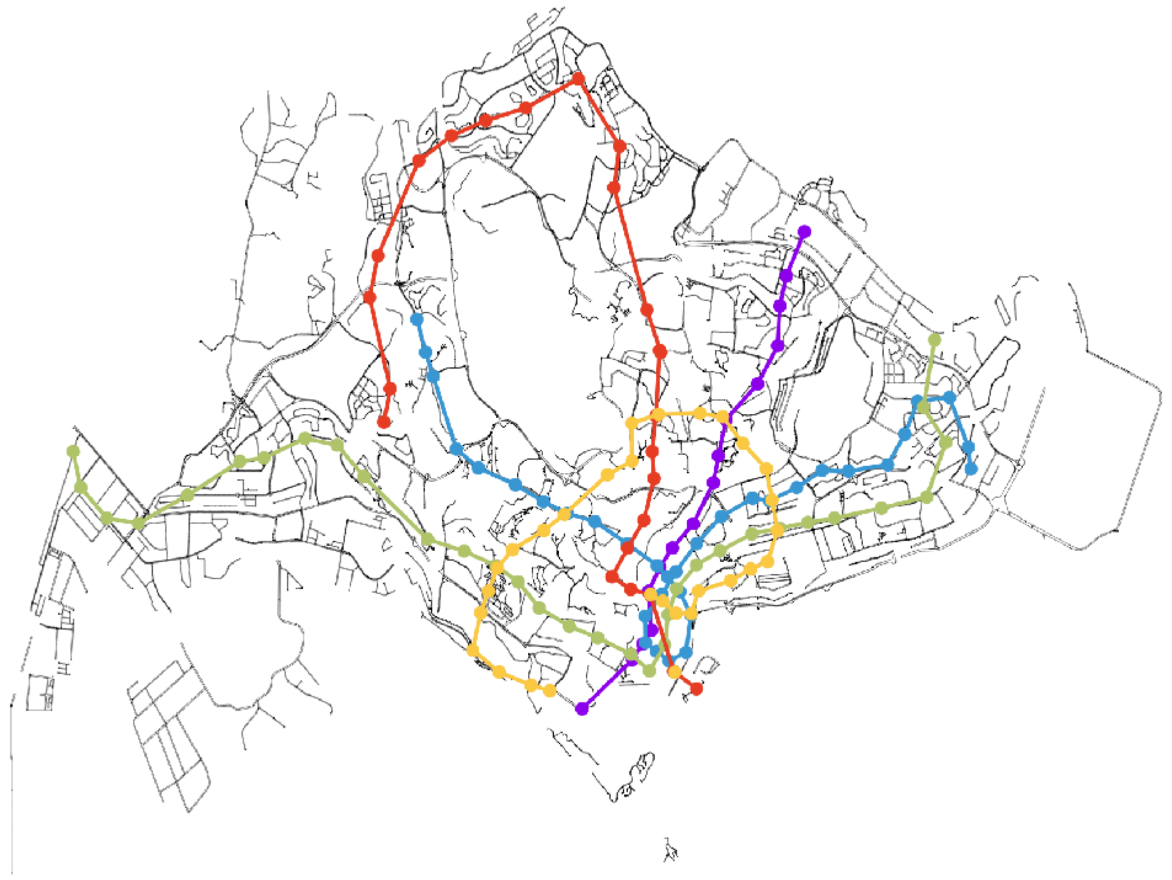
CSCI 357: Algorithmic Game Theory

Lecture 16: Competitive Equilibrium

Shikha Singh

Announcements and Logistics

- **Assignment # 4** due this Friday at noon
- HW 7: Practice question on Bayes Nash (no need to turn it in, solutions will be posted)
- **Midterm # 2** will be on April 29
  - Similar to Exam 1: closed book but can bring up to 5 pages of notes
  - **Extra** office hours this week: **Thursday 1-3 pm, Friday 9.30-11.30 am**
  - **Change of Room** to give more space to spread out: **Wach 015 (Downstairs)**

Questions?

Midterm 2 in this Room Downstairs

- Exam time: 1.10 - 2.25 pm, Wachenheim 015
- Please arrive 5 minutes early. Rohit will be proctoring the exam.

Questions?

Topics: Midterm 2

- **Mostly focused** on everything covered after **Midterm 1**
    - One-sided and two-sided matching/ stable matchings, voting, fair division and decentralized matching markets with money
    - Until today's lecture (Competitive equilibrium/Market-clearing prices), Until HW 7 and Assignment 4
    - No question about on paper eval # 3 but questions about top-trading cycle (topic of paper eval # 2) are fair game
- From markets with money: Bayes Nash and revenue equivalence will be included
- Need to remember and know how to use fundamental definitions (dominant-strategyproof, Nash equilibrium, Condorcet consistency, etc.)

