CS 357: Algorithmic Game Theory

Lecture 5: VCG Mechanism

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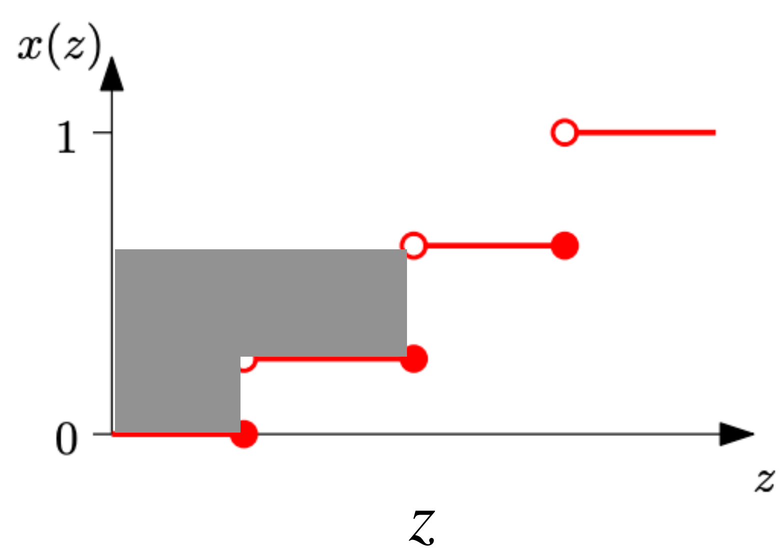
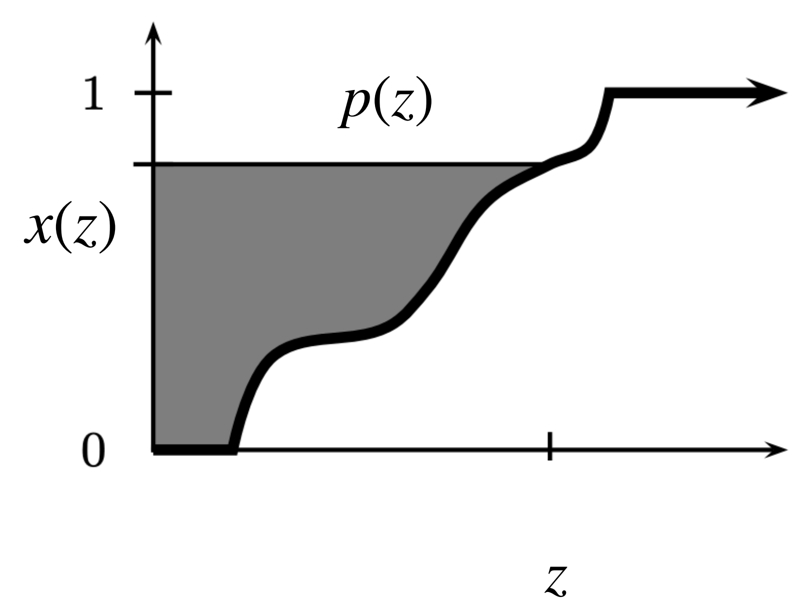
Announcements

- Pick up HW 3, due Tues in class
  - Short examples to practice Myerson payment rule
- Paper evaluation 1 (due next Fri): Case study of internet ad auctions
  - **Part A**: Submit a google form individually
  - **Part B**: Work on technical analysis in groups of 4
  - Each group must turn in their **write up** of at least 3 out of 5 proofs in class and **present** one of them on the board
- Updated help hours (TCL 304/ CS croom):
  - **Mon and Wed 1.30-3 pm**, Friday 9.30-10.30 am

Recap from Last Time

- **Myerson Lemma:** powerful characterization of dominant-strategyproof algorithms ("*mechanisms*") for single-parameter settings
  - Allocation can be made dominant-stratetyproof iff it is monotone
  - Unique payment rule: $p_i(z, \mathbf{b_{-i}}) = z \cdot x_i(z,\ \mathbf{b_{-i}}) - \int_0^z x_i(z,\ \mathbf{b_{-i}})\ dz$
  - For piece-wise constant allocation with ℓ points at which the allocation "jumps" before bid z, the payment at bid z $p_i(z) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i \text{ at } z_j]$





Sponsored Search Review

- Reindex bidders s.t. $b_1 \geq b_2 \ldots \geq b_n$
- Allocate ith bidder to ith slot for $i = 1, 2, \ldots, k$
- Charge ith bidder a payment p_i given by Myerson's rule:

$$\sum_{j=i}^{k} \left(b_{j+1} \cdot (\alpha_j - \alpha_{j+1}) \right) = b_{i+1}(\alpha_i - \alpha_{i+1}) + p_{i+1}(\mathbf{b})$$

Recursive definition might help think about it!

Welfare Maximization is Monotone

- Allocation rule to maximize welfare: $\mathbf{x}(\mathbf{b}) = \text{argmax}_{x_1,\ldots,x_n \in X} \sum_{i=1}^{n} b_i x_i$
  - Means pick $x_1, \ldots, x_n$ such that they are feasible (in X) and they maximize the sum $\sum_i b_i x_i$ for a given bid vector $\mathbf{b}$
- Show that this rule is monotone for single-parameter domains: Assignment 2
  - Myerson's Lemma always applies to these
- Challenge: Welfare maximization might be an NP hard problem
  - Example: Knapsack auctions

Knapsack Auction

- **Classic NP hard optimization problem**: Given n items with a weight w_i and value v_i and a knapsack with capacity W, find the subset of items with maximum value that fit in the Knapsack.
- Now consider the same problem where the n items are buyers with publicly known weights and private values
- Want a dominant-strategyproof mechanism to allocate to buyers in a feasible way (fits in Knapsack) and maximizes welfare
- This is NP hard, need approximations

Knapsack Approximation

A Greedy Knapsack Heuristic

1. Sort and re-index the bidders so that
$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.$$ [6]
2. Pick winners in this order until one doesn't fit, and then halt.[7]
3. Return either the solution from the previous step or the highest bidder, whichever has larger social welfare.[8]

- Exercise: Show that this is a 2-approximation of Knapsack
- Question. Without Step 3, this is not a 2-approximation

