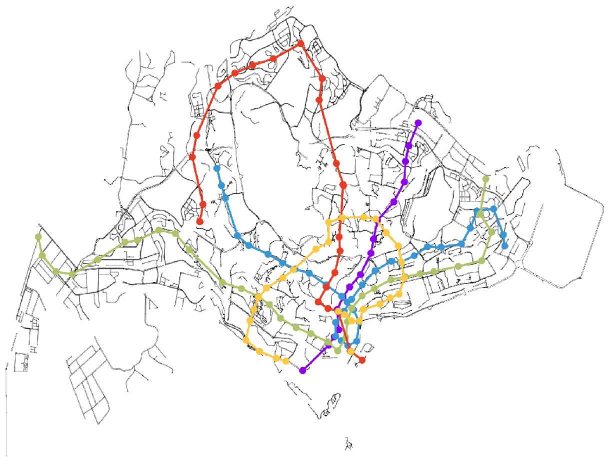
CS 357: Algorithmic Game Theory

Lecture 7: Revenue Equivalence

Shikha Singh



Announcements

- Hand in **HW 3**
- Pick up **HW 4**, due next Tuesday in class
- **Assignment 2** due Friday (March 7) at noon
  - Partner assignment: **submit joint PDF** on Gradescope
- Assignment 1 grading in progress, solutions are posted on GLOW
- **Exam 1** will be held in class on **March 14**
  - Short-"ish" questions on topics covered until the week before
  - Composed of mostly HW style questions with 1/2 open-ended
- Results on the first-price class auction as well as discussion of analysis
  - Today!

Any questions?

Recap from Last Time

- Great discussion on GSP and its analysis
  - Paper reading and proof writing practice
  - Proof writing is all about convincing others (your group, your classmates, me)
- Proofs of all five parts will be posted on GLOW for review
- Wrap up sponsored search auctions today

Locally Envy Free

- Does such a bid b_i always exist?
  - As long as $b_{i+1} \leq v_i$ and $\alpha_i < \alpha_{i-1}$, then yes ($b_{i+1} < b_i \leq v_i$)

$$\underbrace{\alpha_i(v_i - b_{i+1})}_{\text{utility current position}} \geq \underbrace{\alpha_{i-1}(v_i - b_i)}_{\text{utility in case of retaliation}}$$