Questions?

Review

- A **preferred-item graph** (given prices $\mathbf{p}$) where nodes are items and buyers and there is an edge between buyers and their preferred items (items that maximize utility)
- A selection of prices $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is **market-clearing** if:
  - **Condition 1.** There is a **matching** in the preferred-item graph such that **all buyers are matched**
  - **Condition 2.** If an item j is not matched to any buyer, then its price $p_j = 0$, in other words, every item with non-zero price $p_j > 0$ must get sold
- **First Welfare Theorem (Max-weight matching).** If $(M, \mathbf{p})$ is a competitive equilibrium, then M is a matching with maximum total value, that is,
  - $\sum_{i=1}^{n} v_{iM(i)} \geq \sum_{i=1}^{n} v_{iM'(i)}$ for every matching M'

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

$$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)} \quad \text{for every bidder } i$$

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$

M^* can assign each bidder at most one item

- Summing up the inequality in blue over all bidders

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

  $v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)}$ for every bidder i

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Reorganizing this inequality, we get that value of $M \geq$ value of M^* ∎

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

Competitive Eq: Existence

- **Theorem.** In every market where at most one good is assigned to each buyer, there is at least one competitive equilibrium.
- **Corollary.** Market-clearing prices are guaranteed to exist.
- We prove this constructively through an mechanism that shows how such prices might emerge organically in a market
- Intuition idea behind our **"ascending-price auction"**
  - If a set of k items is preferred by more than k buyers at its current price, then the prices of these items should rise
  - Keep identifying such "constricted sets" and increasing prices until the market clears

Ascending-Price Mechanism

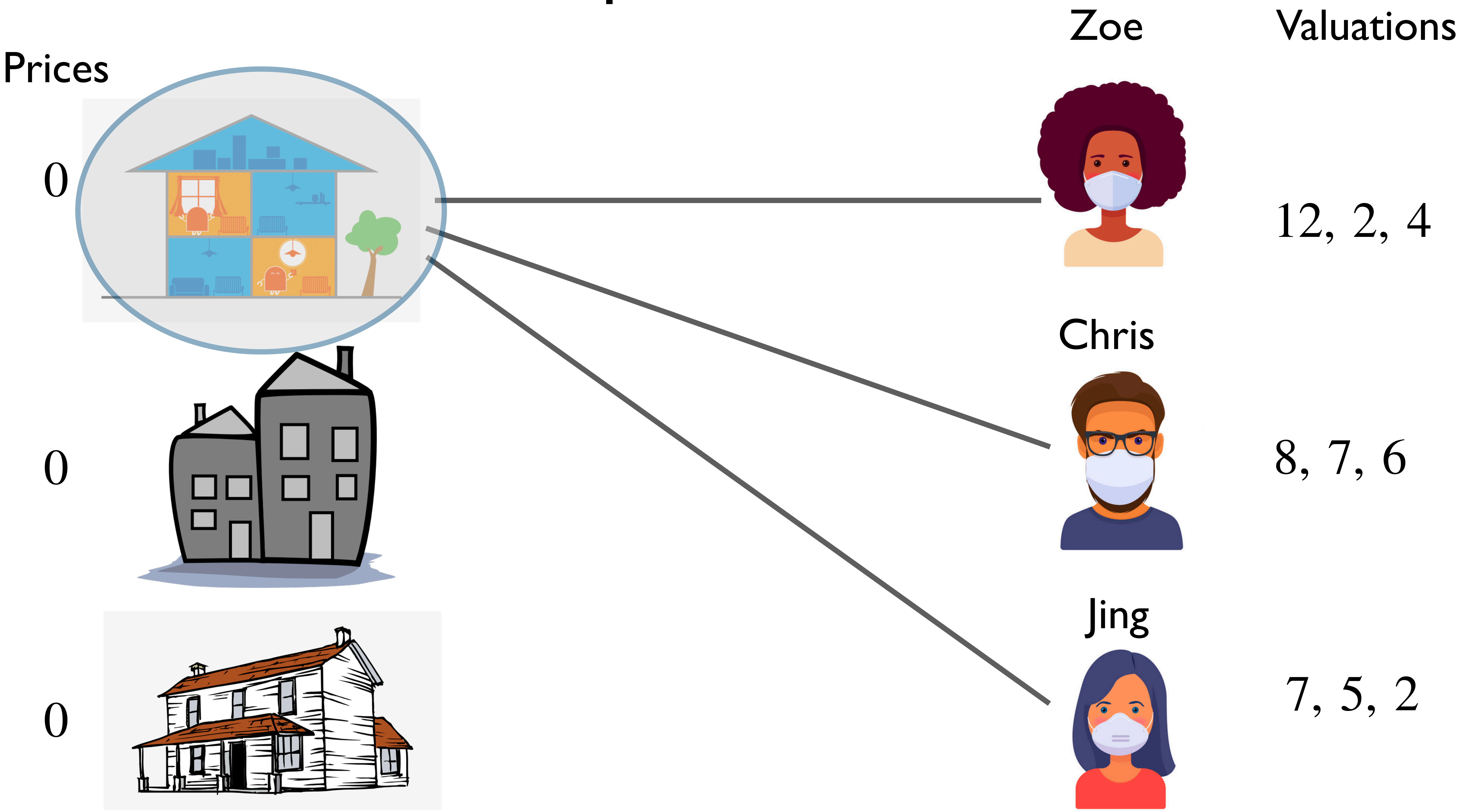
- Start with prices of all items $p_j = 0$
- Assume all valuations are integers $v_{ji} \in \mathbb{Z}$ (simplifying assumption)
- **Step 1.** Check if the current prices are market clearing, if so we are done
  - build the preferred graph, check if there is a buyer-perfect matching
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1, \ldots, n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**