Knapsack Approximation

A Greedy Knapsack Heuristic

1. Sort and re-index the bidders so that

$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}.^{6}$$

2. Pick winners in this order until one doesn't fit, and then halt.[7]

3. Return either the solution from the previous step or the highest bidder, whichever has larger social welfare.[8]

- To apply Myerson, need to check if approximation rule is monotone
- Payment rule: each winner pays "critical bid"

Myerson and Externality

- When restricted to 0-1 allocations and welfare maximization in single-parameter environments, derive an alternate to Myerson's payment
- An agents **externality** is the change in social surplus excluding the agent, resulting from the agent's participation in the auction

$$\max_{(x_i=0,\mathbf{x}_{-i})\in X} \sum_{j\neq i} x_j b_j \quad - \quad \max_{(x_i=1,\mathbf{x}_{-i})\in X} \sum_{j\neq i} x_j b_j$$

Maximum possible welfare when i is absent (or $x_i = 0$)

Maximum possible welfare (by other winners) when i is present (and $x_i = 1$)

Myerson and Externality

- Myerson's payment for i in 0-1 allocations:

  critical bid $b_i^*(\mathbf{b}_{-i})$, $=$ **agent** i**'s externality**

- An agent must pay for the welfare loss it inflicts on others
- You will prove this in Assignment 2

$$\max_{(x_i=0,\mathbf{x}_{-i})\in X} \sum_{j\neq i} x_j b_j \quad - \quad \max_{(x_i=1,\mathbf{x}_{-i})\in X} \sum_{j\neq i} x_j b_j$$

Maximum possible welfare when i is absent (or $x_i = 0$)

Maximum possible welfare (by other winners) when i is present (and $x_i = 1$)

VCG Mechanism for General Mechanism Design

General Mechanism Design

- So far we have focused on single-parameter mechanism design
- Bidders can have valuations for any subset of allocations
- Direction revelation is even more challenging:
    - Asking bidders for up to $2^{|S|}$ values in the worst case

n buyer with private valuations over all possible allocations

Multiple items S

Unit Demand Case

- Matching markets to match buyers to items
    - n buyers and m items
    - Each buyer i has a valuation v_{ij} for item each j
    - Each buyer wants only one item **(unit demand)**
- Note that this is more general than the single-parameter domain
    - Each buyer has a valuation profile (not a single number)
- Many applications: housing markets, matching renters to rooms etc
- Auctioning off government licenses or construction projects etc

Housing Matching Market

Valuation profile

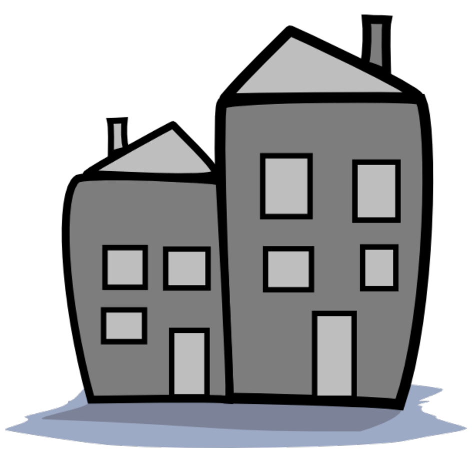
Houses

Welfare maximizing allocation assuming value is known?

