CSCI 357: Algorithmic Game Theory

Lecture 4: Myerson's Lemma

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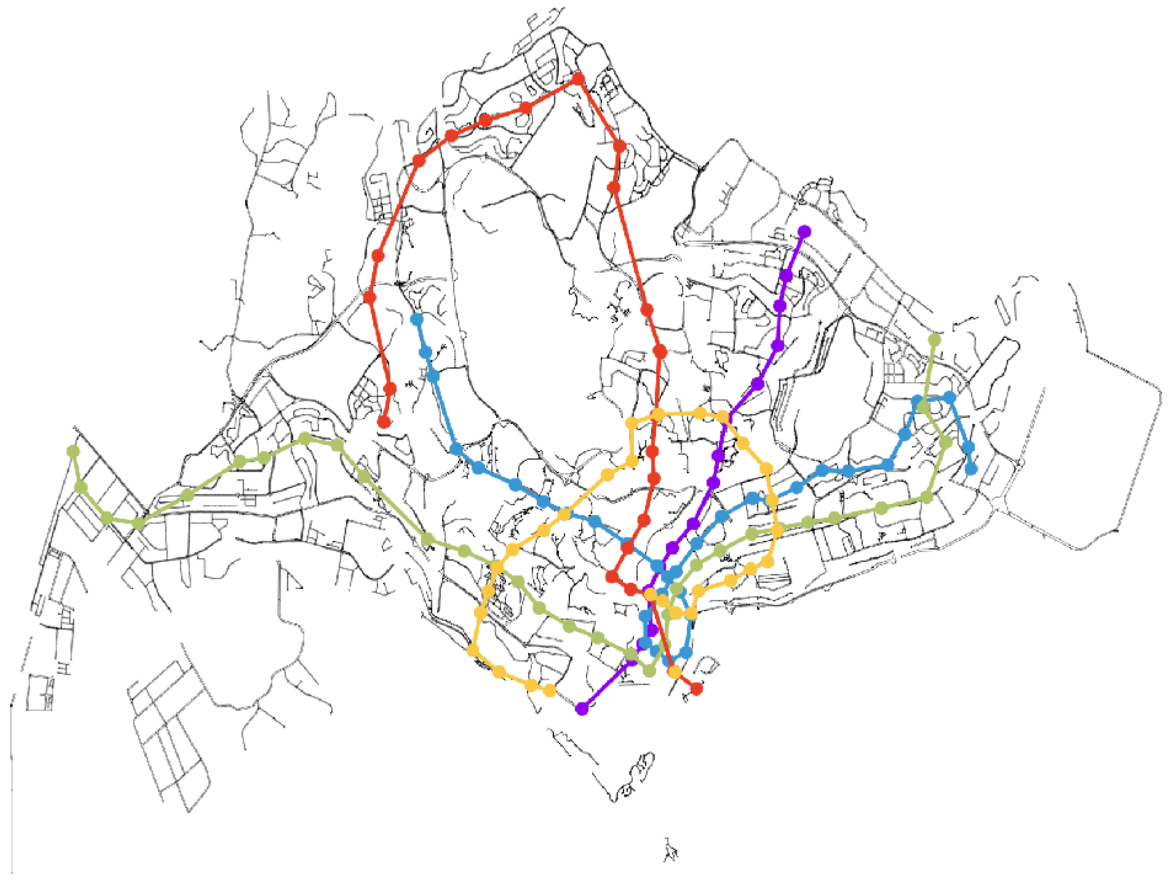


Announcements and Logistics

- Hand in **Homework 2**
- **Paper evaluation 1 (due next Fri)**: Case study of internet ad auctions
  - Read the research paper
  - **Part A**: Submit a google form individually
  - **Part B**: Work on technical analysis in groups of 4
  - Each group must turn in their write up of at least 3 out of 5 proofs in class and present one of them on the board
- **Assignment 2** will be released on Mon and due the following week

Questions?

Last Time

- Discussed single item (sealed bid) auctions
- Second price (Vickrey auctions) are dominant strategyproof and maximize welfare in linear time
- Ran a first price auction:
    - We will discuss the results next week, stay tuned!

Single-Parameter Mechanism Design

- Multiple items but each agent has a single valuation for their allocation

n buyer with private valuations which can be described by a **single number** v_i

Multiple items

Example: k identical goods

- Simple example of single-parameter setting: we have k copies on an item
- Feasible allocation is then $X = (x_1, \ldots, x_n) \subseteq \{0,1\}^n$, where $x_i = 1$ if bidder gets an copy; 0 otherwise and $\sum_{i=1}^{n} x_i \leq k$

Example: Single Subset Case

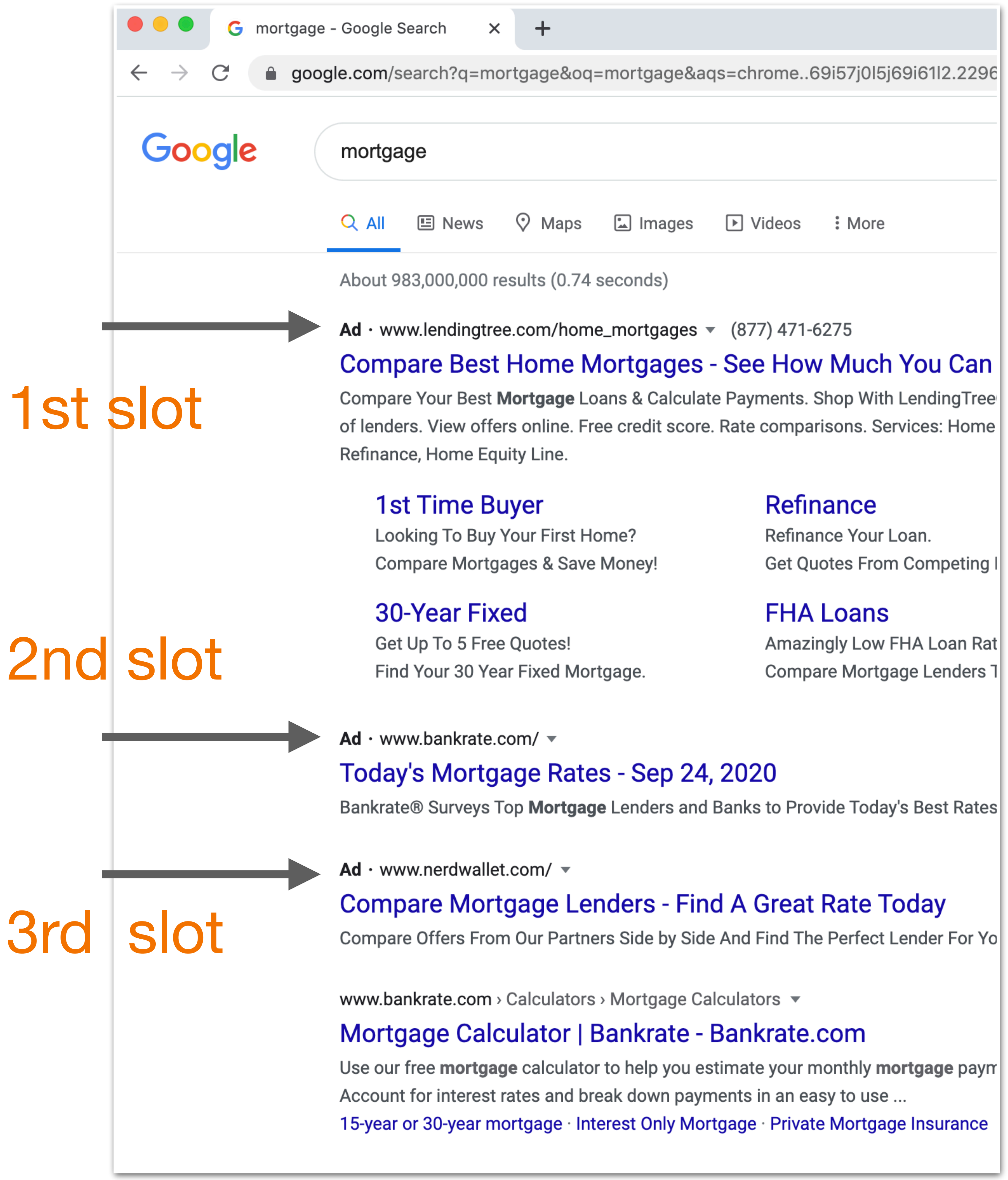
- Each buyer i has value v_i for a certain subset $S_i \subseteq S$, 0 other others
- Feasible allocation is $X = (T_1, \ldots, T_n)$ where each $T_i \subseteq S$

Multiple items

n buyers but each buyer only wants a certain subset

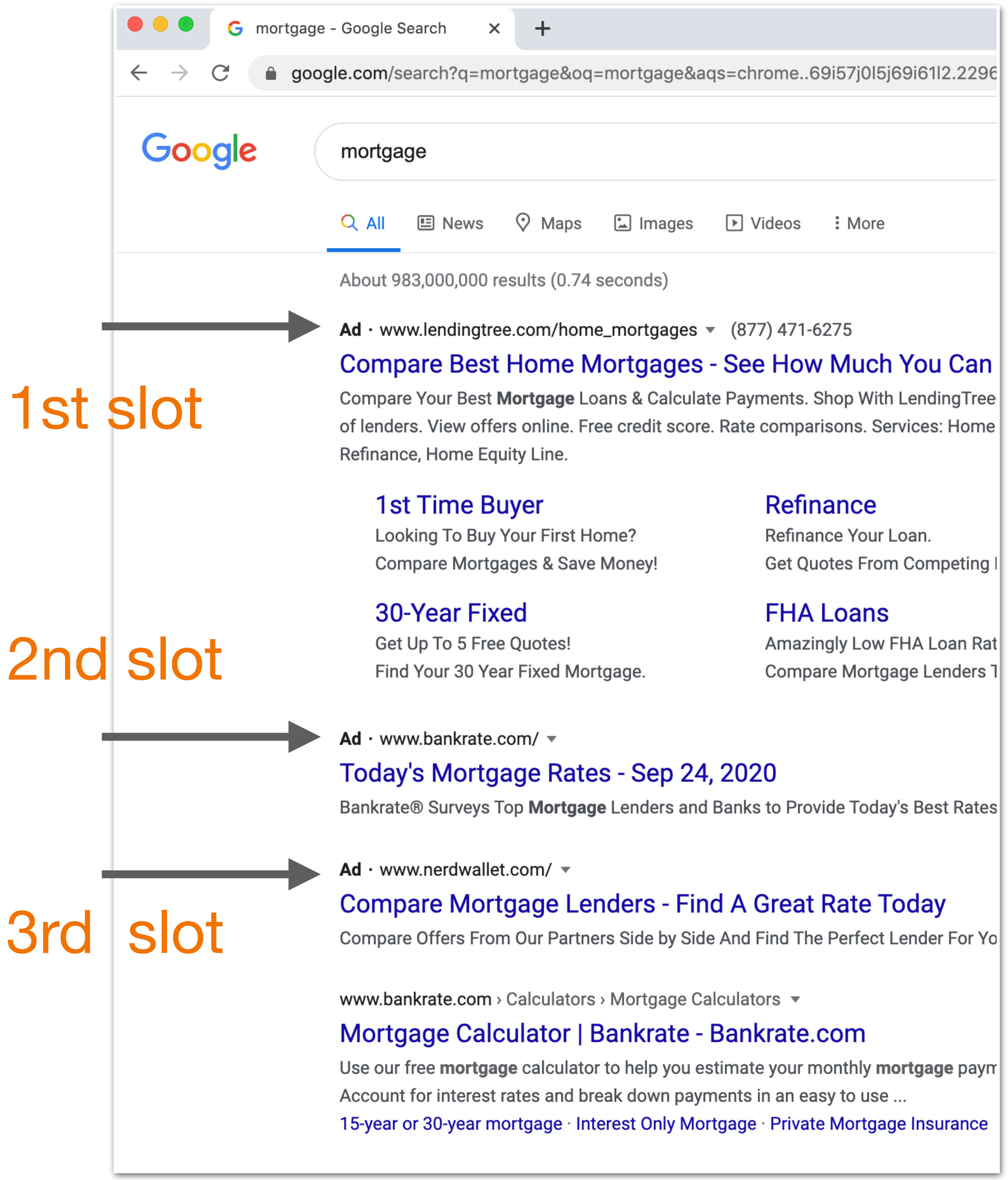
Sponsored Search Model [Edelman & Varian]