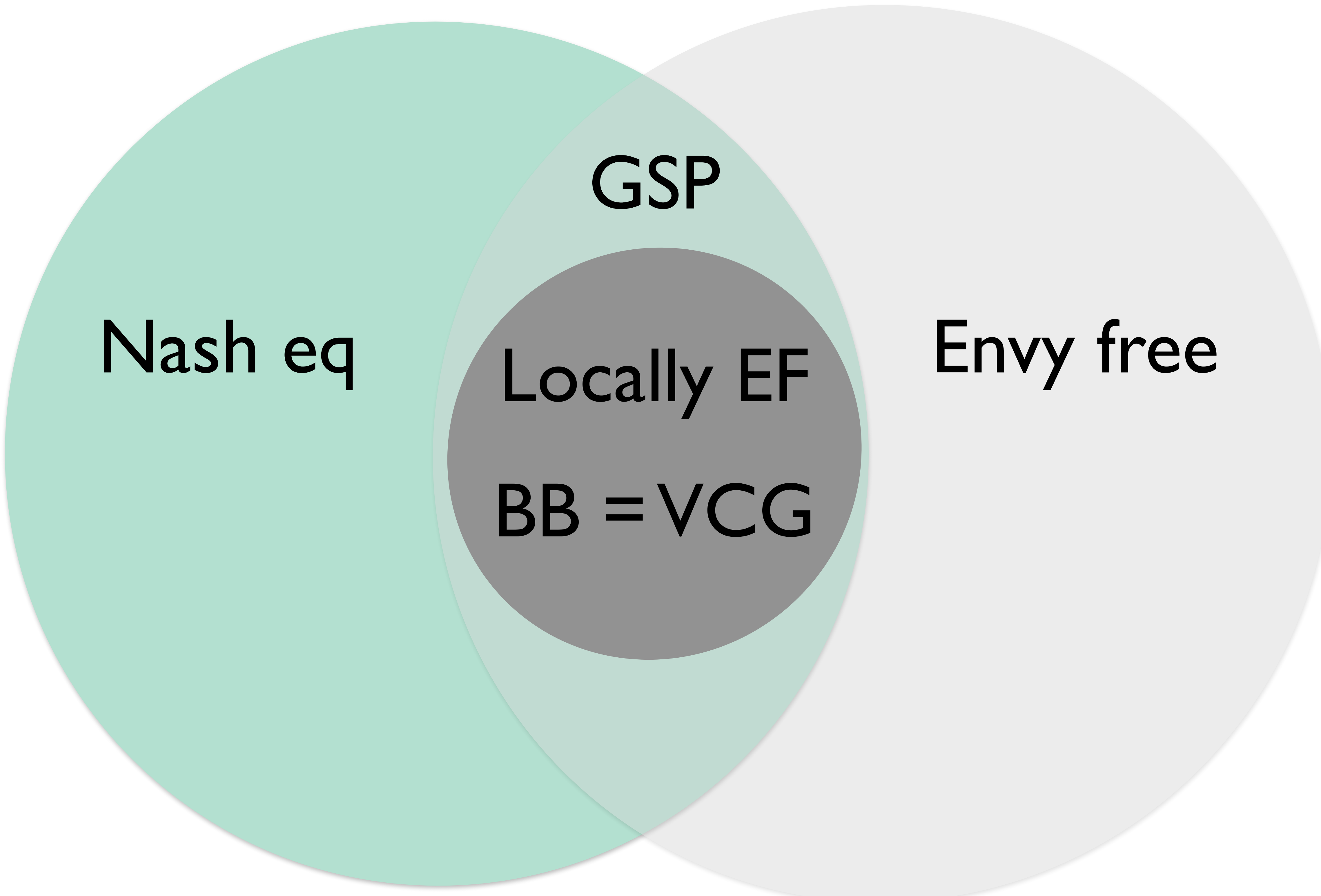
Balanced Bidding

- A bid profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ satisfies the balanced bidding if
  - For bidder i for $2 \leq i \leq k$

$$\underbrace{\alpha_i(v_i - b_{i+1})}_{\text{utility current position}} = \underbrace{\alpha_{i-1}(v_i - b_i)}_{\text{utility in case of retaliation}}$$

  - Any unassigned bidder bids their true value
- For value ordered bids, the **balanced bidding requirement defines a unique bid profile** (up to the indifference of the top bidder)

Locally Envy Free Implies Envy Free



Takeaways

1. **[Equivalence of various auction formats].** As long as the allocation is rule, payments can be irrelevant: payment charged as a function of bids does not determine revenue: it is determined by bidder's strategies.
2. **[DSE vs Equilibrium outcome].** VCG ("Front-load the work" on the designer---payments enforce a truthful DSE rule or GSP (let the bidder's "fight" it out)
3. **[Role of Information].** Complete information is a strong assumption, incomplete information is harder to analyze
4. **[Theory vs Empirical behavior].** Analyzing bidder behavior requires understanding the equilibrium; equilibrium not always "reached", but best response dynamics tend to converge to it

Design Trade Offs

- VCG is used for contextual non real-time advertising, e.g. by X & Facebook
- Switch from GSP to VCG :
  - 2012, Google switched from GSP to VCG for its ad network AdSense
  - 2015: Yandex search engine
- **Reasons to prefer VCG over GSP?**
  - Truthful behavior: no need for bidders to strategize
  - Easier for sellers to estimate revenue
  - Enables faster experimentation: seeing how reserve prices effect revenue, etc.
  - Flexibility: VCG auction is highly configurable to different preferences and contexts

Back to First Price

- Ad exchanges moved from second-price sealed bid to first-price sealed-bid, with Google switching during 2019
- **Transparency.** Some businesses are both sellers and buyers
- Composability with different types of ads

	Non real-time	**Real-time (programmatic)**
Sponsored search	∘ Google and Bing **GSP** ∘ Yandex **VCG**	n/a
Contextual	Own inventory ∘ Twitter and Facebook **VCG** 3rd party inventory (ad networks) ∘ Google AdSense, FB Audience Network **VCG** ∘ Microsoft Audience Ads **GSP**	3rd party inventory (ad exchanges) ∘ AppNexus, Twitter, MoPub, and Google DoubleClick **First price** (was second price)

Why Do Competitive Markets Converge to First-Price Auctions?