1

Zoe

12, 2, 4



2

Chris

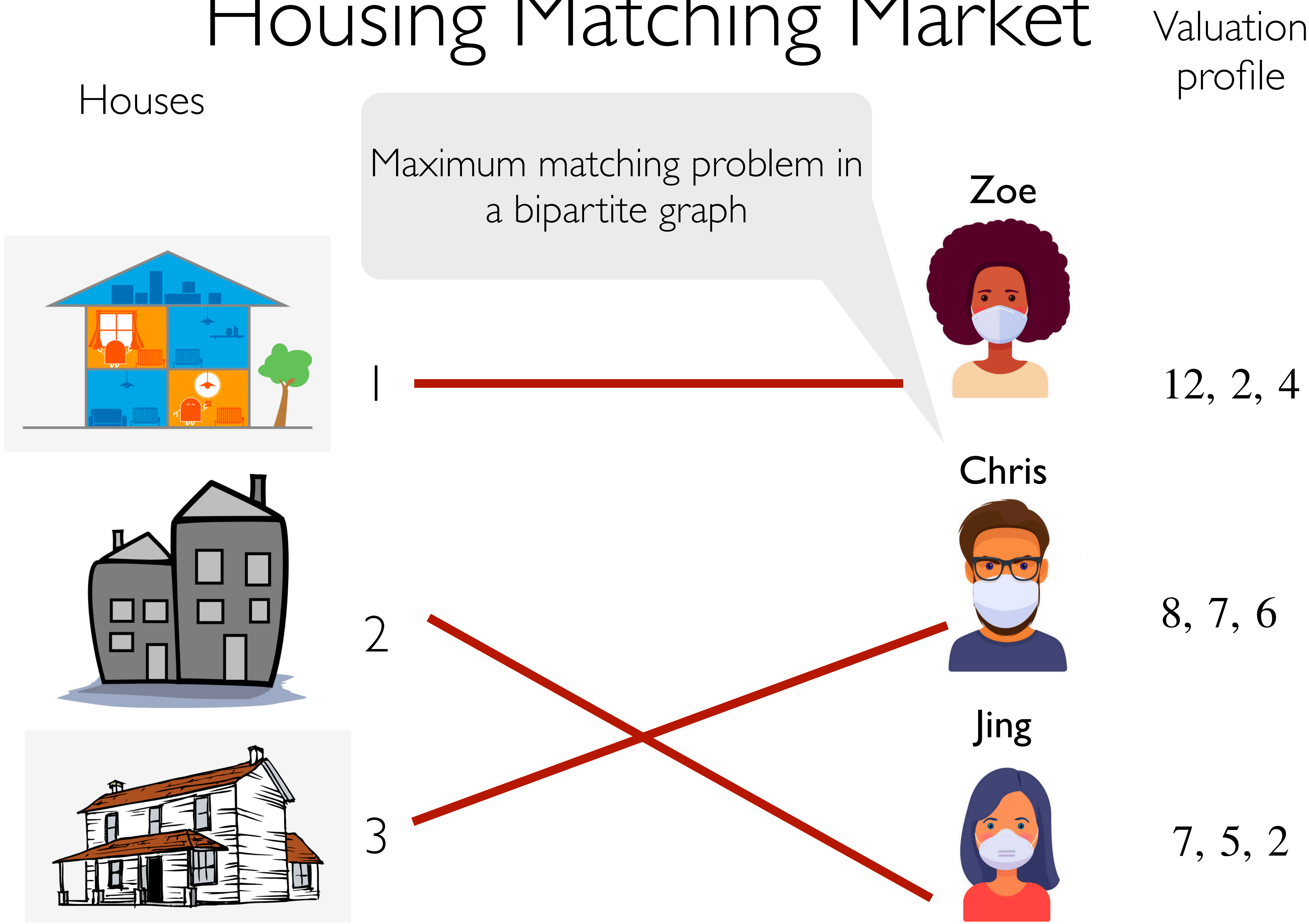
8, 7, 6

3

Jing

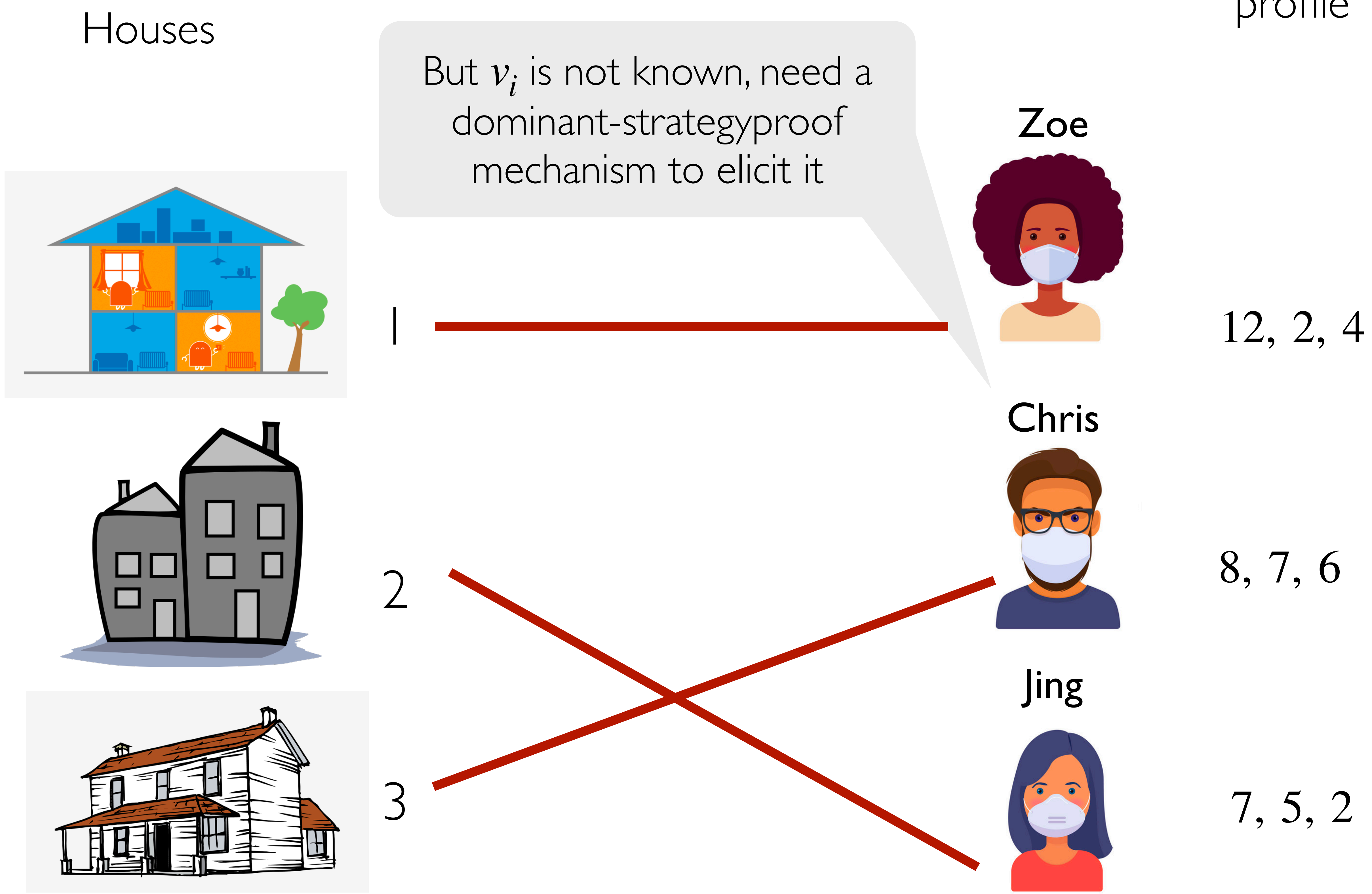