- Every time someone searches a query, an auction **is run in real time** to decide: which advertisers links are shown, in what order, and how they are charged
- We look at a simplified but effective model to study sponsored search auction
- Items for sale are k slots for sponsored links on a page
- Bidders (advertisers) have a standing bid on a keyword that was searched on
- Slots higher up on the page are more valuable than low
  - Users more likely to click on them



Sponsored Search Model [Edelman & Varian]

- Slots higher up on the page more likely to be clicked
  - Quantified through **click-through-rates (CTRs)**
  - CTR α_j of a slot j is the probability of clicks it is expected to receive
  - Reasonable to assume $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$
- **Simplifying assumption.** CTR of a slot is independent of its occupant, that is, doesn't depend on the quality of the ad
- Assume advertisers have a private valuation v_i **for each click on its link**: value derived from slot j by advertiser i is $v_i \cdot \alpha_j$



Example: Sponsored Search

- A **feasible allocation** is an assignment of bidders to slots, such that each slot is assigned to at most one bidder and each bidder is assigned at most one slot, that is, $X = (x_1, x_2, \ldots, x_n)$
  - where $x_i = \alpha_j$, the click through of slot j if bidder i is assigned to it; otherwise $x_i = 0$ if bidder is unassigned

n buyers, each has private value of v_i "per click" they get

k slots, with different click-through rates α_j

Sealed-Bid Mechanism

- We will focus on sealed-bid mechanisms that
    - Collect bids/reports $\mathbf{b} = (b_1, \ldots, b_n)$
    - Choose a feasible allocation rule $\mathbf{x}(\mathbf{b}) \in X \subseteq \mathbb{R}^n$
    - Choose payments $\mathbf{p}(\mathbf{b}) \in \mathbb{R}^n$
- Such mechanisms are called **direct-revelation mechanism**
    - Mechanisms that ask agents to report their private value up front
- **Quasilinear utility:** $u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$ on the bid profile $\mathbf{b}$
- We will focus on payment rules that satisfy
    - $p_i(\mathbf{b}) \geq 0$: sellers **can't pay the bidders**
    - $p_i(0, \mathbf{b}_{-i}) = 0$: a zero bid leads to a zero payment

Design Approach

- Our goal is to **maximize surplus** **argmax**$_{(x_1,\ldots,x_n)\in X} \sum_{i=1}^{n} v_i x_i$
- **Challenge**: jointly design two pieces: who gets what, and how much do they pay
  - Not enough to figure out who wins, if don't charge them the right amount
- Usually, the recipe we will follow:
  - **Step 1.** Assume **truthful bids**, and decide how to allocate so as to **maximize surplus** (in polynomial time)
  - **Step 2.** Using the allocation in step 1, decide how to charge payments so as that the mechanism is strategyproof (DSIC)

k identical goods: Allocation

- Collect sealed bids
- Who should we give the k items to maximize surplus (assuming truthful bids)
  - Top k bidders
- **Question**. What should we charge them so that truth telling is dominant strategy?

Sponsored Search: Allocation

- How do we do we assign slots to maximize $\sum_{i=1}^{n} b_i x_i$?
  - Greedy allocation is optimal (can be showed by an exchange argument)
  - Recall that CTR rates $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$
  - Sort and relabel bids $b_1 \geq b_2 \geq \cdots \geq b_n$
  - Assign jth highest bidder to jth highest slot
- Can we create a payment rule (an analog of second-price rule) that makes the greedy allocation incentive compatible?

Towards a General Characterization

- **Question.** Can any allocation rule be paired with a payment rule such that the mechanism is strategyproof (truthtelling is a dominant strategy)?
    - When is this possible and how should we design the payment rule?
- Myerson's lemma gives a **general characterization** of allocation rules that can be turned into a truthful mechanism
- And tells us exactly how to design payment rules to achieve that

Myerson's Lemma: Informal

- In a fixed-parameter setting,
  - an allocation rule $\mathbf{x}$ can be made dominant-strategy incentive compatible if and only if $\mathbf{x}$ is **monotone (non decreasing)**, and
  - if $\mathbf{x}$ is monotone, there is a **unique payment** rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is dominant strategyproof.

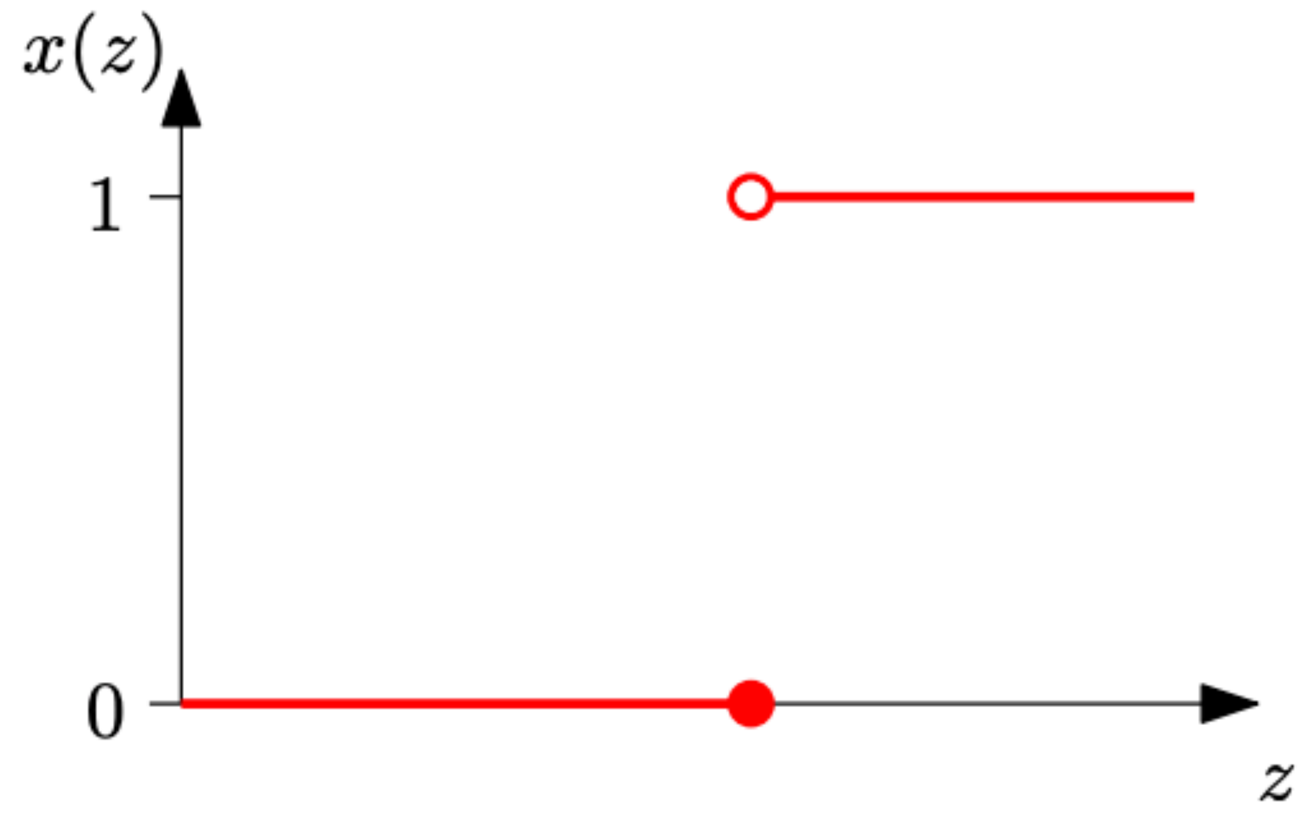
Implications of Myerson's Lemma

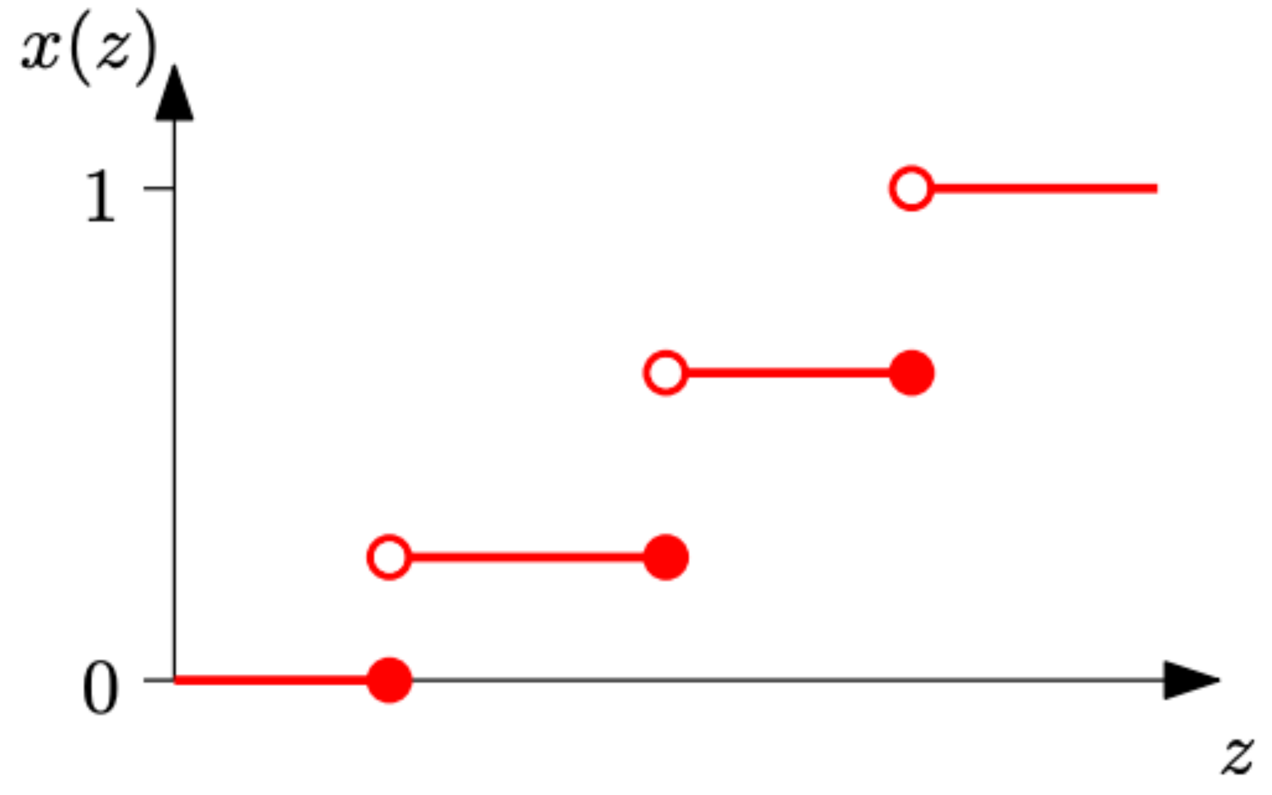
- Very powerful characterization
- Our initial design dilemma: can we make some allocation rule $\mathbf{x}$ dominant strategyproof by pairing it with an appropriate payment rule?
- Myerson's lemma takes this question and turns into one that is more wieldy and operational: checking if $\mathbf{x}$ is monotone
- If an allocation rule is monotone, the lemma says there is **exactly one way to assign payments** to make it dominant strategyproof
    - A direct formula for the payments