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ABSTRACT

We consider a setting in which bidders participate in multiple auctions run by different sellers, and optimize their bids for the *aggregate* auction. We analyze this setting by formulating a game between sellers, where a seller's strategy is to pick an auction to run. Our analysis aims to shed light on the recent change in the Display Ads market landscape: here, ad exchanges (sellers) were mostly running second-price auctions earlier and over time they switched to variants of the first-price auction, culminating in Google's Ad Exchange moving to a first-price auction in 2019. Our model and results offer an explanation for why the first-price auction occurs as a natural equilibrium in such competitive markets.

"Moving to a first-price auction puts Google at parity with other exchanges and SSPs (supply-side platforms) in the market, and will contribute to a much fairer transactional process across demand sources." : Scott Mulqueen

First Price Auctions

Takeways

- Winners were mostly the same (bids were proportional to value)
- Bidders shade their bid down in first price
- Competition drives the bids up!
    - More number of bidders means more revenue
    - If sellers care about revenue, need to get more participation
- It is difficult for bidders to reason about equilibrium strategies

Questions

- What is the theoretical equilibrium that bidders are supposed to reach?
- Does our class auction match what theory says?
- Which auction (first price or second price) generates more revenue?

First vs Second Price Auctions

Both the first-price and second-price auction (at equilibrium) generate the same (expected) revenue!

To show this, need to analyze first-price auction, which is an incomplete-information or "Bayesian game"

All Pay Auction

- Single item auction where highest bidder wins
- Each bidder (even those who lose) pay their bid
- Question. Does this auction make more revenue (at equilibrium) compared to first/second price auctions??

All auctions which have the **same allocation** (at equilibrium) make the exact **same revenue** (at equilibrium)

Same as VCG/GSP connection!! Both auctions, the winners are always the top k bidders

Incomplete Information Game

- Complete information game: utility structure (payoff matrix) is common knowledge
- Auctions are games of incomplete information: bidders values (and thus utilities) are private
- No dominant strategy equilibrium in first-price auction, thus we need a variant of Nash for incomplete information games
  - Called Bayesian Nash Equilibrium

Simplifying Assumptions

- Assume bidders have private values that are drawn independently and identically from the distribution G
  - We say values are drawn i.i.d from G
- Distribution G is common knowledge (called "common prior")
  - Every bidder knows the distributions and knows that others know it as well
- For first-price auction: we will further assume values are drawn i.i.d from the uniform distribution on $[0,1]$

BNE of First Price Auctions

Bayesian Nash Equilibrium

- A strategy or plan of action for each player (as a function of values) should be such that it maximizes each players expected utility
  - expectation is over the private values of other players
- Given a Bayesian game with independent private values v_{-i} , i's expected utility for a strategy profile $s = (s_1, \ldots, s_n)$ is

$$\mathbb{E}[u_i(s)] = \sum_{v_{-i}} u_i(s) \cdot \Pr(v_{-i})$$

- A strategy profile s is a **Bayes Nash equilibrium** if no player can increase their expected utility by unilaterally changing their strategy s_i

Strategy Assumptions

- Recall: strategy s_i is a function that maps their value to their bid b:
  - $s_i(v_i) = b_i$
- We assume that the strategy of all bidders in the auctions we study
  - Is a strictly increasing differentiable function: gives us that the bidder with higher value will also provide a higher bid (no ties)
  - $s_i(v_i) \leq v_i$ for all v_i and bidders i: that is, bidders can "shade" down their bids but will never bid above their true values
    - Also implies $s_i(0) = 0$
- These assumptions are just to simplify analysis

BNE of First Price Auctions

- Guess-and-check approach: guess an equilibrium strategy and verify
- Starting guess: Each bidder shade their bids down proportional to their value $s_i(v_i) = b_i = \alpha v_i$ for each bidder i
- To check if it is a symmetric BNE, fix s_{-i} and analyze what is the best response bid for bidder i : bid that maximizes expected $u_i(v_i, b_{-i})$
- Will use **law of total expectation**: $E[X] = \sum_i E[X \mid A_i] \Pr[A_i]$

  where events A_i are partitions of the sample space

First-Price Auction: 2 Bidders

- Suppose v_1, v_2 are i.i.d. from the uniform distribution on $[0,1]$
- Consider how bidder 1 should set their best response