7, 5, 2

Housing Matching Market



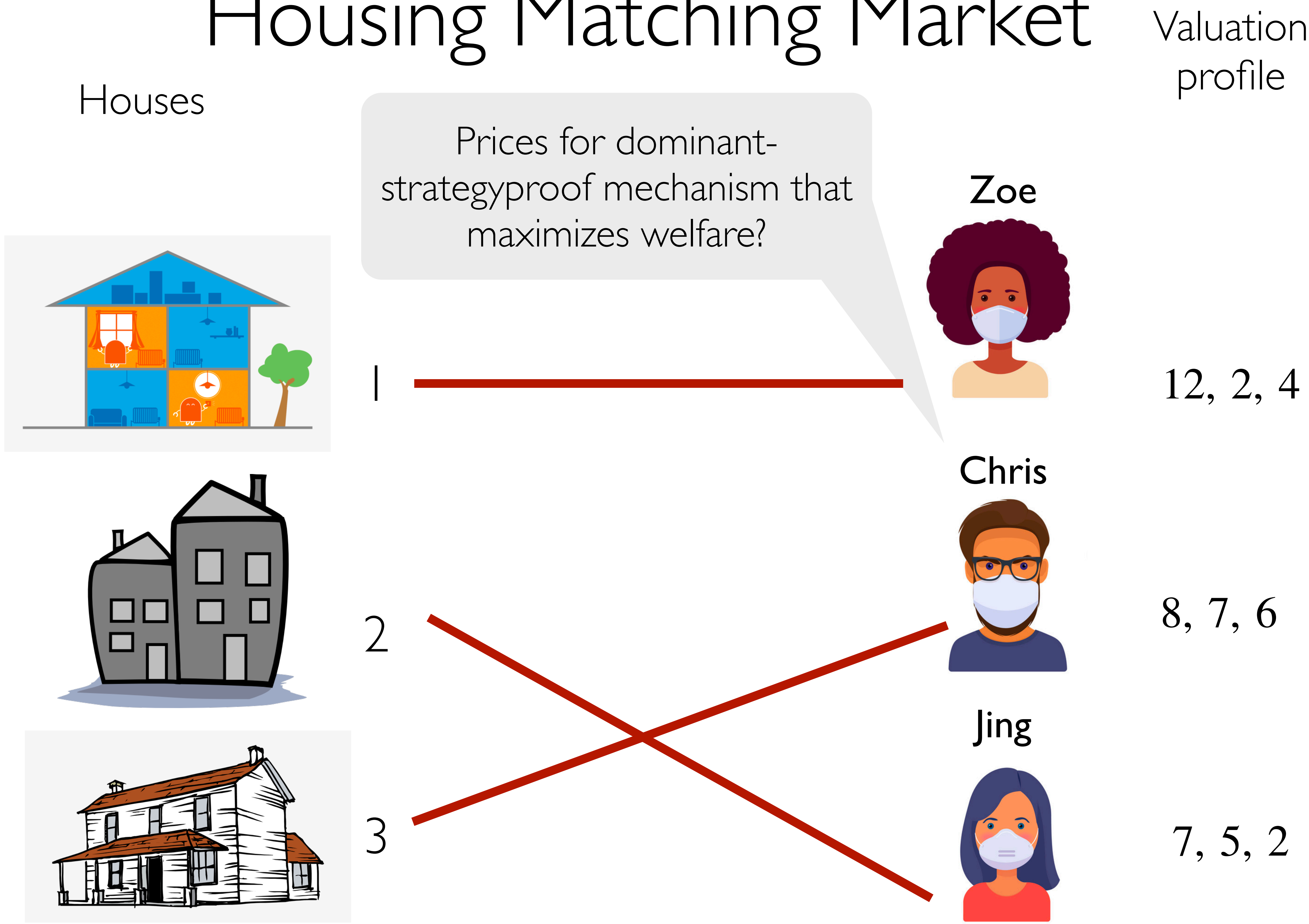
Allocation profile: (1,3,2)