Monotone Allocation Rule

- **Definition.**

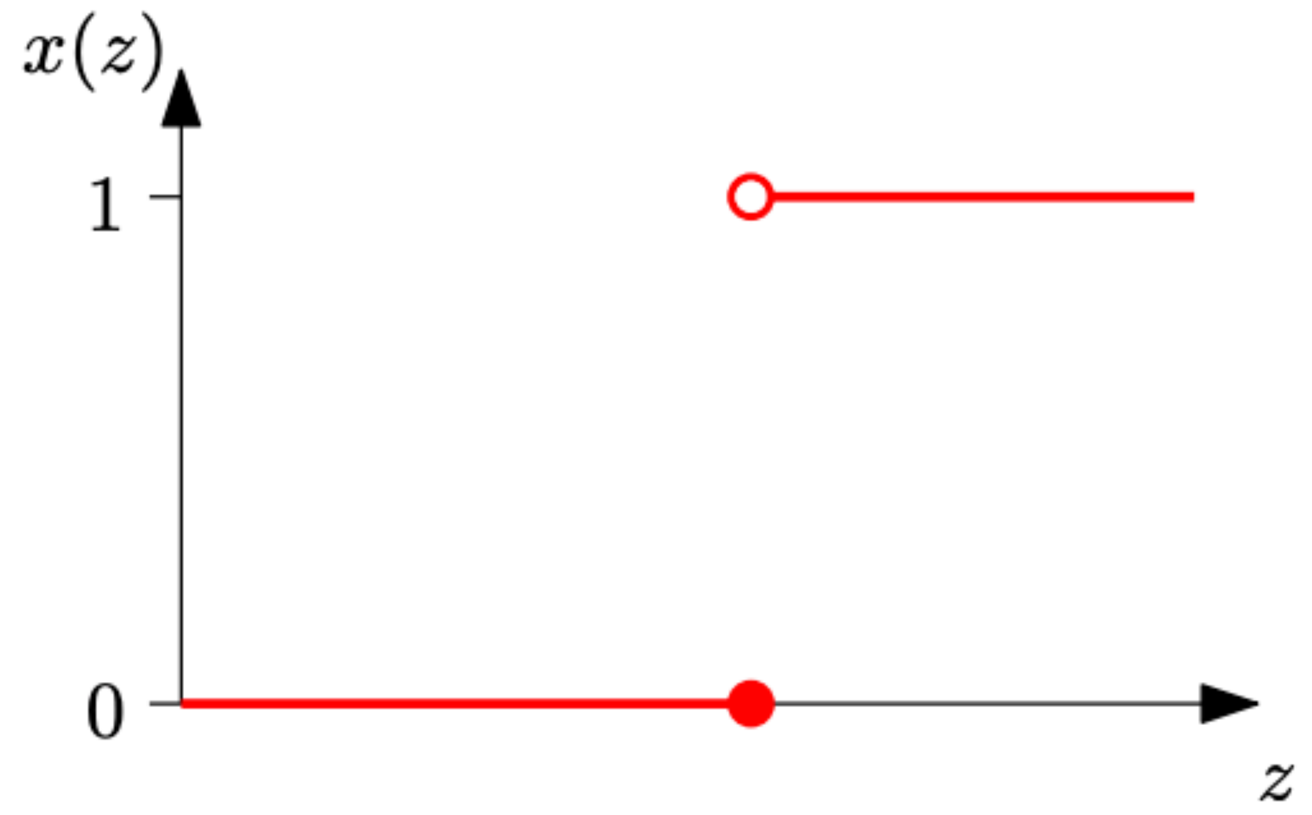
  An allocation rule $\mathbf{x} = (x_1, \ldots, x_n)$ for a single-parameter domain is monotone-non-decreasing if for every bidder i and bids $\mathbf{b}_{-i}$ of other bidders, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is non-decreasing in its bid z.

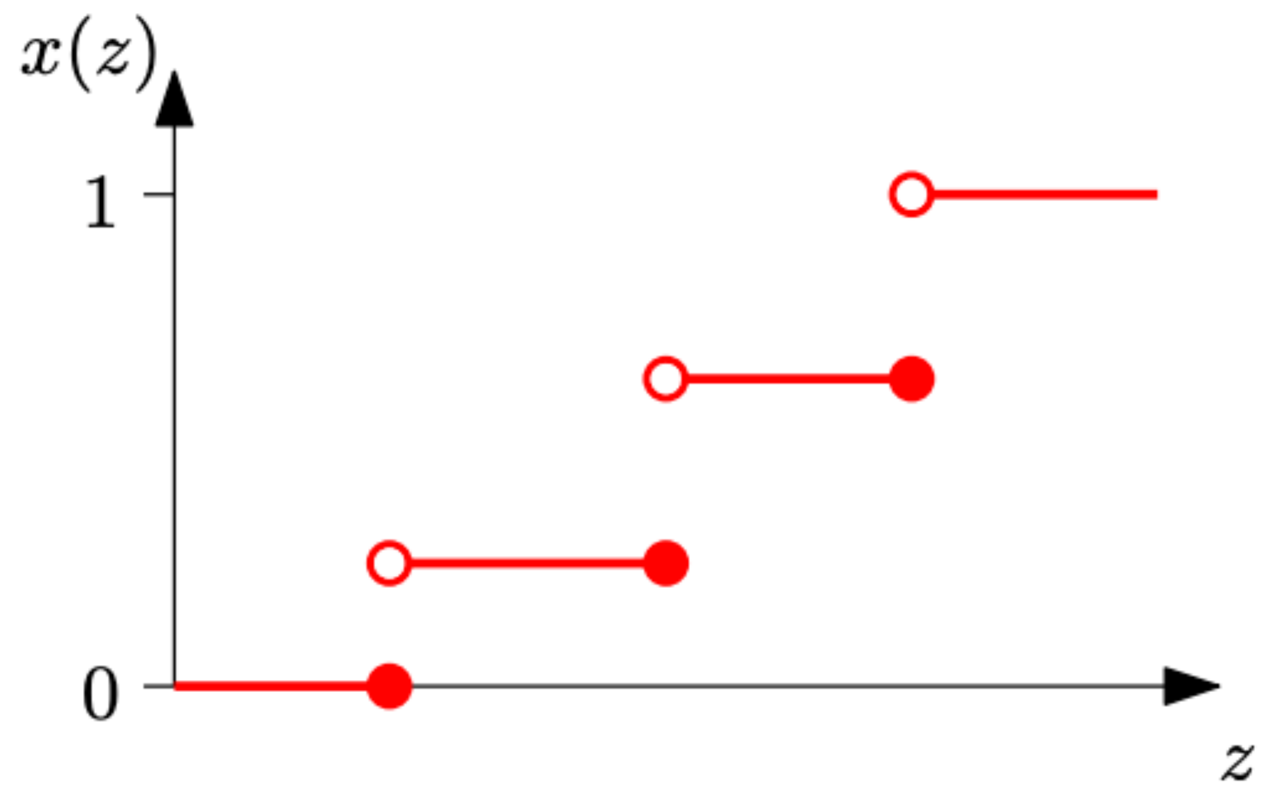




Monotone Allocation Rule

- That is, in a monotone allocation rule, bidding higher can only get you "more" stuff
- Example of a monotone allocation rule?
- Example of a non-monotone allocation rule?





Myerson’s Lemma: Proof

- **Part 1:** An allocation $\mathbf{x}$ rule can be made dominant strategyproof only if $\mathbf{x}$ is monotone
- **Part 2:** A mechanism $(\mathbf{x}, \mathbf{p})$, where is $\mathbf{x}$ is monotone, is dominant strategyproof only if $\mathbf{p}$ is given by the expression in Myerson’s lemma
- **Part 3:** Finally, we show that if the allocation $\mathbf{x}$ is monotone and the payment rule $\mathbf{p}$ is as given by the expression in the lemma then, $(\mathbf{x}, \mathbf{p})$ is dominant strategyproof.

Myerson's Lemma: Proof

- **Recall dominant strategyproof condition**:
  - for every agent i, every possible private valuation v_i, every set of bids $\mathbf{b}_{-i}$ by the other agents, $i's$ utility is maximized by bidding truthfully
- Fix an arbitrary player i and bid profile of others $\mathbf{b}_{-i}$
- Let $x(z)$ and $p(z)$ be shorthand for $i's$ allocation $x_i(z, \mathbf{b}_{-i})$ & payment $p_i(z, \mathbf{b}_{-i})$
- Throughout the proof, we will vary the bid z and see how it changes the allocation

Myerson's Lemma: Proof Part 1

- **Part 1.** An allocation rule $\mathbf{x}$ can be made dominant-strategy incentive compatible only if $\mathbf{x}$ is monotone non-decreasing
- If player i (with value v) deviates and bids as if she has value z, then her utility is $v \cdot x(z) - p(z)$
    - Notice: no control over your value v
- For truth telling to be a (weakly) dominant strategy for all values, must be that
    - $v \cdot x(v) - p(v) \geq v \cdot x(v^\dagger) - p(v^\dagger)$ for all $v, v^\dagger$
- We consider two possible values z_1, z_2 with $z_1 < z_2$
    - Case 1 (Underbidding): $v = z_2$, $v^\dagger = z_1$
    - Case 2 (Overbidding): $v = z_1$, $v^\dagger = z_2$

Myerson's Lemma: Proof Part 1

- In case (a), where $v = z_2$ and player underbids z_1

  $z_2 \cdot x(z_2) - p(z_2) \geq z_2 \cdot x(z_1) - p(z_1)$ — (Ineq 1)

- In case (b), where $v = z_1$ and player overbids z_2

  $z_1 \cdot x(z_1) - p(z_1) \geq z_1 \cdot x(z_2) - p(z_2)$ — (Ineq 2)

- Adding both: $z_2 \cdot x(z_2) + z_1 \cdot x(z_1) \geq z_2 \cdot x(z_1) + z_1 \cdot x(z_2)$

- Rearranging: $(z_2 - z_1) \cdot (x(z_2) - x(z_1)) \geq 0$

  - Does this imply something about the allocation rule $\mathbf{x}$?

