $b_i = b_{i+1} + 0.01$. Furthermore, their valuations per click satisfy $v_i \geq v_{i+1}$, i.e., the assignment of bidders to positions are efficient if they are mutually vindictive to each other.*

4.1 (Non) Existence of PSNE with 3 Players

Suppose there are 3 players with bids $b_1 > b_2 > b_3$ and there VPC v_1, v_2, v_3 are larger than the 4th highest bid b_4. For NE, $b_2 = b_1 - 0.01$, and $b_3 = b_2 - 0.01$ since these three players are vindictive to each other. By the definition of NE, player 2 will not benefit by bidding $b_1 + 0.01$ or $b_3 - 0.01$, thus

$$\begin{aligned} c_2(v_2 - b_3) &\geq c_1(v_2 - b_1), \quad (1) \\ c_2(v_2 - b_3) &\geq c_3(v_2 - b_4). \quad (2) \end{aligned}$$

Let $c_2' = c_2/c_1, c_3' = c_3/c_1$, then $0 < c_3' \leq c_2' < 1$. Eq. (1) implies that

$$\begin{aligned} (1 - c_2')v_2 + c_2'b_3 &\leq b_1 = b_3 + 0.02 \\ \Rightarrow \quad b_3 &\geq v_2 - \frac{0.02}{1 - c_2'}. \end{aligned}$$

Eq.(2) implies that

$$b_3 \leq (1 - \frac{c_3'}{c_2'})v_2 + \frac{c_3'}{c_2'}b_4.$$

By combining the above two inequalities we obtain

$$\begin{aligned} v_2 - \frac{0.02}{1 - c_2'} &\leq (1 - \frac{c_3'}{c_2'})v_2 + \frac{c_3'}{c_2'}b_4 \\ \Rightarrow \quad v_2 &\leq b_4 + \frac{c_2'}{c_3'(1 - c_2')}0.02. \quad (3) \end{aligned}$$

Let $v_1 = v_2 = v_3 = v = 0.3$, $b_4 = 0.1$, $c_2' = 0.8$, $c_3' = 0.5$, then Eq.(3) implies that $0.3 \leq 0.1 + 0.16$, a contradiction. So that for this special case, there is no pure strategy NE.

Pure strategy Nash equilibria may exist if we choose parameters appropriately with three vindictive players. Given three vindictive bidders with bids $b_1 = b_2 + 0.01$, $b_2 = b_3 + 0.01$, NE is equivalent to

$$\begin{aligned} c_1(v_1 - b_2) &\geq c_3(v_1 - b_4) \quad (4) \\ c_3(v_3 - b_4) &\geq c_1(v_3 - b_1) \quad (5) \end{aligned}$$

together with Eq.(1), (2).

Let $b^* = b_3$, then $b_1 = b^* + 0.02$, $b_2 = b^* + 0.01$. Eq.(4) implies that

$$b^* \leq (1 - c_3')v_1 + c_3' b_4 - 0.01.$$

Eq.(5) implies that

$$(1 - c_3')v_3 + c_3' b_4 - 0.02 \leq b^*.$$

So Nash equilibrium is equivalent to the following inequalities:

$$\max\left\{(1 - c_3')v_3 + c_3' b_4 - 0.02, v_2 - \frac{0.02}{1 - c_2'}\right\} \leq b^*$$
$$\leq \min\left\{(1 - \frac{c_3'}{c_2'})v_2 + \frac{c_3'}{c_2'} b_4, (1 - c_3')v_1 + c_3' b_4 - 0.01\right\}. \quad (6)$$

As an example, let $v_1 = v_2 = v_3 = v = 0.18$, $b_4 = 0$, $c_2' = 0.9$, $c_3' = 0.5$, then NE inequalities (6) are equivalent to $b^* \in [0.07, 0.08]$. In other words, for the discrete version where each bidding price is an integer factor of 1 cent, there are two PSNE, with (b_1, b_2, b_3) equals to $(0.09, 0.08, 0.07)$ and $(0.10, 0.09, 0.08)$ respectively.