Housing Matching Market

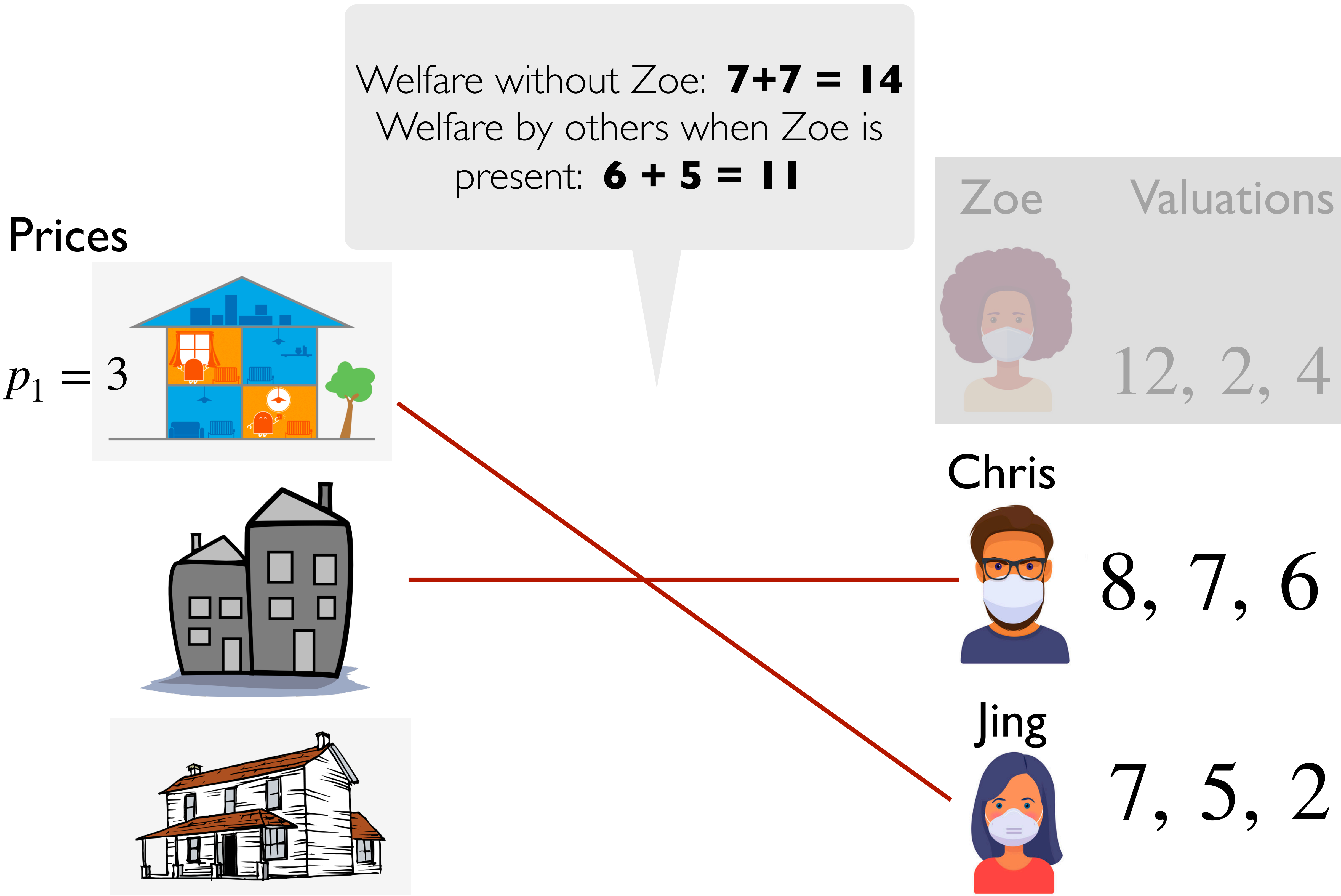


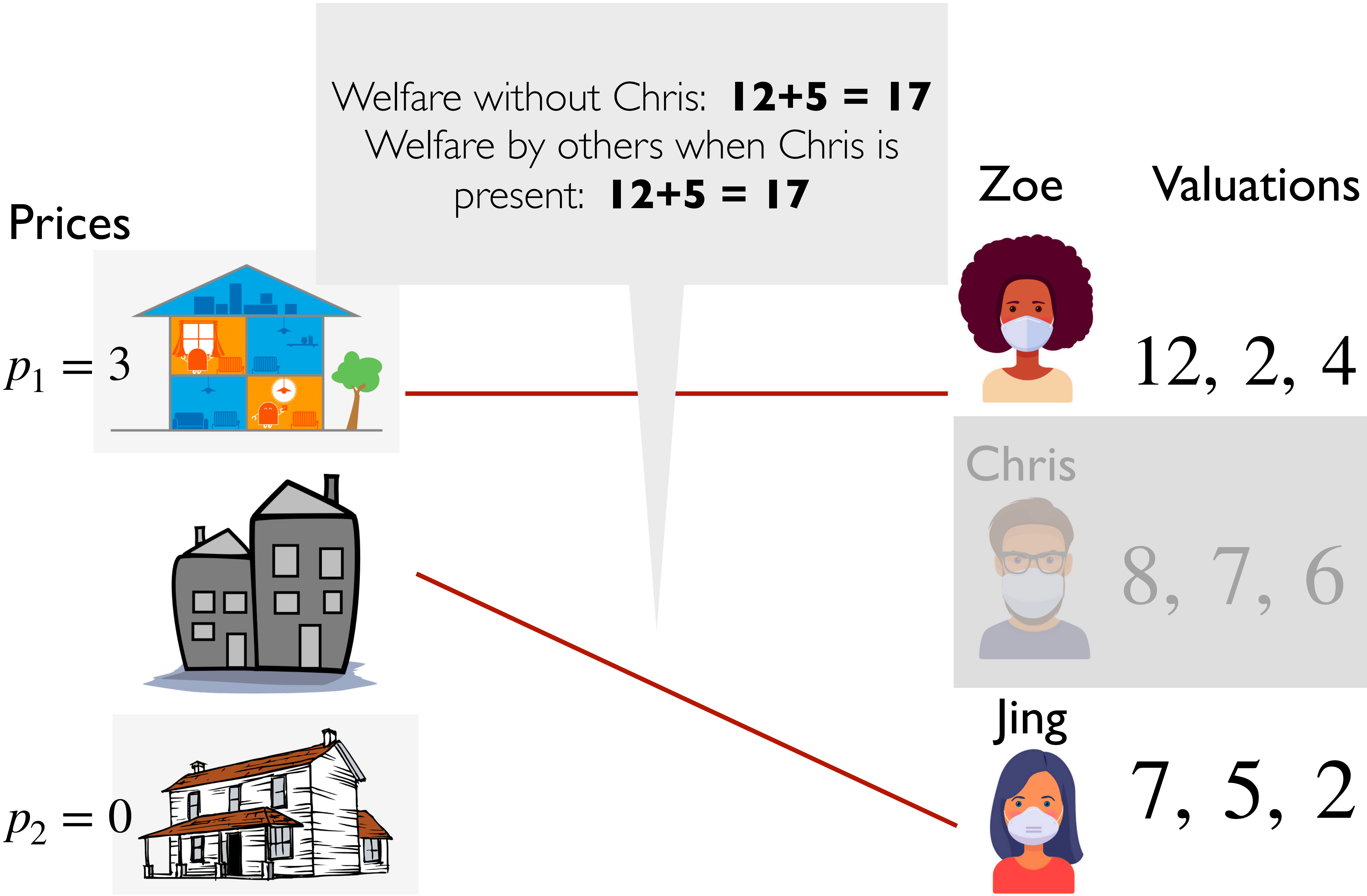
Allocation profile: $(1,3,2)$

Housing Matching Market



Allocation profile: (1,3,2)


Welfare without Zoe: **7+7 = 14**
Welfare by others when Zoe is present: **6 + 5 = 11**
Zoe
Valuations
12, 2, 4
Prices
$p_1 = 3$
Chris
8, 7, 6
Jing
7, 5, 2


Welfare without Chris: **12+5 = 17**
Welfare by others when Chris is present: **12+5 = 17**
Prices
$p_1 = 3$
$p_2 = 0$
Zoe
Valuations
12, 2, 4
Chris
8, 7, 6
Jing
7, 5, 2

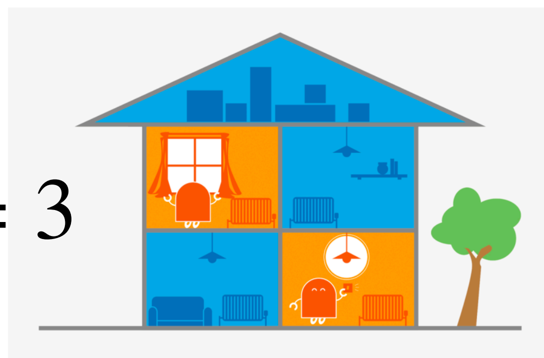
Welfare without Jing: **12+7 = 19**
Welfare by others when Jing is present: **12+6 = 18**