- Since $z_2 > z_1$,this only holds if $x(z_2) \geq x(z_1)$: thus $\mathbf{x}$ must be monotone non-decreasing ∎ (Part 1)

All pictures are from [Hartline's Book on Mechanism Design]()

Myerson's Lemma: Proof Part 2

- **Part 2.** Suppose mechanism $(\mathbf{x}, \mathbf{p})$ dominant-strategyproof, where $\mathbf{x}$ is monotone, let's derive $\mathbf{p}$

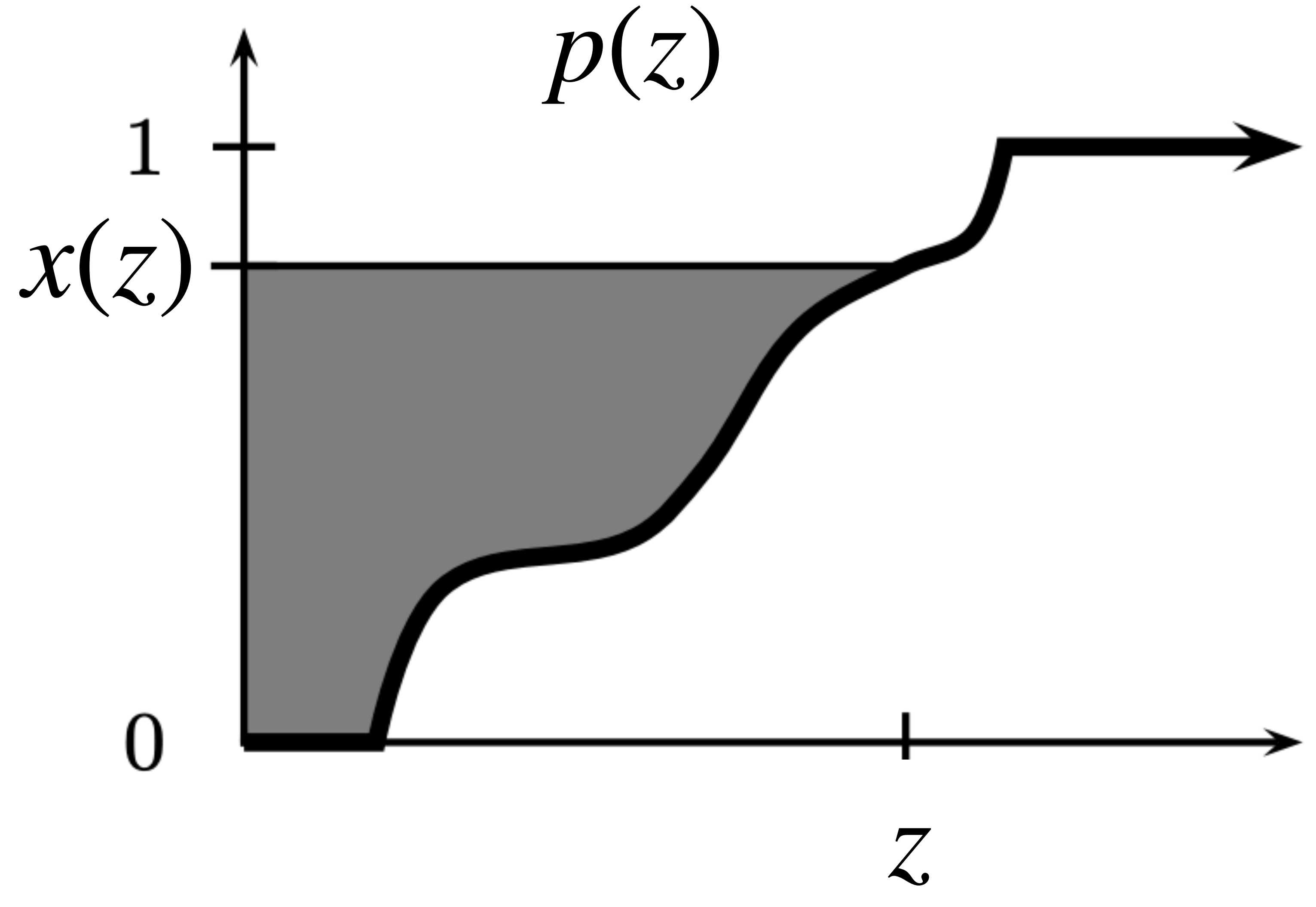
- We reuse the inequalities from part 2 of the proof:

$$z_2 \cdot x(z_2) - p(z_2) \geq z_2 \cdot x(z_1) - p(z_1) \text{ — (Ineq 1)}$$

$$z_1 \cdot x(z_1) - p(z_1) \geq z_1 \cdot x(z_2) - p(z_2) \text{ — (Ineq 2)}$$

- We can upper and lower bound $p(z_2) - p(z_1)$ using them as

$$z_2 \cdot (x(z_2) - x(z_1)) \geq p(z_2) - p(z_1) \geq z_1 \cdot (x(z_2) - x(z_1))$$



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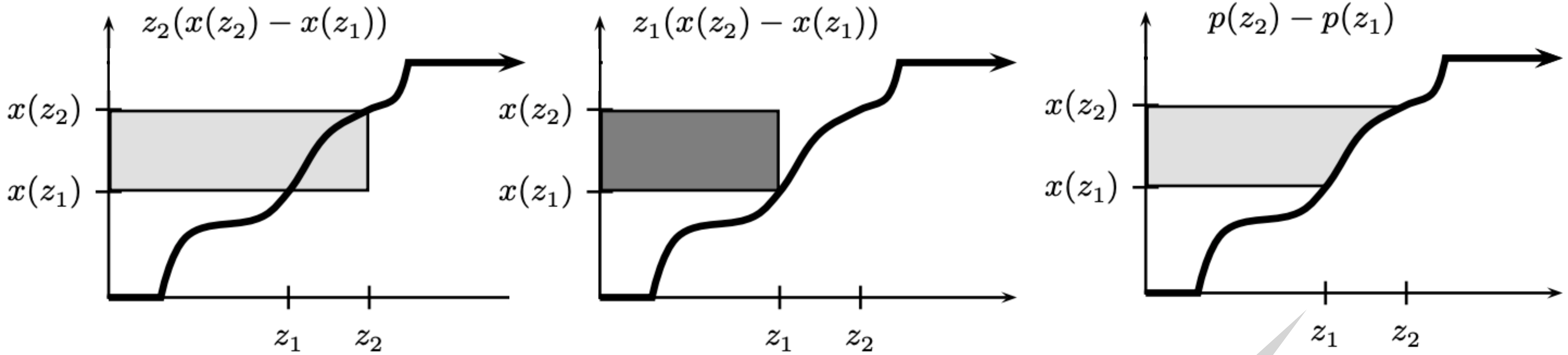
Myerson’s Lemma: Proof Part 2

$$z_2 \cdot (x(z_2) - x(z_1)) \geq p(z_2) - p(z_1) \geq z_1 \cdot (x(z_1) - x(z_1))$$



(a) Payment upper bound

(b) Payment lower bound

To finish,
set $z_1 = 0$ and $z_2 = v$

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Myerson Payment Rule

This payment rule is given by the following expression for all i:

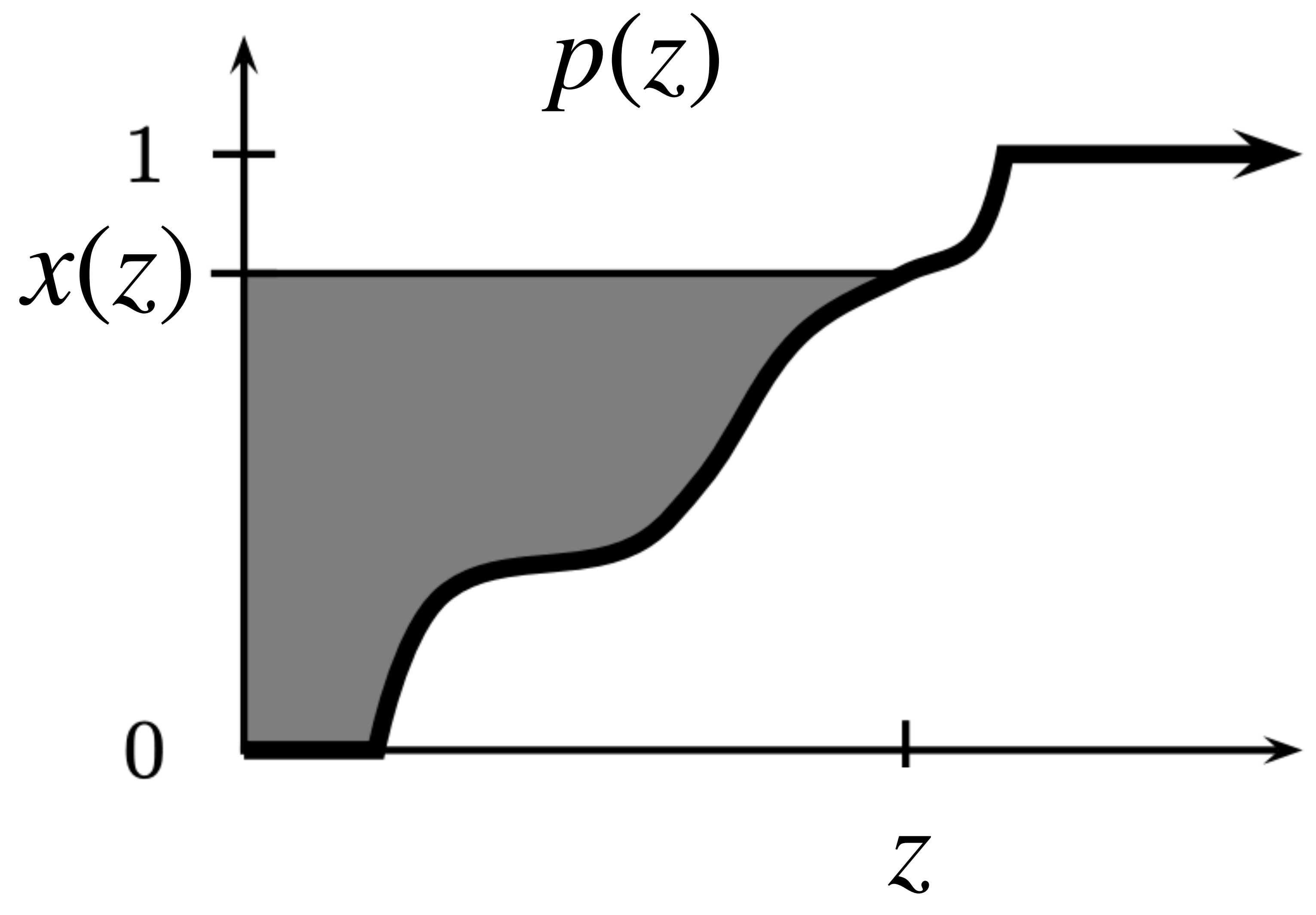
$$p_i(z, \mathbf{b_{-i}}) = z \cdot x_i(z, \mathbf{b_{-i}}) - \int_0^z x_i(z, \mathbf{b_{-i}})\ dz$$

where player i bids z.

Keeping $\mathbf{b}_{-i}$ fixed, we can simplify:

$$p_i(z) = z \cdot x_i(z) - \int_0^z x_i(z)\ dz$$

Assuming that $p_i(0) = 0$.