Pr$(b_1 < b_2)$

Pr$(b_1 \geq b_2)$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: 2 Bidders

- Guess a BNE strategy profile: say both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

$\Pr(b_1 < b_2)$

$\Pr(b_1 \geq v_2\alpha)$

v_1 loses & pays 0

Utility: 0

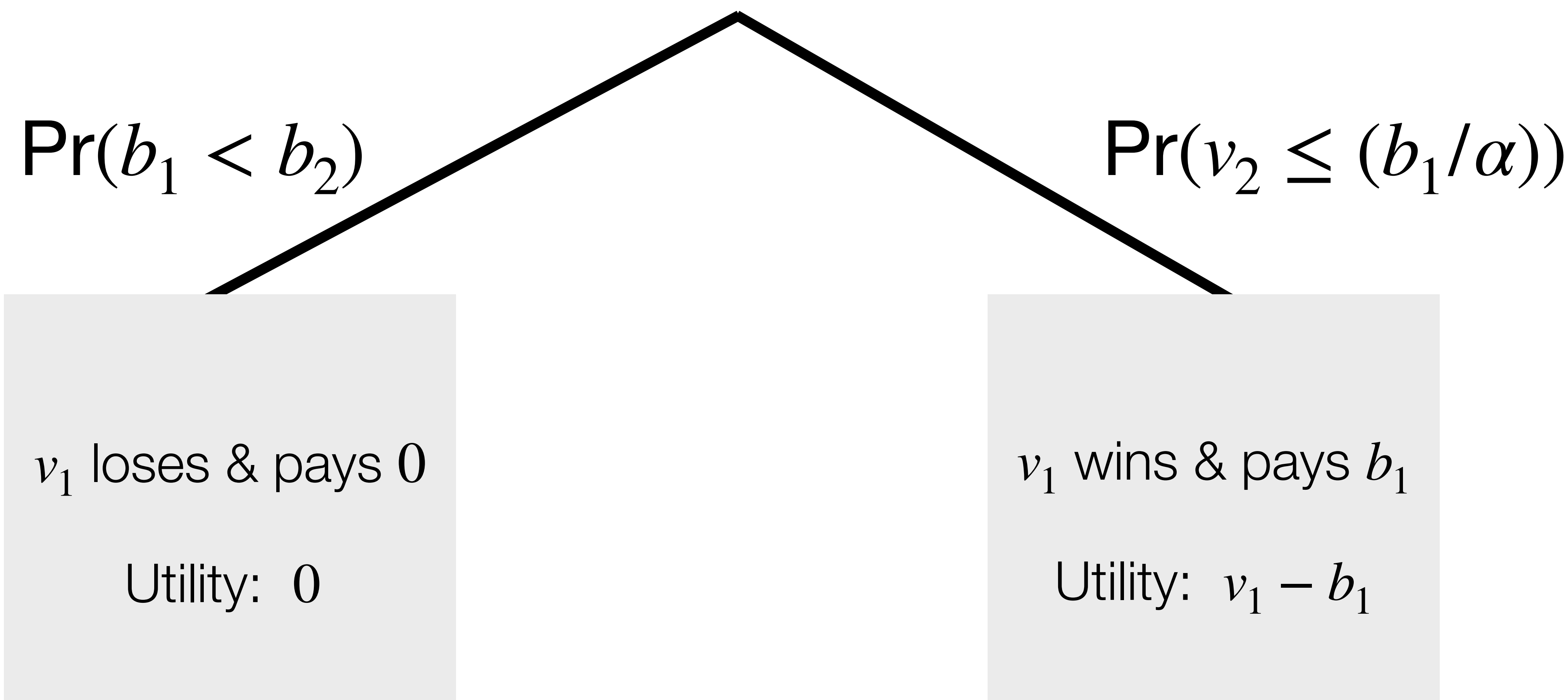
v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: 2 Bidders

- Guess a BNE strategy profile: say both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