We summarize the results into the following theorem:

THEOREM 2. *If there is only one pair of vindictive bidders, there always exists a pure strategy Nash equilibrium. If there are three or more bidders who are vindictive to each other, pure strategy Nash equilibria may or may not exist.*

5. COMPUTING PSNE WITH VINDICTIVE BIDDING

In this section we show how to compute PSNE (if one exists) with the assumption of complete knowledge of bidders' valuations per click and the set of vindictive bidding pairs. We restrict to Nash equilibria satisfying the following property, which corresponds to assortative matching in stable matching theory.

$$b_i > b_j \text{ if } v_i > v_j, \text{ for all bidders } i, j. \quad (7)$$

Eq.(7) makes the task of computing NE quite easy. Without loss of generality, assume that

$$v_1 > v_2 > \ldots > v_k > v_{k+1} \geq \ldots \geq v_n.$$

If there are bidders with the same VPC value, we can break the tie arbitrarily. The first k bidders will win the auction, and the i-th bidder will obtain the i-th position, for $1 \leq i \leq k$. For each vindictive bidding pair i, j with $i > j$, if $i > j+1$, we ignore, since bidder j can only affect its immediate above neighbor directly; if $i = j + 1$, then add the constraint

$$b_j = b_i - 0.01. \quad (8)$$

Since the set of bids consist of a Nash equilibrium, by the definition of NE, we have

$$c_i(v_i - b_{i+1}) \geq c_j(v_i - b_{j+1}), \quad \forall i < j \leq k,$$
$$c_i(v_i - b_{i+1}) \geq c_j(v_i - b_j), \quad \forall j < i \leq k.$$

Furthermore, for each bidder $i > k$, since it does not win the auction, its utility is 0. Since it could not obtain a positive utility by switching to one of the top k positions, thus

$$b_k \geq v_{k+1}.$$

In summary, we obtain a set of linear constraints, with some linear equality constraints, and some linear inequality constraints, and the total number of linear constraints is $O(k^2)$. We can use a standard linear programming (LP) solver to solve it in time polynomial of n, k. If the LP solver returns with no solution, it implies that the keyword auction problem with the corresponding set of vindictive bidding pairs has no PSNE. If the LP solver returns a solution, the solution then translates into a PSNE for the corresponding auction problem.

Even though we can compute a PSNE given a set of vindictive bidding pairs assuming that bidding prices satisfy the assortative matching property, it is not necessary for all NE to satisfy this property. Thus to check whether there exists a PSNE, we have to enumerate all possible orderings of bidders based on their bidding prices, which is in the order of $n! = \Omega((n/e)^n)$. So the naive approach takes exponential time and it is not feasible with even 20-30 bidders. We leave it as an open problem whether it is NP-hard to compute a PSNE given a set of vindictive bidding pairs if one exists. We believe the answer is affirmative given that the assortative matching property is economically efficient. Another related problem is to either prove or disprove that if there exists a PSNE, then there exists a PSNE satisfying the assortative matching property.

5.1 Computing Symmetric Nash Equilibria

A subclass of Nash equilibria was studied independently by Varian [10] and Edelman et al. [6], and it was termed *symmetric Nash equilibria* by Varian and *locally envy free equilibria* by Edelman et al. We will follow the terminology by Varian and call it *symmetric Nash equilibria* (SNE). Suppose that $b_1 > b_2 > \ldots > b_n$, then the set of bids corresponds to a SNE if and only if

$$c_i(v_i - b_{i+1}) \geq c_j(v_i - b_{j+1}), \quad \forall i \neq j, 1 \leq i, j \leq n. \quad (9)$$