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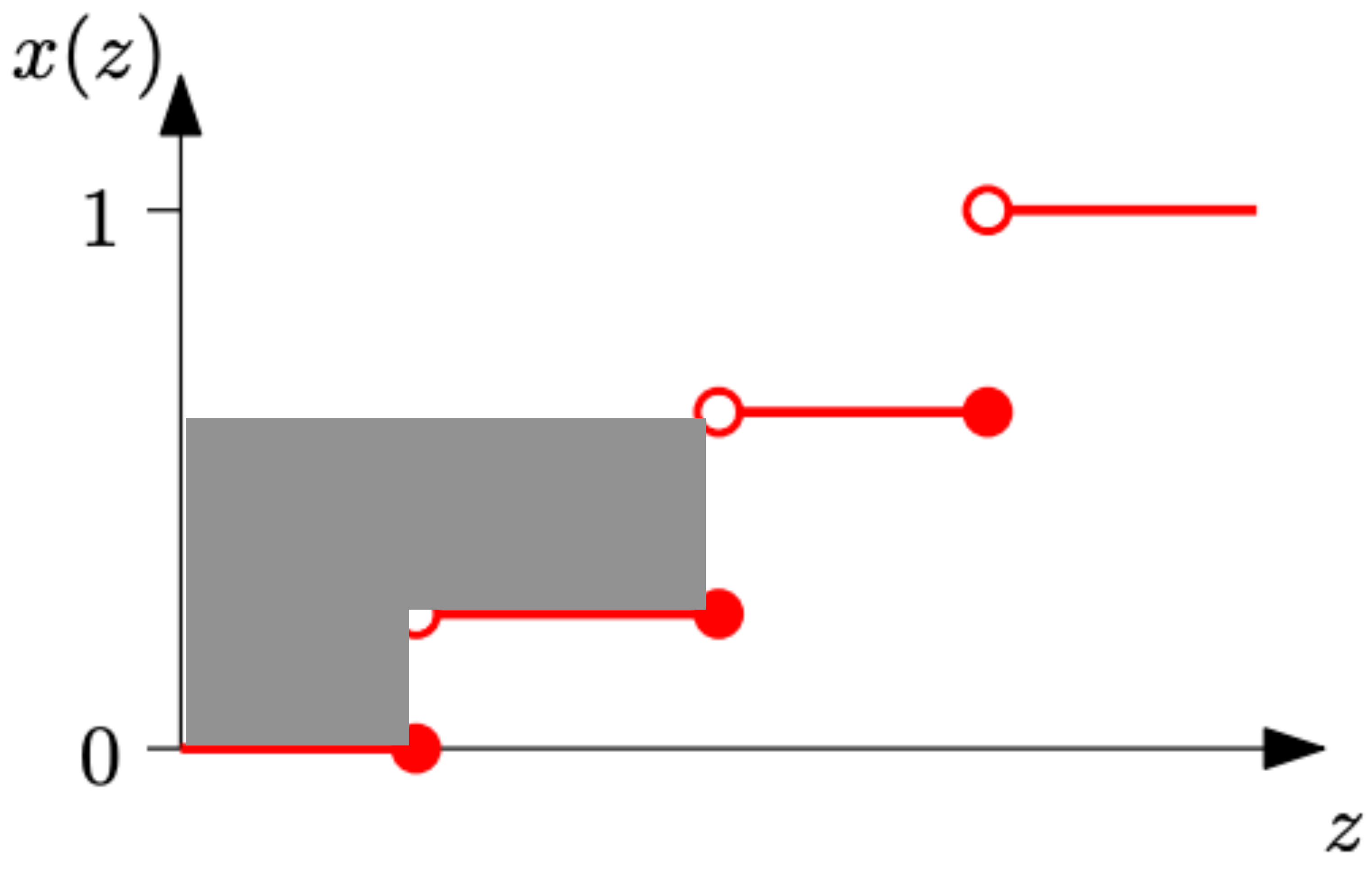
Myerson Payment Rule

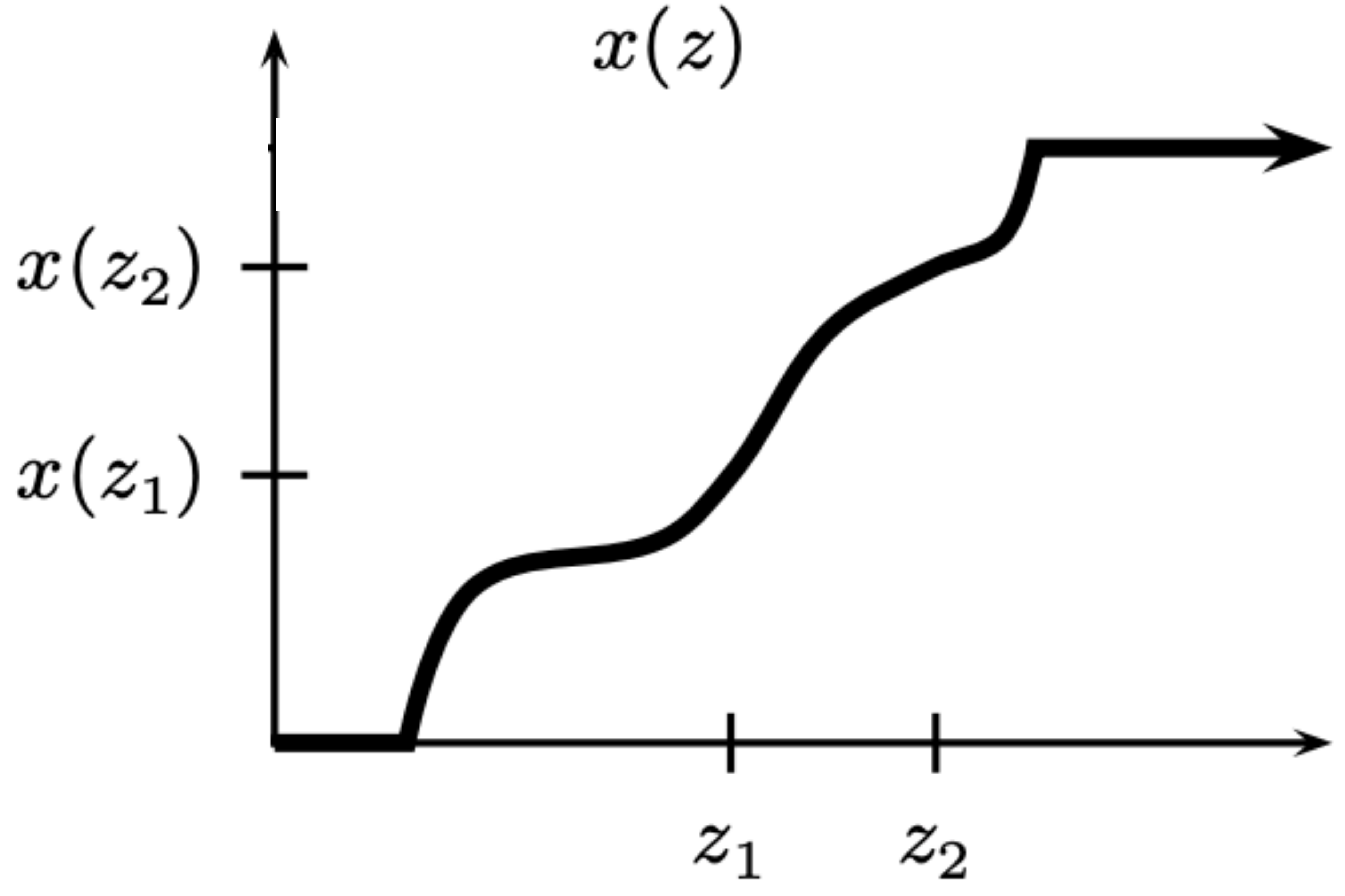
- Suppose $\mathbf{x}$ is piecewise constant
- If there are ℓ points at which the allocation "jumps" before bid z, the payment at bid z

$$p_i(z) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i \text{ at } z_j]$$



Myerson's Lemma: Proof Part 3

- **Part 3.** If the allocation $\mathbf{x}$ is monotone and the payment rule $\mathbf{p}$ is as given by the expression in the lemma then, $(\mathbf{x}, \mathbf{p})$ is dominant strategyproof
- Suppose Alice's value is $v = z_2$, and she **underbids** $v^\dagger = z_1$
- We will compare utilities $v \cdot x(v) - p(v)$ and $v \cdot x(v^\dagger) - p(v^\dagger)$



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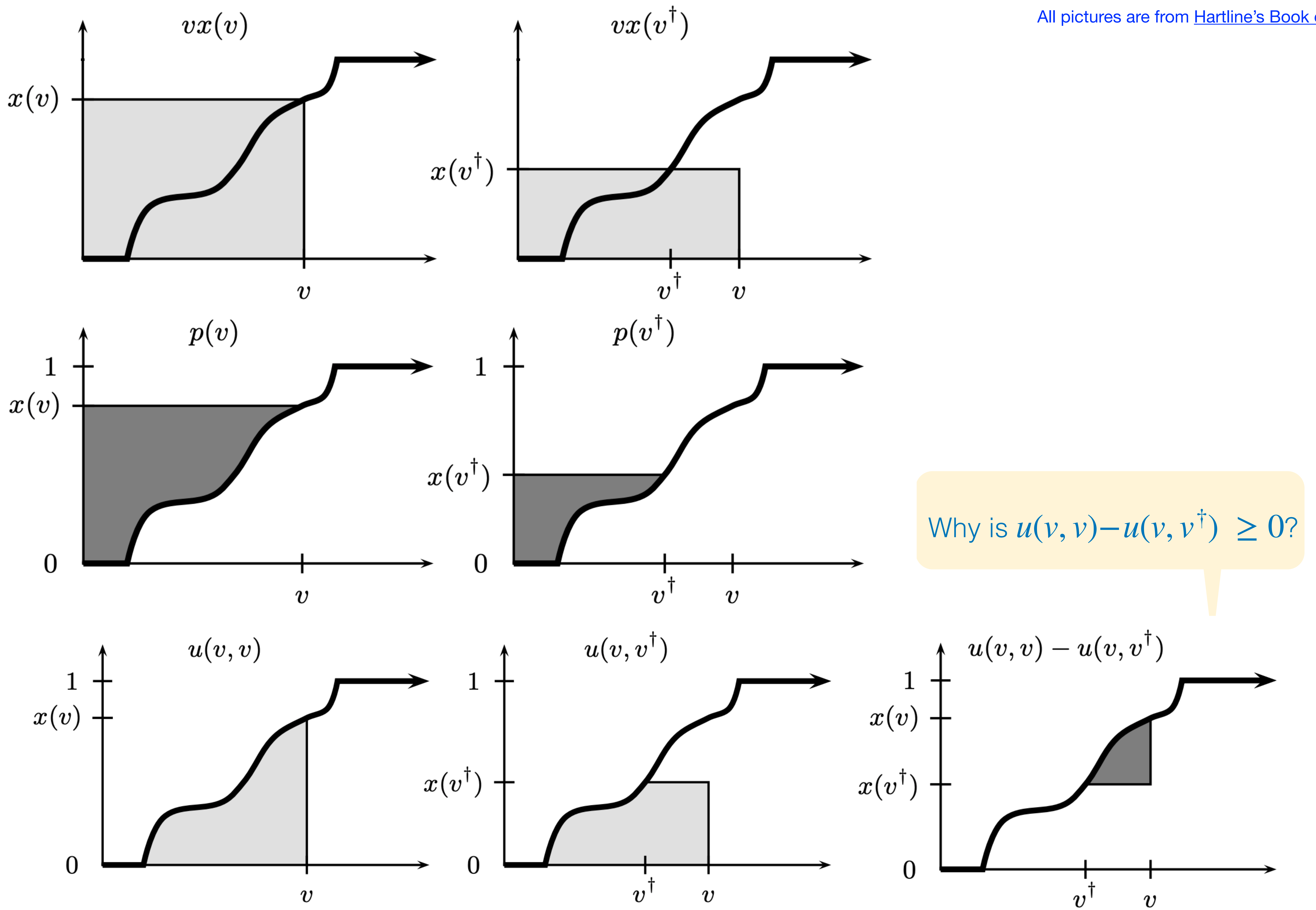
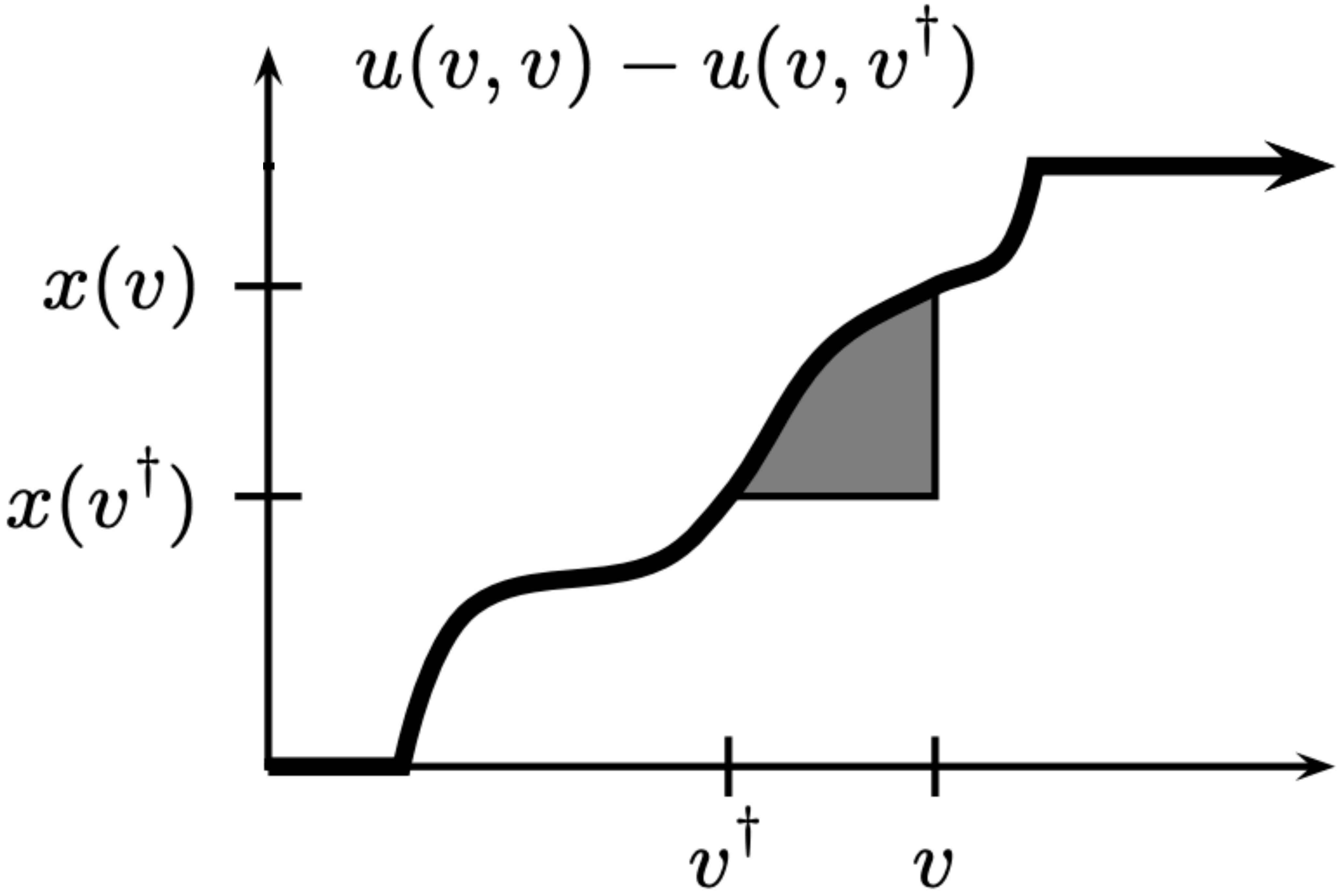