Single Item Case

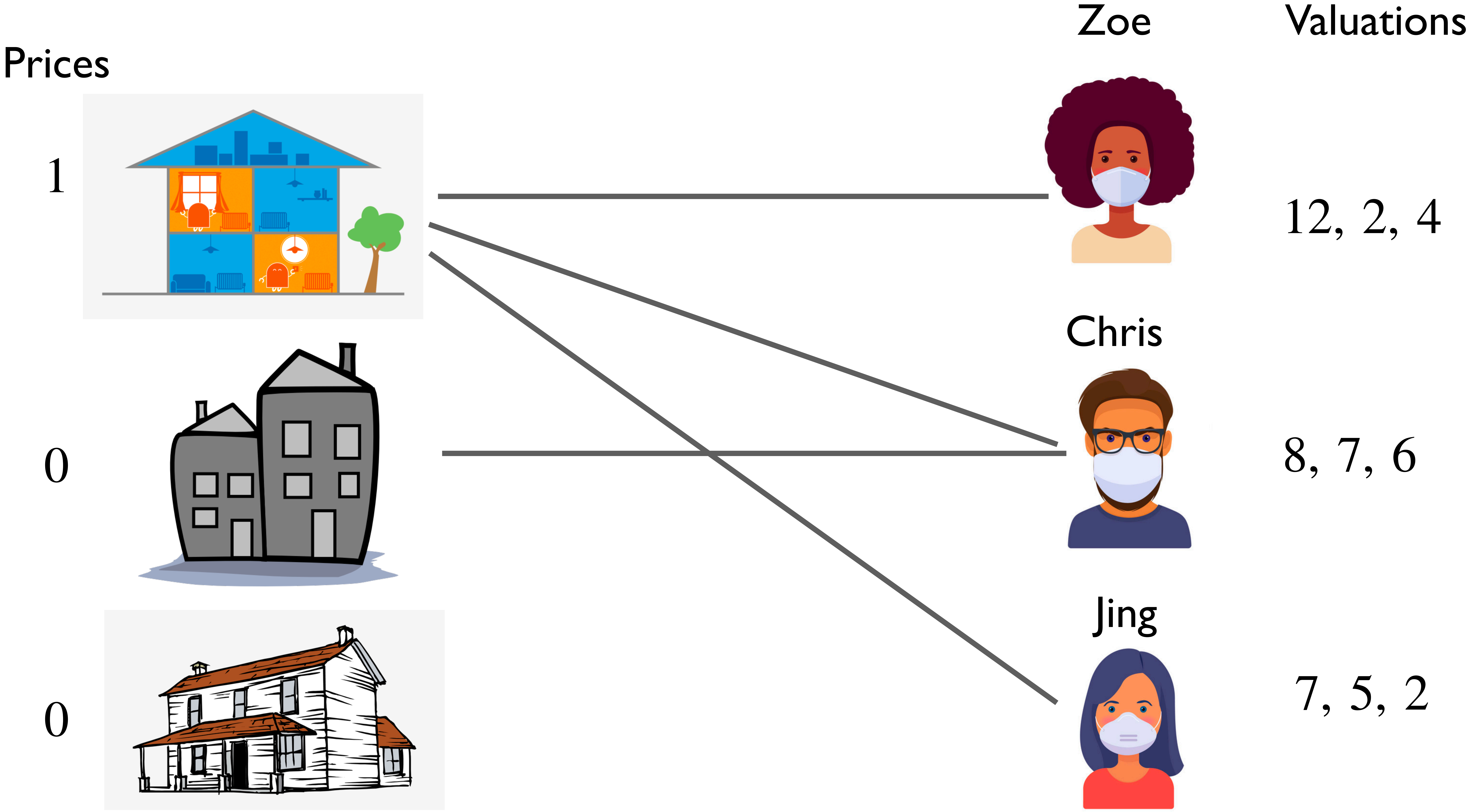
- A single item (labelled 1) for which each buyer has a value $v_i > 0$
- Add $n - 1$ dummy items $(2, \ldots, n)$ that everyone values at 0
- At the beginning preferred-item graph has edges from each buyer to item 1
- Thus, $\{1\}$ is our minimal constricted set
- We need to keep raises the price of item 1 until all except one buyer has a preferred edge to at least one item in $\{2, 3, \ldots, n\}$
- At what price does this happen?
  - Exactly when $p_1 =$ second-highest valuation
  - The person with the highest valuation is matched to item 1