SNE corresponds to a stable matching if you treat the auction problem as matching advertisers to positions of the search results page. The maximum revenue of NE is the same as the maximum revenue of SNE, and the SNE with minimum revenue corresponds to the VCG payment scheme with truthful bidding agents. Among all other properties, SNE satisfies the assortative matching property stated in Eq. (7). Based on the discussion in the previous section, this implies that we can compute a pure strategy SNE in polynomial time. We can even compute a pure strategy SNE much faster, as SNE is equivalent to the following simplified conditions:

$$c_i(v_i - b_{i+1}) \geq c_{i+1}(v_i - b_{i+2}), \quad \forall 1 \leq i < n$$
$$c_i(v_i - b_{i+1}) \geq c_{i-1}(v_i - b_i), \quad \forall 1 < i \leq n$$

There are only $2n$ inequalities here. Together with Eqs. (8) obtained from the input of vindictive bidding pairs, we obtain the necessary and sufficient conditions for SNE. We can solve the set of linear inequalities using a standard LP solver, and here the number of linear inequalities is $O(n + k)$. In summary, we have the following theory:

THEOREM 3. *We can compute a pure strategy symmetric Nash equilibrium if one exists by solving $O(n + k)$ linear inequalities. By using the algorithm of Karmarkar [7], it can be computed in time $O(n + k^{3.5})$.*

6. OPEN PROBLEMS

Our work is an ongoing research activity, and it raises as many questions as it settles. In the following, we propose some intriguing open problems.

Right Amount of Vindictiveness. Intuitively too much vindictiveness is harmful to you since these bidders will also be vindictive to you. It is an intriguing problem for each bidder to decide whether vindictive or cooperative to other players. It is an interesting problem to characterize the set of vindictive bidding pairs within a keyword auction which always accept a PSNE.

Complexity of Computing PSNE. In Section 5, we compute a PSNE with vindictive bidding pairs assuming that assortative matching property holds. We conjecture that if a PSNE exists then a PSNE with the assortative matching property also exists. If the answer is affirmative, then we can focus on classic PSNE with the assortative matching property. It is an open problem whether it is NP-hard to find a PSNE given the set of vindictive bidding pairs. Even though computing Nash equilibria for general games is an outstanding open problem, keyword auction is a special case which may be easy to solve.

Multi-Round Game. Another interesting line of investigation is the multi-round setting, which is obviously of practical importance. For pure strategies, it seems unlikely that the analysis would change very much given the nature of the real auction system. For example, the term "inkjet" is queried approximately 20,000 times per month (on Yahoo), or about 28 times per hour on average. Even assuming a 25% clickthrough rate, which is high, the expected number of clickthroughs per hour is then 7, suggesting a fairly long waiting time between clicks. If a bidder tries to deviate from a one shot PSNE with vindictiveness, it seems likely that the dynamics described in Section 3 would ensue on a time scale faster than the characteristic waiting time between actual clicks (and hence actual payments). Thus, intuitively, the PSNE of the one shot game with vindictiveness could be a good characterization for the multi-period game as well.

Mixed Strategy Nash equilibria. Another question is the affect of vindictiveness on mixed strategy Nash equilibria. In this, for the one shot game, vindictiveness is likely to change the incentives with respect to mixed strategy equilibria without vindictiveness. For example, suppose the top bidder is very high at $20 and the second and third bidders are relatively low at $2.00 and $1.99 (such cases do occur in practice). For this to be a PSNE with vindictiveness, the first and second bidder cannot form a vindictive pair according to our definition of weak vindictiveness. However, if mixed strategies are allowed, it seems possible for the second bidder to hurt the top bidder by randomly (with probability p) raising his bid to be just below the top. This can be very expensive for the top bidder. However, the third bidder can reason similarly for hurting the second bidder. But the randomness decreases the likelihood that the second player is hurt since, roughly speaking, the probability that the second bidder is hurt playing this mixed strategy would be p^2, assuming the third player is playing p as well.

7. CONCLUSION

Vindictive bidding is prevalent in sponsored search keyword auctions, and it leads to instability of most traditional Nash equilibria. It is thus very important to understand and utilize this bidding strategy, and characterize the resulting Nash equilibria. Our work is the first step in this direction.

Acknowledgement. We thank Terence Kelly for comments and technical assistance.

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