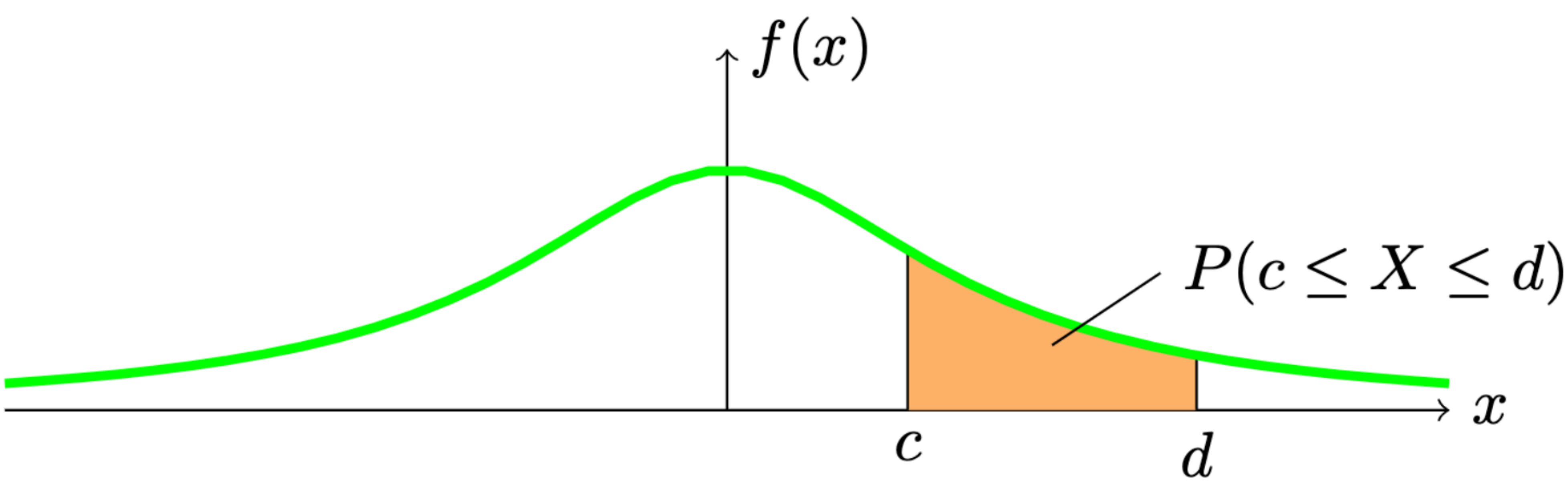


How to set b_1 to maximize expected utility?

Continuous Probability Review

- **(Definition)** A random variable X is continuous if there is a function $f(x)$ such that for any $c \leq d$ we have $\Pr(c \leq X \leq d) = \int_c^d f(x)dx$ where $f(x)$ is the probability density function (pdf)

$$\boxed{P(c \leq X \leq d) = \text{area under the graph between } c \text{ and } d.}$$