Myerson's Lemma: Proof Part 3

- $u(v, v) - u(v, v^\dagger) \geq 0$ because $\mathbf{x}$ is monotone non-decreasing
  - Since $v > v^\dagger$, we have $x(v) \geq x(v^\dagger)$
- A similar argument proves the other case: where $v^\dagger > v$



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Myerson's Lemma Complete

- Fix an single-parameter domain. We state the result for the continuous case.

(a) An allocation rule $\mathbf{x}$ can be made dominant-strategy incentive compatible if and only if $\mathbf{x}$ is monotone (non decreasing).

(b) If $\mathbf{x}$ is monotone, there is a **unique** payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is dominant strategyproof.   This payment rule is given by the following expression for all i:

$$p_i(z, \mathbf{b_{-i}}) = z \cdot x_i(z,\ \mathbf{b_{-i}}) - \int_0^z x_i(z,\ \mathbf{b_{-i}})\ dz$$

where player i bids z. Keeping $\mathbf{b}_{-i}$ fixed, we can simplify:

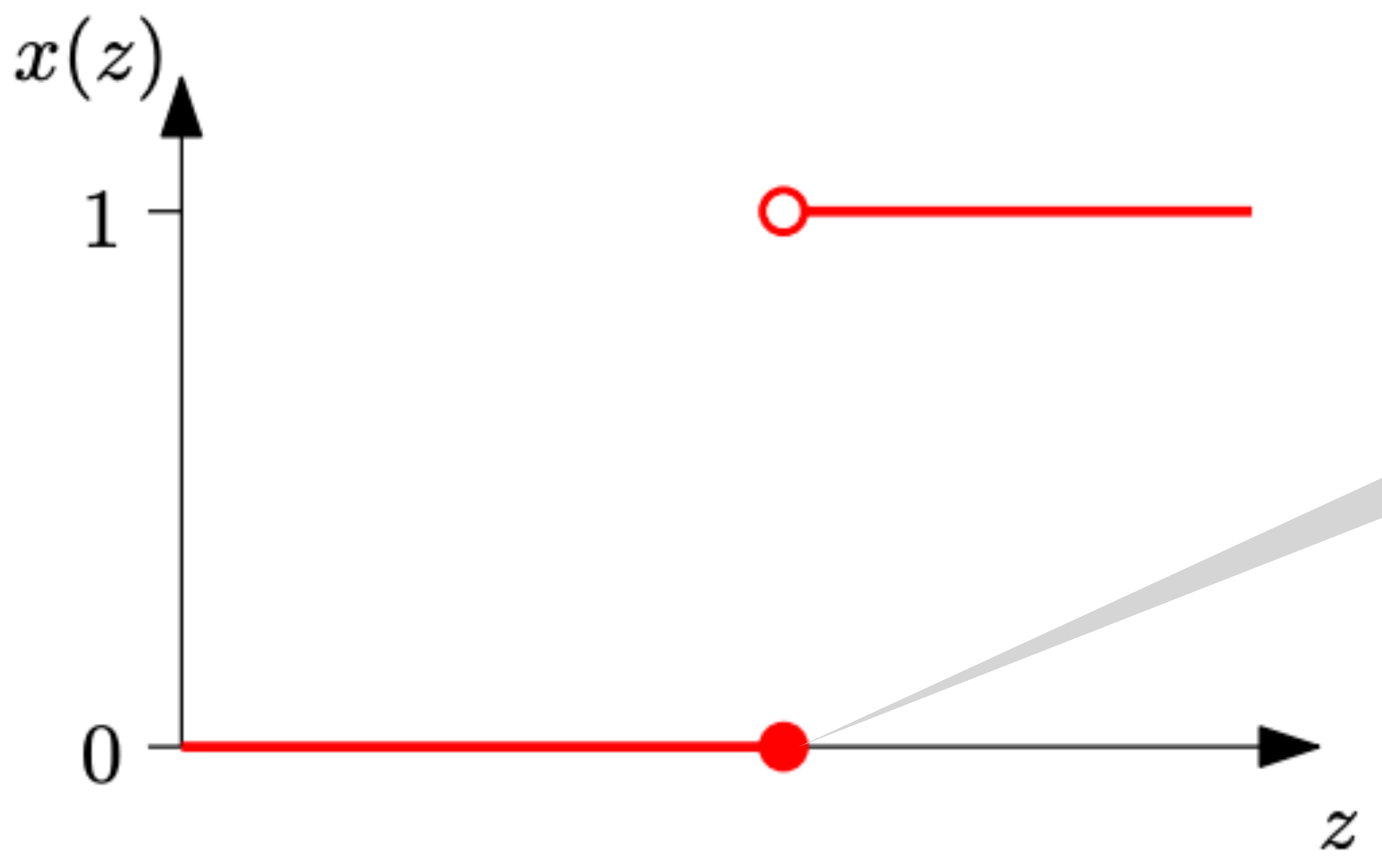
$$p_i(z) = z \cdot x_i(z) - \int_0^z x_i(z)\ dz$$

Assuming that $p_i(0) = 0$.

Examples

Single-Item Auction

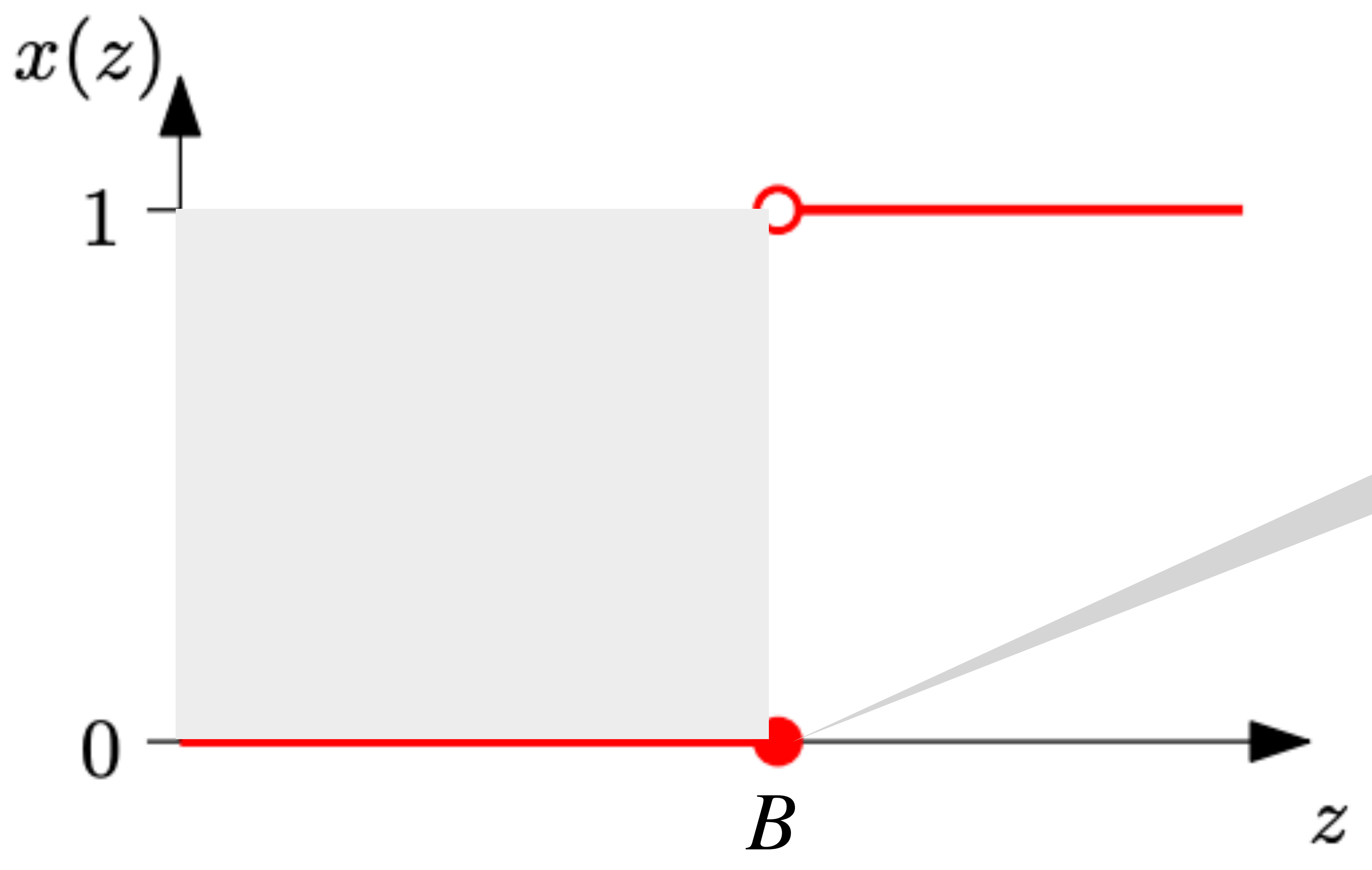
- Let's apply Myerson's lemma to a single item auction that allocates the item to highest bidder
- This allocation rule is monotone: in fact a $0/1$ monotone curve
- Fixing $\mathbf{b}_{-i}$, we can plot bidder i allocation wrt to bid:



This jump occurs exactly at $B = \max \mathbf{b}_{-i}$, called **critical bid**

Single-Item Auction

- If $z < B$: payment is 0
- If $z \geq B$: payment is given by shaded region, that is, B
- We have recreated the Vickrey auction from Myerson's lemma
- Moreover, this payment scheme is the only way to make the allocation rule (giving to highest bidder) truthful!