Prices

Zoe

Valuations

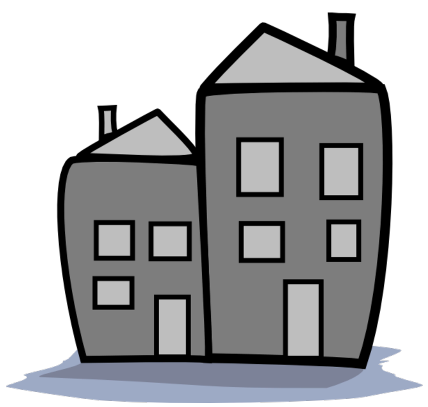
$p_1 = 3$



12, 2, 4

Chris

$p_3 = 1$

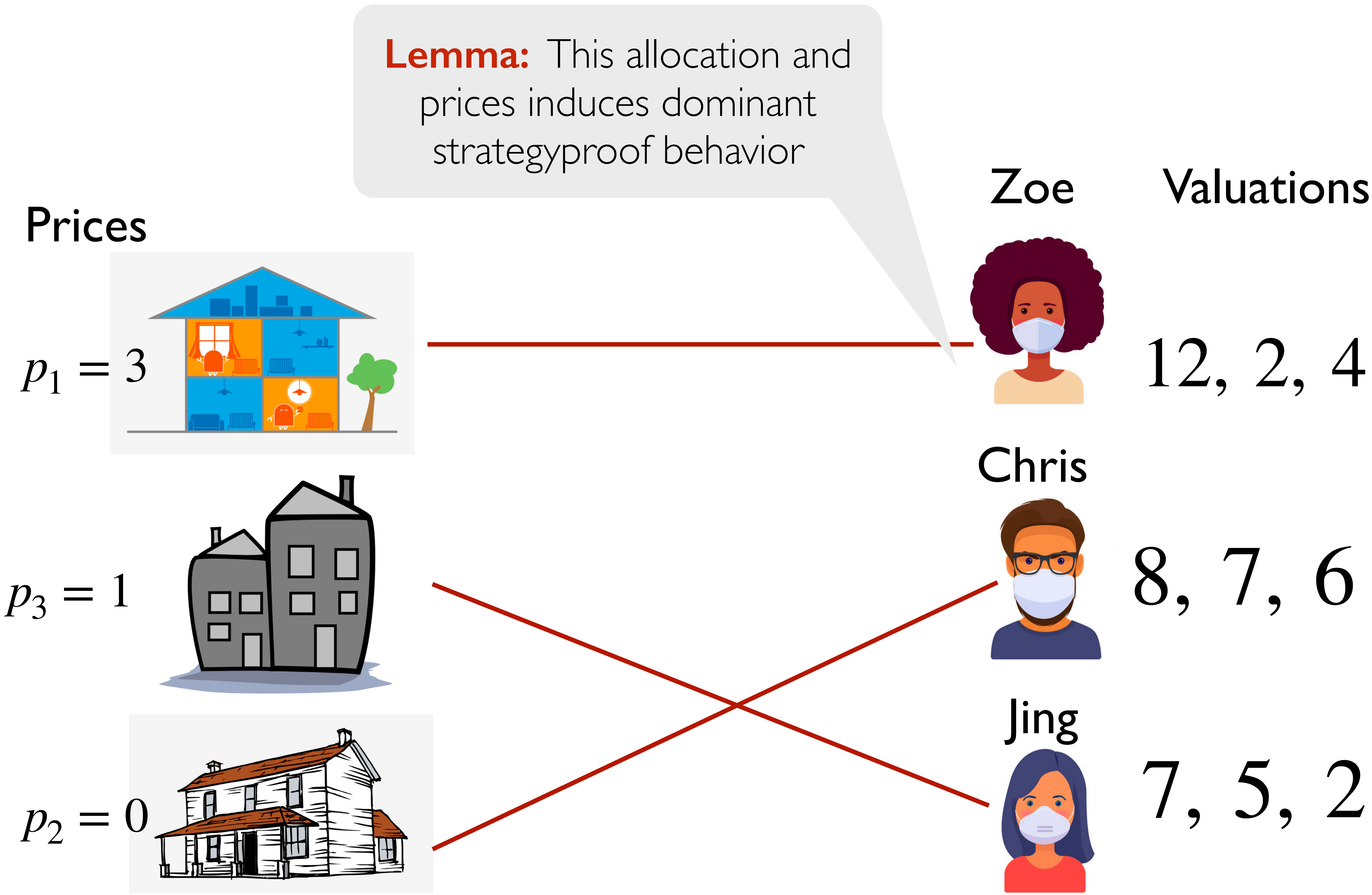


8, 7, 6

Jing

$p_2 = 0$

7, 5, 2



Lemma: This allocation and prices induces dominant strategyproof behavior
Prices
Zoe
Valuations
$p_1 = 3$
12, 2, 4
Chris
$p_3 = 1$
8, 7, 6
Jing
$p_2 = 0$
7, 5, 2

General Mechanism Design

- Combinatorial (multi-parameter auctions): set S of items, and $2^{|S|}$ possible subsets that can be allocated (**outcomes**)
- Ingredients of a multi-parameter mechanism design problem
    - n strategic agents
    - A finite set A of feasible outcomes
    - Each agent i has a private valuation $\mathbf{v}_i(\mathbf{a})$ for each $\mathbf{a} \in A$: each $\mathbf{v}_i$ is now a vector describing values for all possible outcomes
- **Goal:** Dominant-strategyproof, welfare maximizing, polynomial time mechanism

VCG Mechanism

- Surprisingly, there exists a dominant-strategyproof welfare-maximizing algorithm for the general setting.
- **Theorem [Vickrey-Clarke-Groves (VCG) Mechanism]:** The following mechanism is dominant-strategyproof for any general mechanism design problem:
  - Collect sealed bids
  - Allocated based on the surplus maximizing rule
  - Charge each bidder their "**externality**": the welfare loss inflicted on others by their presence
- Turns out the above allocation and payment imposes DSIC behavior

VCG Mechanism

- **Allocation.** Given bids $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ where each $\mathbf{b}_i$ is now a vector indexed by $|A|$, the welfare maximizing allocation is (assuming bids as proxies for valuations)

$$a^*(\mathbf{b}) = \mathrm{argmax}_{a \in A} \sum_{i=1}^{n} \mathbf{b_i}(a)$$

- **Payment.** Charge each bidder their externality:

$$p_i(\mathbf{b}) = \underbrace{\max_{a_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i})}_{\text{without } i} - \underbrace{\sum_{j \neq i} \mathbf{b}_j(a^*)}_{\text{with } i}$$

Where a^* is the welfare maximizing outcome in the presence of i

VCG Mechanism

- **Payment.** Alternate way to look at it:

$$p_i(\mathbf{b}) = \underbrace{\max_{a_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i})}_{\text{without } i} - \underbrace{\sum_{j \neq i} \mathbf{b}_j(a^*)}_{\text{with } i}$$

$$p_i(\mathbf{b}) = \mathbf{b}_i(a^*) - \left(\sum_{i=1}^{n} \mathbf{b}_i(a^*) - \max_{a_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i}) \right)$$