We have recreated the **second-price auction** outcome!

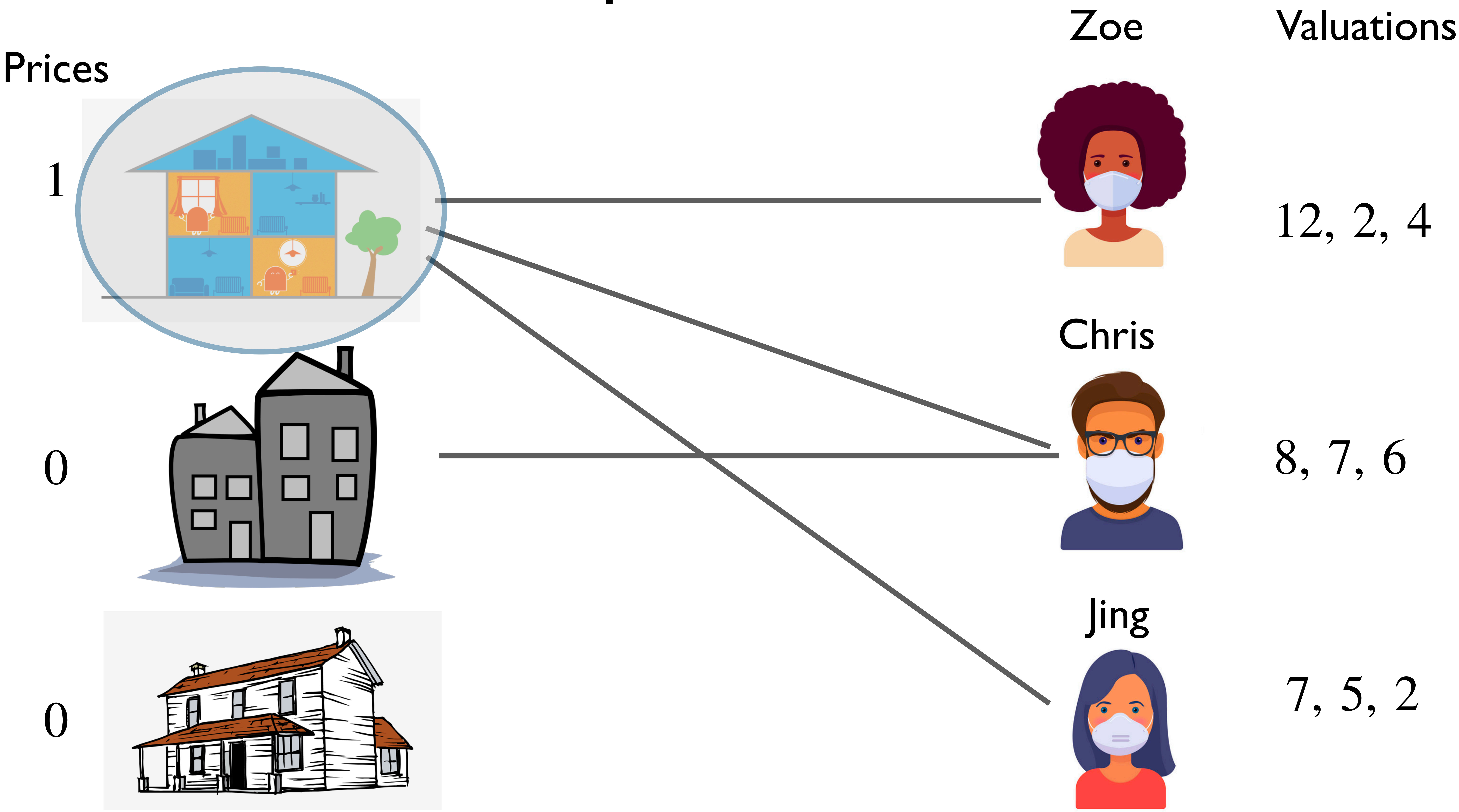