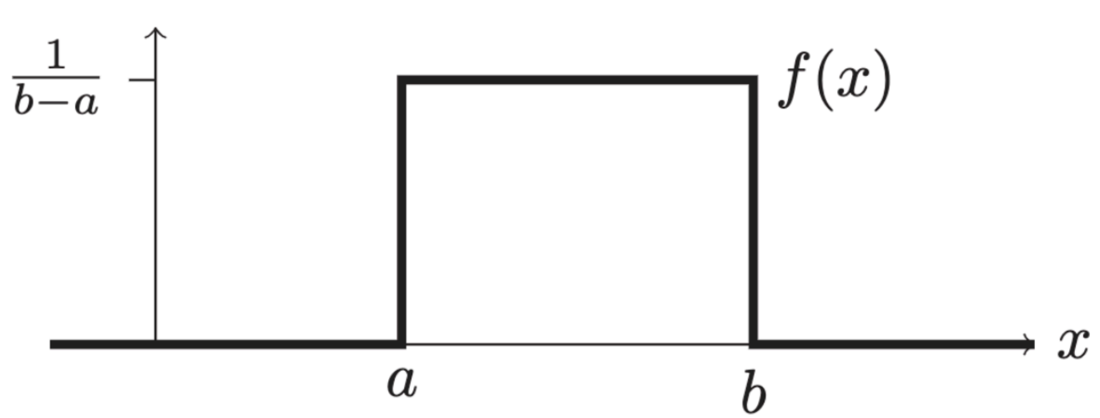


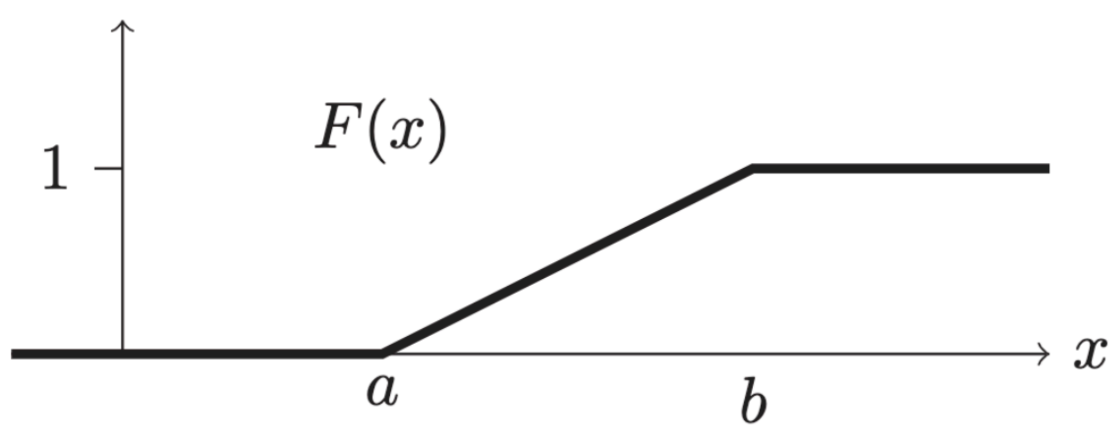
Continuous Probability Review

- **(Definition)** The cumulative distribution function (cdf) F of a continuous random variable X denotes the probability that it is at most a certain value

$$F(k) = \Pr(X \leq k) = \int_{-\infty}^{k} f(x)dx \text{ where } f(x) \text{ is the pdf } X$$

- We often say X has distribution or is drawn from distribution $F(x)$ rather than X has cumulative distribution function $F(x)$