Rebate equal to the welfare generated by i's presence

VCG is Dominant Strategyproof

- **Proof.** Fix i and $\mathbf{b}_{-i}$. Suppose the chosen outcome is $\mathbf{x}(\mathbf{b}) = a^*$
  - Utility of i for outcome a^* is $\mathbf{v}_i(\mathbf{a}^*) - p_i(\mathbf{b})$
  - Term B is a constant (max surplus generated without i)
  - Maximizing i's utility $\iff$ maximizing term A

$$\underbrace{\mathbf{v}_i(a^*) + \sum_{j \neq i} \mathbf{b}_j(a^*)}_{A} - \underbrace{\max_{a_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i})}_{B}$$

VCG is Dominant Strategyproof

- **Proof.** Fix i and $\mathbf{b}_{-i}$. Suppose the chosen outcome is $\mathbf{x}(\mathbf{b}) = a^*$
  - Maximizing i's utility $\iff$ maximizing term A
  - Setting $\mathbf{b}_i = \mathbf{v}_i$ maximizes i's utility under a welfare maximizing allocation.

Bidding truthfully maximizes i's utility

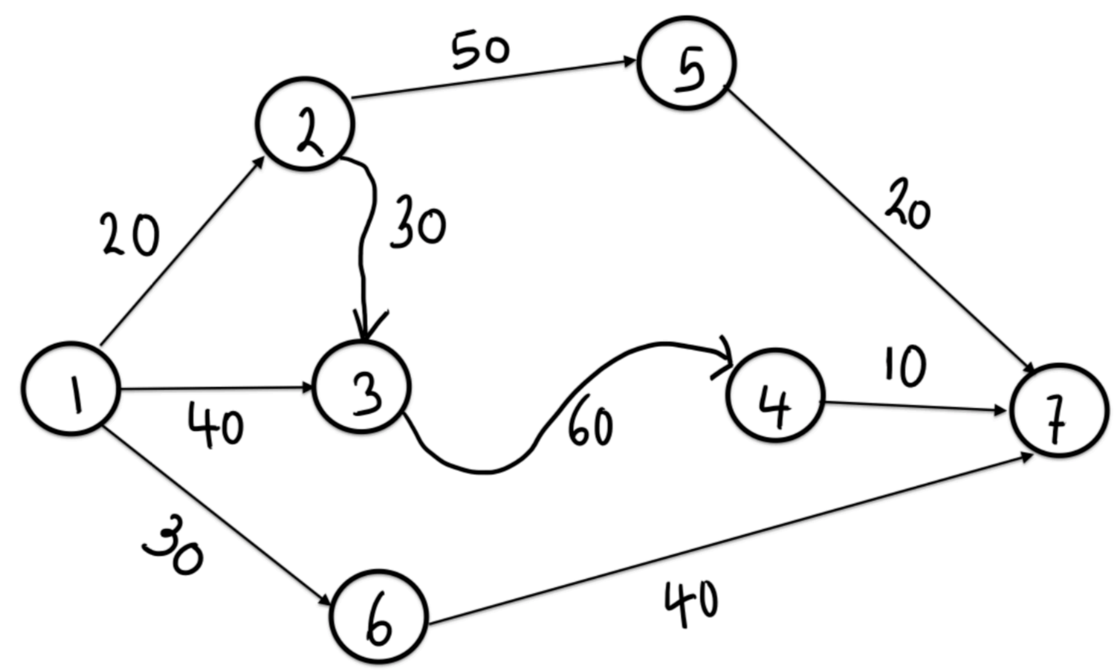
$$\underbrace{\mathbf{v}_i(a^*) + \sum_{j \neq i} \mathbf{b}_j(a^*)}_{A} - \underbrace{\max_{a_{-i} \in A_{-i}} \sum_{j \neq i} \mathbf{b}_j(a_{-i})}_{B}$$

VCG: Sponsored Search

- Single parameter domains are a special case of VCG
- Let us recover Myerson payment rule using VCG

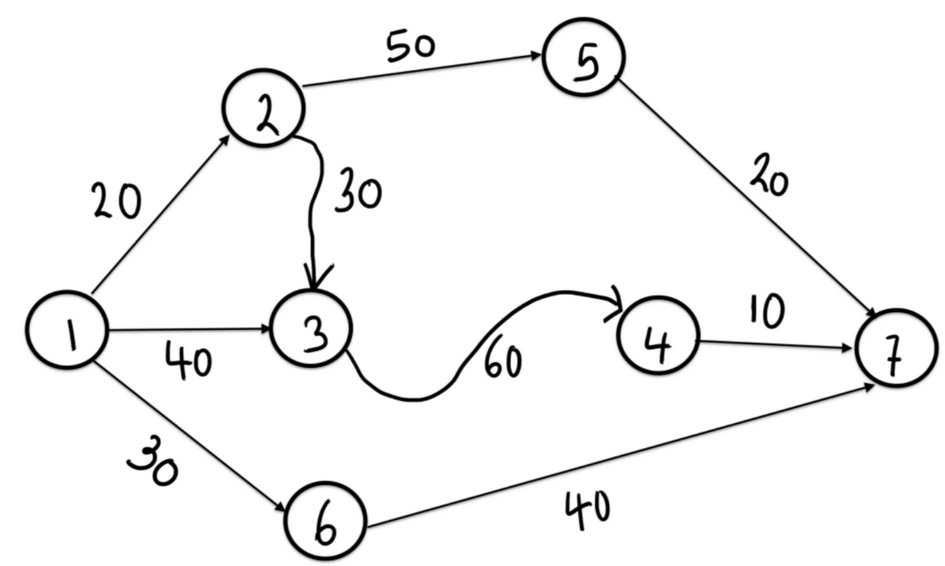
Mechanism Design

- Broader scope of problems than just auctions
- VCG can be used for these domains
- Consider a shortest-path problem
- Each edge is an agent and has private cost c_i is their edge is used
- **Problem**: Find min-cost path from source to destination.