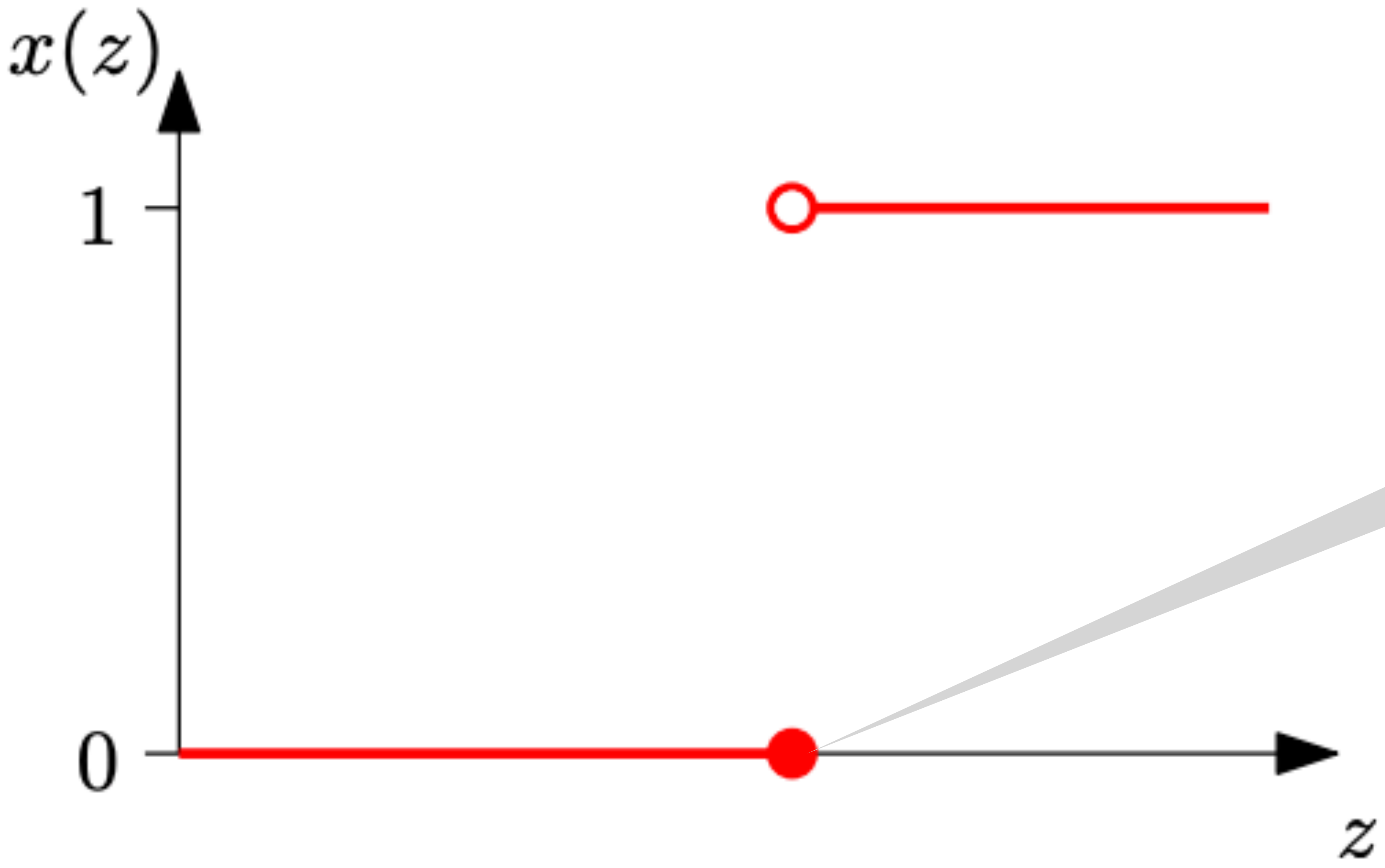


This jump occurs exactly at $B = \max \mathbf{b}_{-i}$, called **critical bid**

Any 0/1 Allocation Mechanism

- In a single-parameter environment, let X be any $0/1$ feasible allocation (each player either wins $x_i = 0$ or loses $x_i = 1$)
  - Example: auctioning k units of the same item to n bidders
- In such auctions, what should the winners pay?

$$p(b_i, \mathbf{b}_{-i}) = \begin{cases} 0 & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 0 \\ b_i^*(\mathbf{b}_{-i}) & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 1 \end{cases}$$



Critical bid: $b_i^*(\mathbf{b}_{-i})$ lowest bid at which i's allocation goes from 0 to 1

Any 0/1 Allocation Mechanism

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  - Example: auctioning k units of the same item to n bidders
- In such auctions, what should the winners pay?
  - $(k+1)^{\text{st}}$ highest bid

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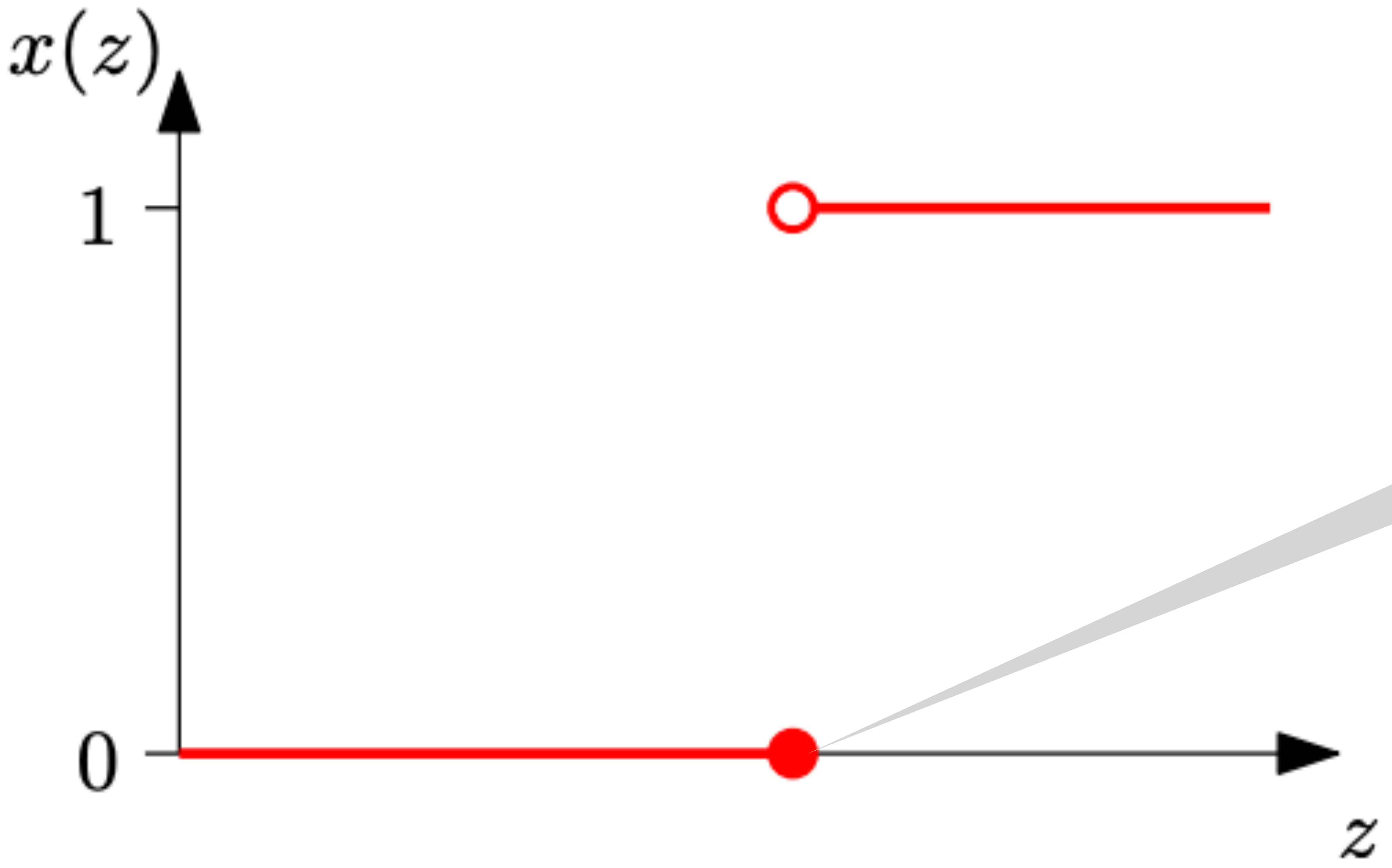
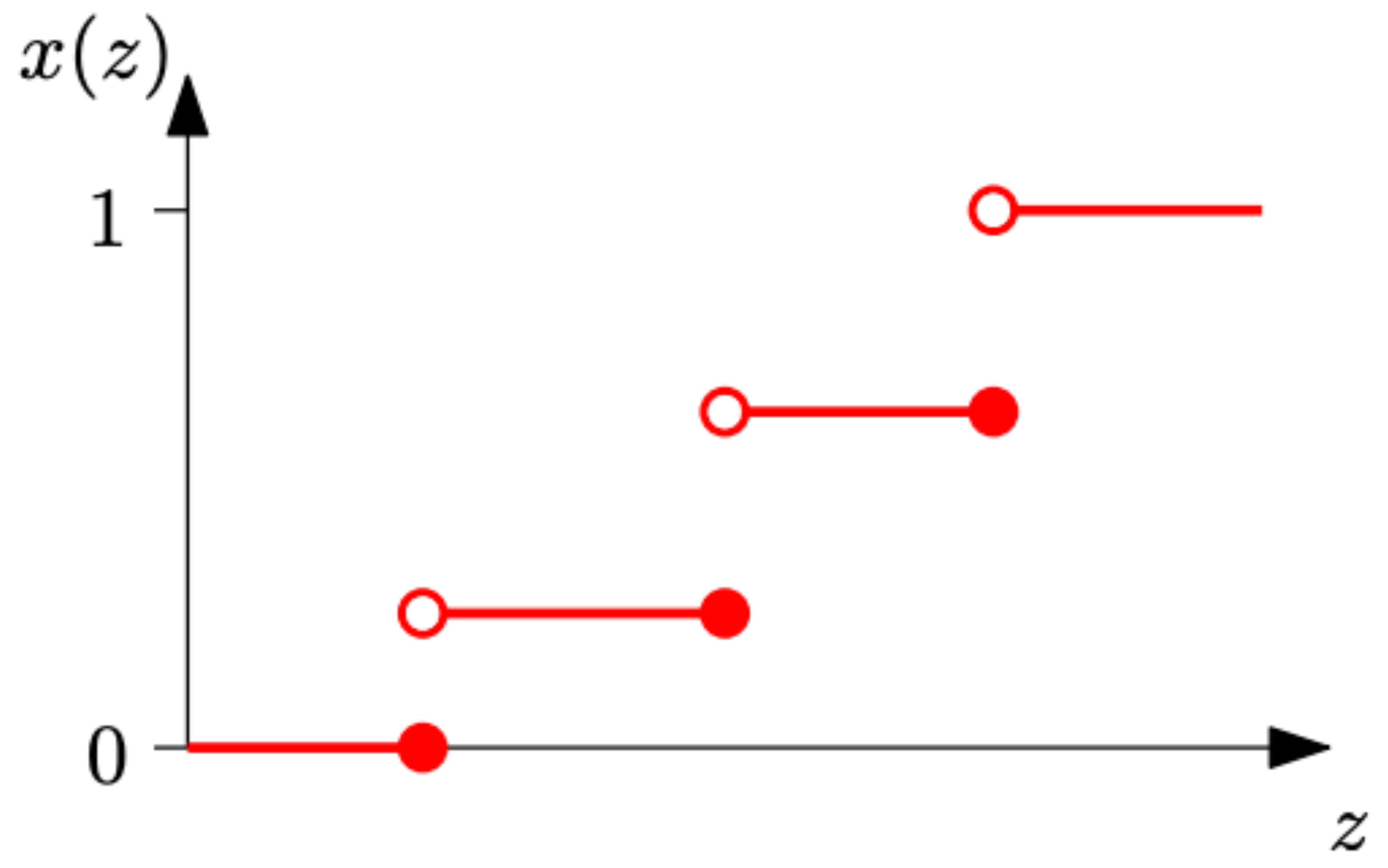