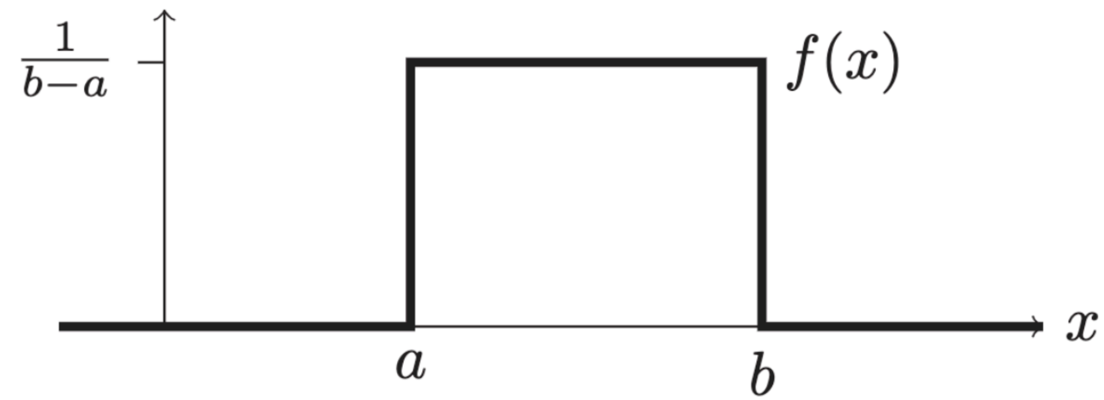




Uniform Distribution on $[a, b]$

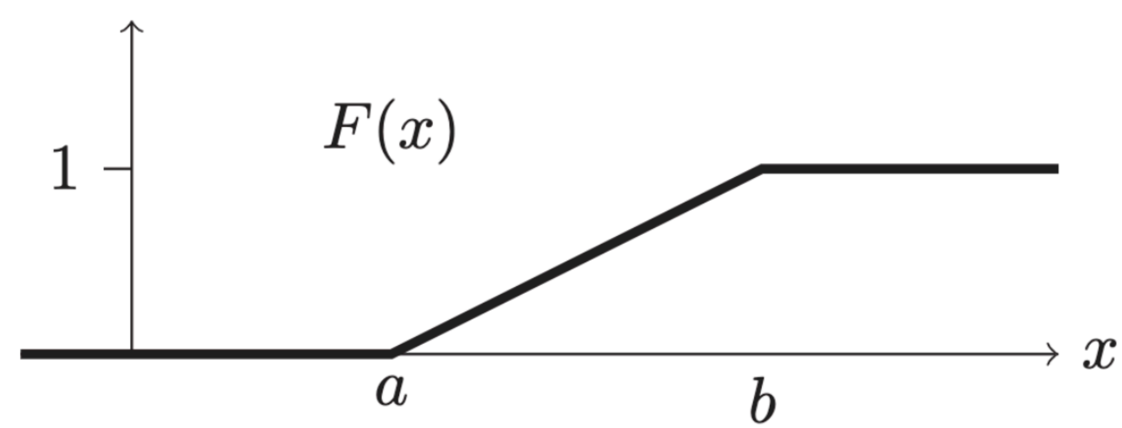
- Models settings where all outcomes in the range are equally likely
- PDF of a continuous uniform distribution on $[a, b]$:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0 & \text{for } x < a \text{ or } x > b \end{cases}$$



- CDF of a continuous uniform distribution on $[a, b]$:

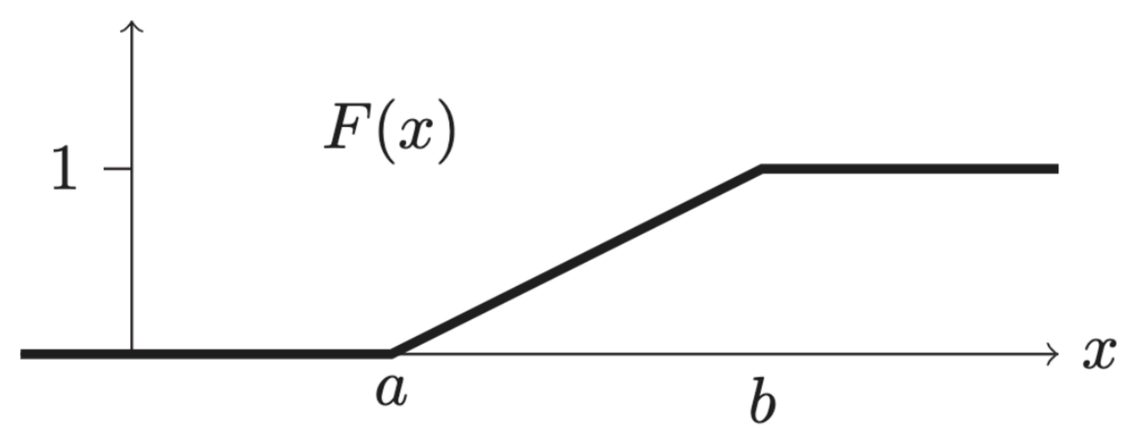
$$\Pr(x \le k) = \frac{k-a}{b-a} \text{ if } a \le k \le b$$



Uniform Distribution on $[0,1]$

- CDF of a continuous uniform distribution on $[0,1]$:

$$\Pr(x \leq k) = k \quad \text{if } a \leq k \leq b$$



First-Price Auction: 2 Bidders

- Guess a BNE strategy profile: say both bidders bid symmetrically some factor of their value $s(v_i) = \alpha \cdot v_i$

Pr$(b_1 < b_2)$

Pr$(v_2 \leq (b_1/\alpha)) = \frac{b_1}{\alpha}$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: 2 Bidders

- $\mathbb{E}[u_1] = (v_1 - b_1)(b_1/\alpha)$: how to set b_1 to maximize expected utility?