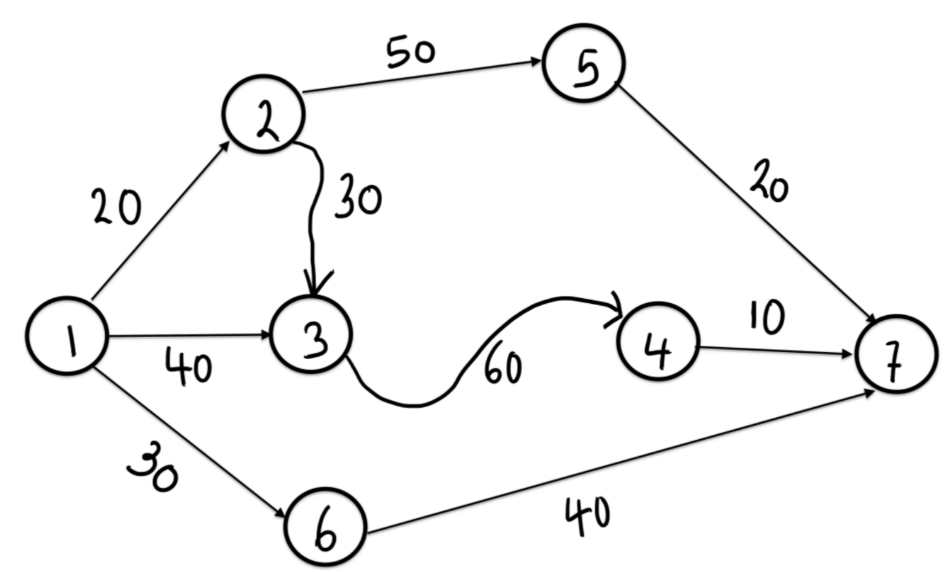
Mechanism Design

- **Goal:** Select a lowest cost path from 1 to 7
- Each edge is an agent with cost $c_i > 0$ if their edge is used $(v_i = -c_i)$
    - Since agent's have costs when used, mechanism may pay them
- $A = \{$all s-t paths$\}$
- $A_{-i} = \{$paths that do not use edge $i\}$
- VCG mechanism selects path with maximum value:
    - Min cost path



Mechanism Design

- Assuming truthful reports, the lowest-cost path is $1 \rightarrow 6 \rightarrow 7$
- What are the payments?
  - For all agents except (1,6) and (6,7): cost is zero
  - For agent (1,6)'s payment
    - What is the lowest cost path without that edge?
      - $1 \rightarrow 2 \rightarrow 5 \rightarrow 7$
    - $p(1\text{-}6) = -90 - (-40) = -50$
  - That is, 1-6 should receive a payment of 50
  - Similarly we can compute 6-7's payment:
    - $p(6-7) = -90 - (-30) = -60$

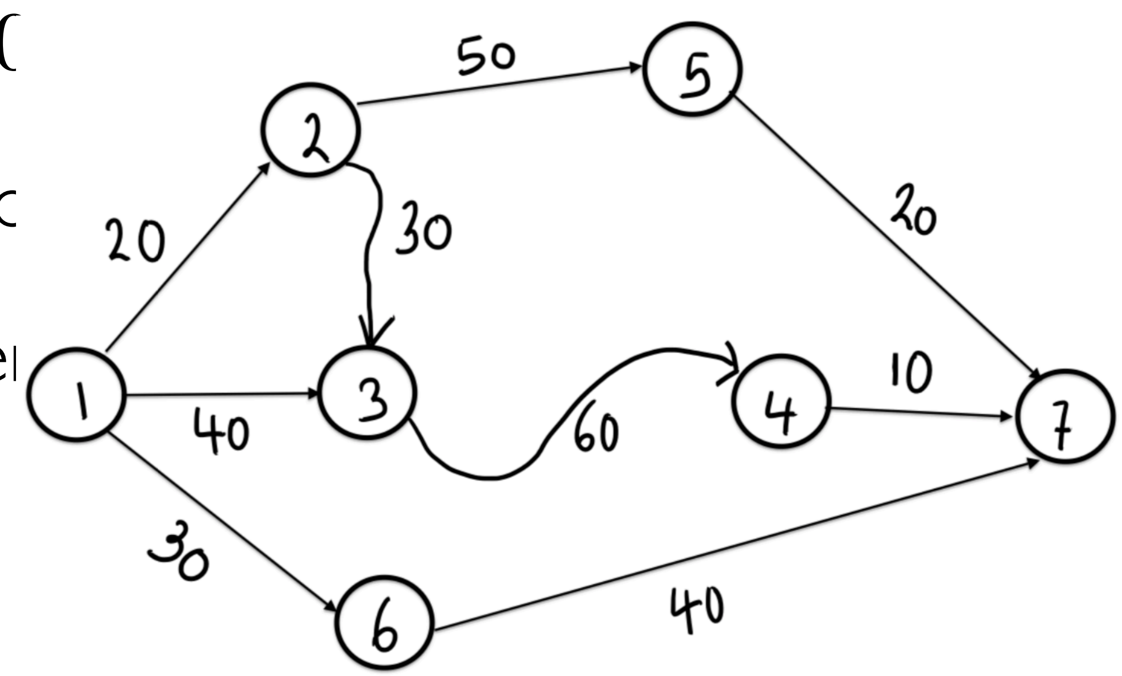


Mechanism Design

- Assuming truthful reports, the lowest-cost path is $1 \rightarrow 6 \rightarrow 7$
- What are the payments?
  - For all agents except (1,6) and (6,7): cost is zero
  - For agent (1,6)'s payment

The agents receive as payment the **maximum cost** they *could have reported* and still been on the selected path!

    - $p(1\text{-}6) = -90 - (-40) = -5($
  - That is, 1-6 should receive a payment c
  - Similarly we can compute 6-7's paymer
    - $p(6-7) = -90 - (-30) = -$



VCG Challenges

- Suffers from **collusion**, same way as second-price auctions
- **Intractability** of welfare maximization
    - This is a challenge even when restricted to a single-parameter setting
- **Budget balance**: If an agent has a **negative value** (say a seller who has a **cost** involved with outcomes) then the mechanism may not generate enough revenue to compensate the seller
        - Positive payments may not equal negative payments
    - That is, the VCG mechanism may incur a **budget deficit**
- **Non- monotonity of revenue:** It may generate worse revenue when the competition increases!