Critical bid: $b_i^*(\mathbf{b}_{-i})$ lowest bid at which i's allocation goes from 0 to 1

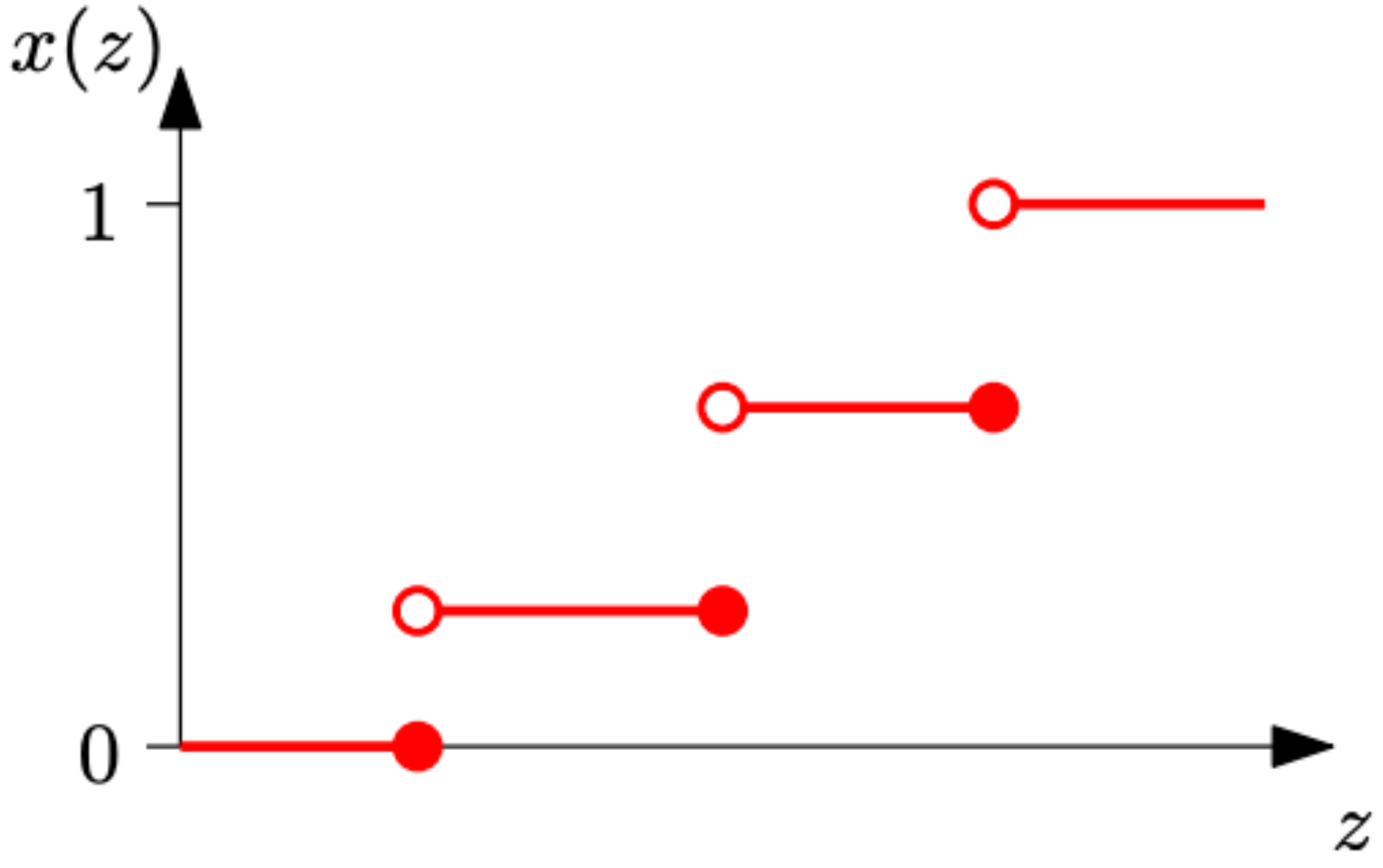
Sponsored-Search Auctions

- Sort bids $b_1 \geq b_2 \geq \ldots \geq b_n$ (reorder bidders in this order)
- Assign slot 1 to bidder 1, slot 2 to bidder 2, etc.
- That is, CTR α_j of slot j gets assigned to bidder j
- What does the graph of such an allocation rule look like?
    - For intuition fix $\mathbf{b_{-i}}$ and think of yourself as bidder 1 slowing raising your value from 0



Sponsored-Search

- If you get no slot, you pay zero
- If you get last slot, you pay the "critical" bid that you beat out to get the slot (the bid of the person just below you in sorted order)
- If you get a lower slot j better than k, what do you pay?
    - **Exercise:** come with the expression for the payment p_j of bidder who wins slot j using Myerson's rule?
- We will come back to this!



Sponsored-Search

- If you get no slot, you pay zero
- If you get last slot, you pay the "critical" bid that you beat out to get the slot (the bid of the person just below you in sorted order)
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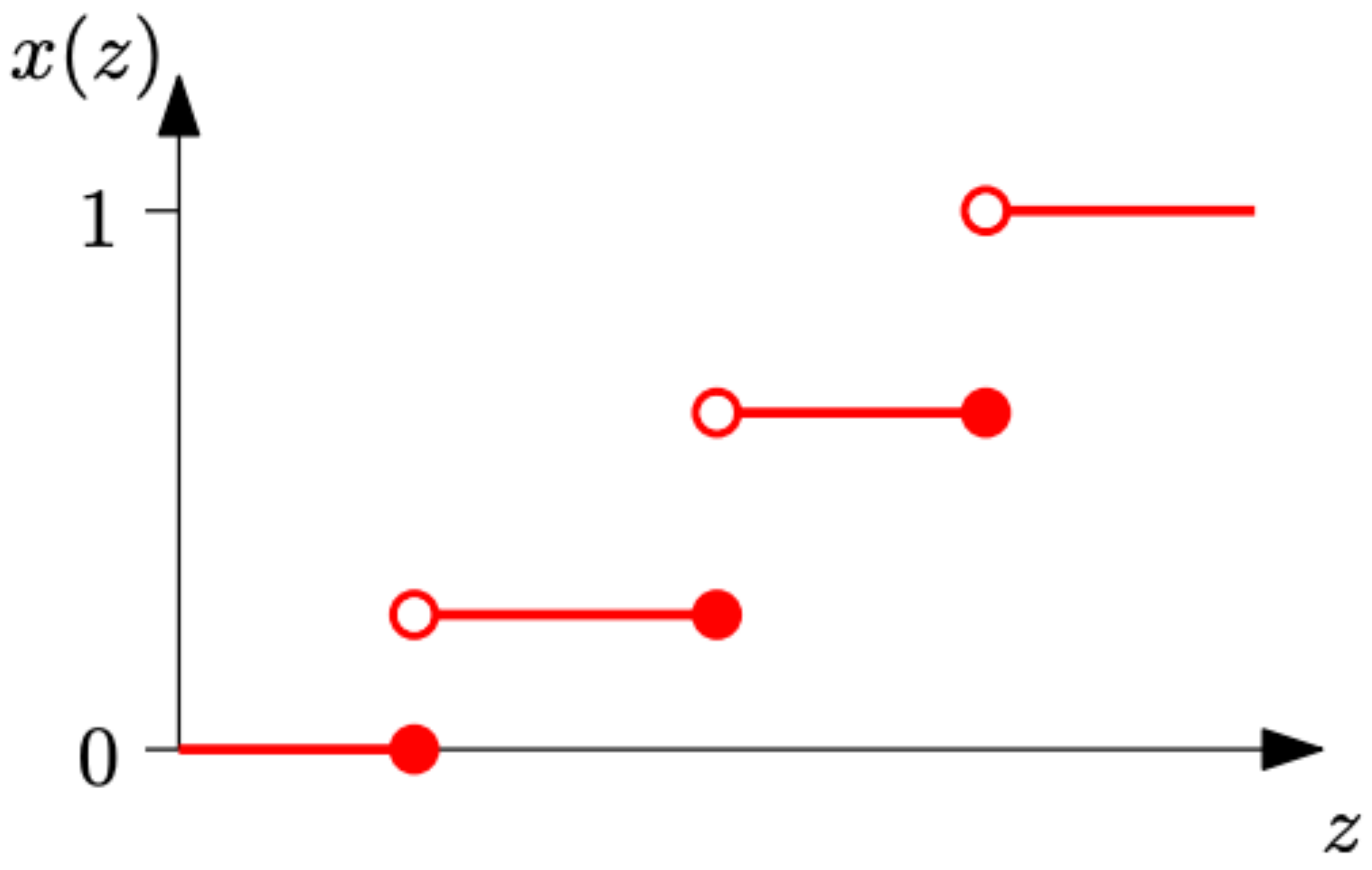

Sponsored-Search

- Myerson's payment rule of monotone piece-wise constant allocation
- If there are ℓ points at which the allocation "jumps" before bid z, the payment at bid z

$$p_i(z) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i \text{ at } z_j]$$

Sponsored-Search

- Using Myerson's lemma, the ith highest bidder (who wins slot i) should pay:
    - $p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \cdot (\alpha_j - \alpha_{j+1})$ where $\alpha_{k+1} = 0$
- The **"per click payment"** of bidder i who is in slot i is $\sum_{j=i}^{k} b_{j+1} \cdot \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}$
- Payments have a nice interpretation:
    - If you win, you pay a suitable convex combination of lower bids!

Question. Are sponsored-search auctions in real life based on our (Myerson's) theory?

Generalized Second Price Auctions

- By "historical accident," the sponsored search auctions in real life (called generalized-second price auction or GSP) are not DSIC
- In GSP, the **allocation rule is the same**
    - Allocate slots to highest bidders
- Payment rule: a bidder wins slot i pays the per-click bid of the winner of slot $i-1$ or 0 if $i=k$ (rather than a convex combination of lower bids)
    - Some say Google incorrectly implemented Myerson's lemma
    - Most likely reason is that the payment rule of GSP is much easier to explain to advertisers and share-holders
- Which one is better for revenue?
    - We'll explore this question next week