Pr$(b_1 < b_2)$

Pr$(v_2 \leq (b_1/\alpha)) = \dfrac{b_1}{\alpha}$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First-Price Auction: 2 Bidders

- $\mathbb{E}'[u_1] = (1/\alpha)(v_1 - 2b_1) = 0$, that is, $b_1 = v_1/2$

Pr$(b_1 < b_2)$

Pr$(v_2 \leq (b_1/\alpha)) = \dfrac{b_1}{\alpha}$

v_1 loses & pays 0

Utility: 0

v_1 wins & pays b_1

Utility: $v_1 - b_1$

How to set b_1 to maximize expected utility?

First Price: Two Bidders

- **Theorem.** Assume two bidders with their values drawn i.i.d. from Uniform $[0,1]$, then the strategy $s(v_i) = v_i/2$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.

- Proof. Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$

- We calculate agent 1's expected utility who has value v_1 and bid b_1

  - $E[u_1] = (v_1 - b_1) \cdot \text{Pr}[\text{1 wins with bid } b_1]$

    $= (v_1 - b_1) \cdot \text{Pr}[b_2 \leq b_1]$

    $= (v_1 - b_1) \cdot \text{Pr}[v_2/2 \leq b_1]$

    $= (v_1 - b_1) \cdot \text{Pr}[v_2 \leq 2b_1]$

    $= (v_1 - b_1) \cdot F(2b_1) = (v_1 - b_1) \cdot 2b_1$

First Price: Two Bidders

- Proof (Cont). Assume agent 2 bids using $s(\,.\,)$, that is, $b_2 = v_2/2$
- Agent 1's expected utility who has value v_1 and bid b_1 when she wins
  - $E[u_1] = (v_1 - b_1) \cdot 2b_1 \; = 2v_1b_1 - 2b_1^2$
- Agent 1 with value v_1 should set b_1 to maximize $2v_1b_1 - 2b_1^2$ as a function of b_1
  - Differentiate and set derivate to zero (also check second order condition)
  - $E'[u_1] = 2v_1 - 4b_1 = 0$, that is, $b_1 = v_1/2$

Symmetric analysis for bidder 2

First Price: n Bidders

- **Goal:** Symmetric Bayes Nash equilibrium for n bidders in first price auctions where *each bidders value is independently and identically distributed (i.i.d) from the uniform distribution on* $[0,1]$
- Suppose every bidder $j \neq 1$ uses strategy $s_j = \alpha(n) \cdot v_j$
- Let's write the expression for expected utility of bidder 1 and figure out what value of b_1 maximizes it
  - Fix b_1, v_1, write $\mathbb{E}(u_1)$ as a function of them
  - Each v_j for $j \neq 1$ is a random variable i.i.d. in uniform $[0,\ 1]$
- Deduce the value of $\alpha(n)$ from this

First Price: n Bidders

$$\mathbb{E}(u_1) = (v_1 - b_1) \cdot \Pr(1 \text{ wins with bid } b_1) + 0 \cdot \Pr(1 \text{ loses with bid } b_1)$$

$$= (v_1 - b_1) \cdot \Pr[b_1 \geq \max_{i=2}^{n} b_i]$$

$$= (v_1 - b_1) \cdot \Pr(b_1 \geq b_2 \cap b_2 \geq b_3 \cdots \cap b_1 \geq b_n)$$

Set $b_i = \alpha \cdot v_i$ for each $i = 2, \ldots, n$. As values are independent, we get:

$$\mathbb{E}(u_1) \ = (v_1 - b_1) \cdot \Pr(v_2 \leq \frac{b_1}{\alpha}) \cdots \Pr(v_n \leq \frac{b_1}{\alpha}) = (v_1 - b_1) \cdot \left(\frac{b_1}{\alpha}\right)^{n-1}$$

To find the bid b_1 that maximizes this utility, can differentiate wrt b_1 and set to zero, which gives us $b_1 = \dfrac{n-1}{n} \cdot v_1$

First Price: n Bidders

- **Theorem.** Assume each of the n bidders have values drawn i.i.d. from uniform distribution on $[0,1]$. Then, the strategy $s(v_i) = \frac{n-1}{n} \cdot v_i$ is a symmetric Bayes Nash equilibrium of the sealed-bid first price auction.
- **Proof.** Verify by confirming that this in fact maximizes expected utility.
- **Takeaway**: the more the competition, the more the bidders need to bid closer to their value if they want to win.