Preferred-Item Graph



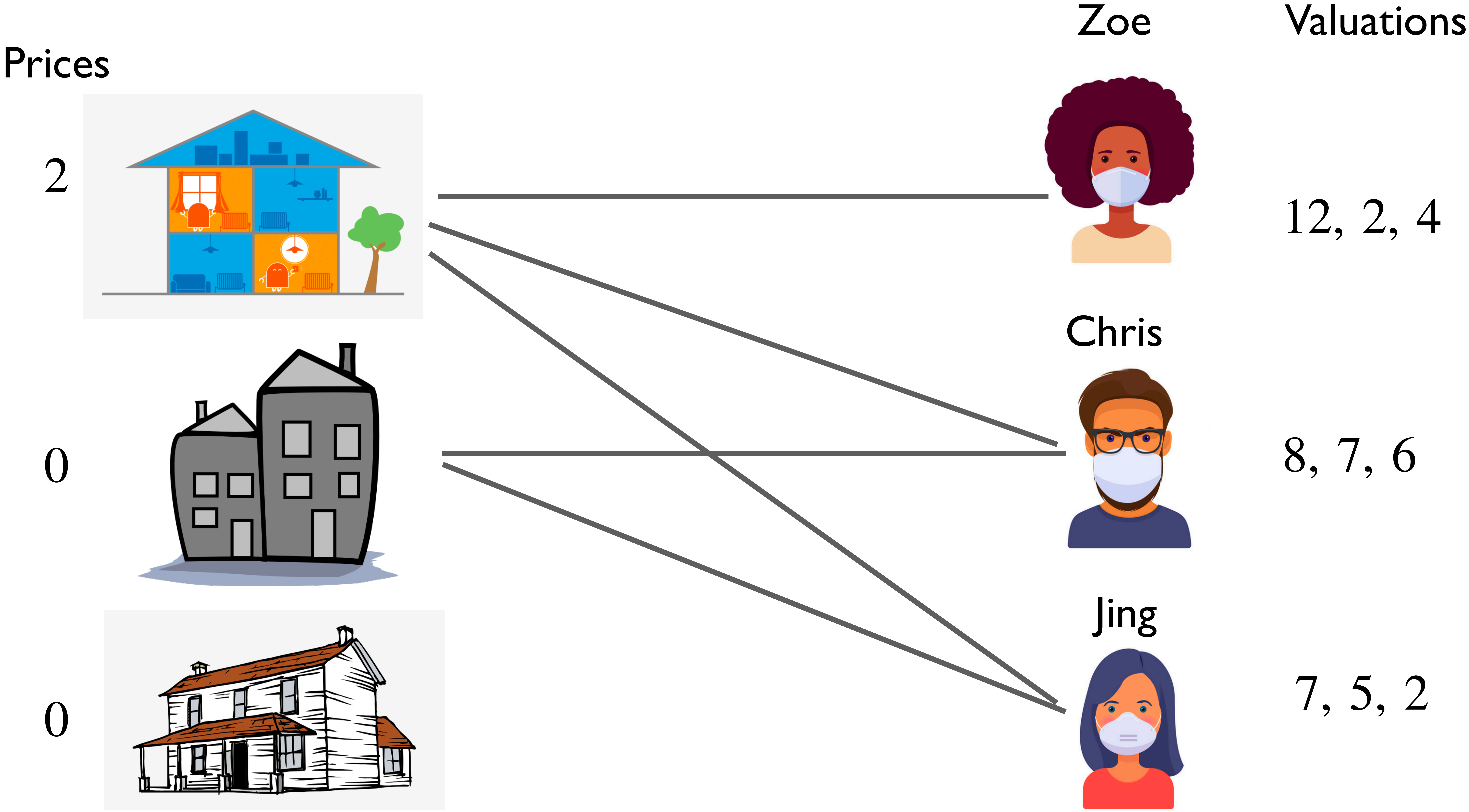
Preferred-Item Graph



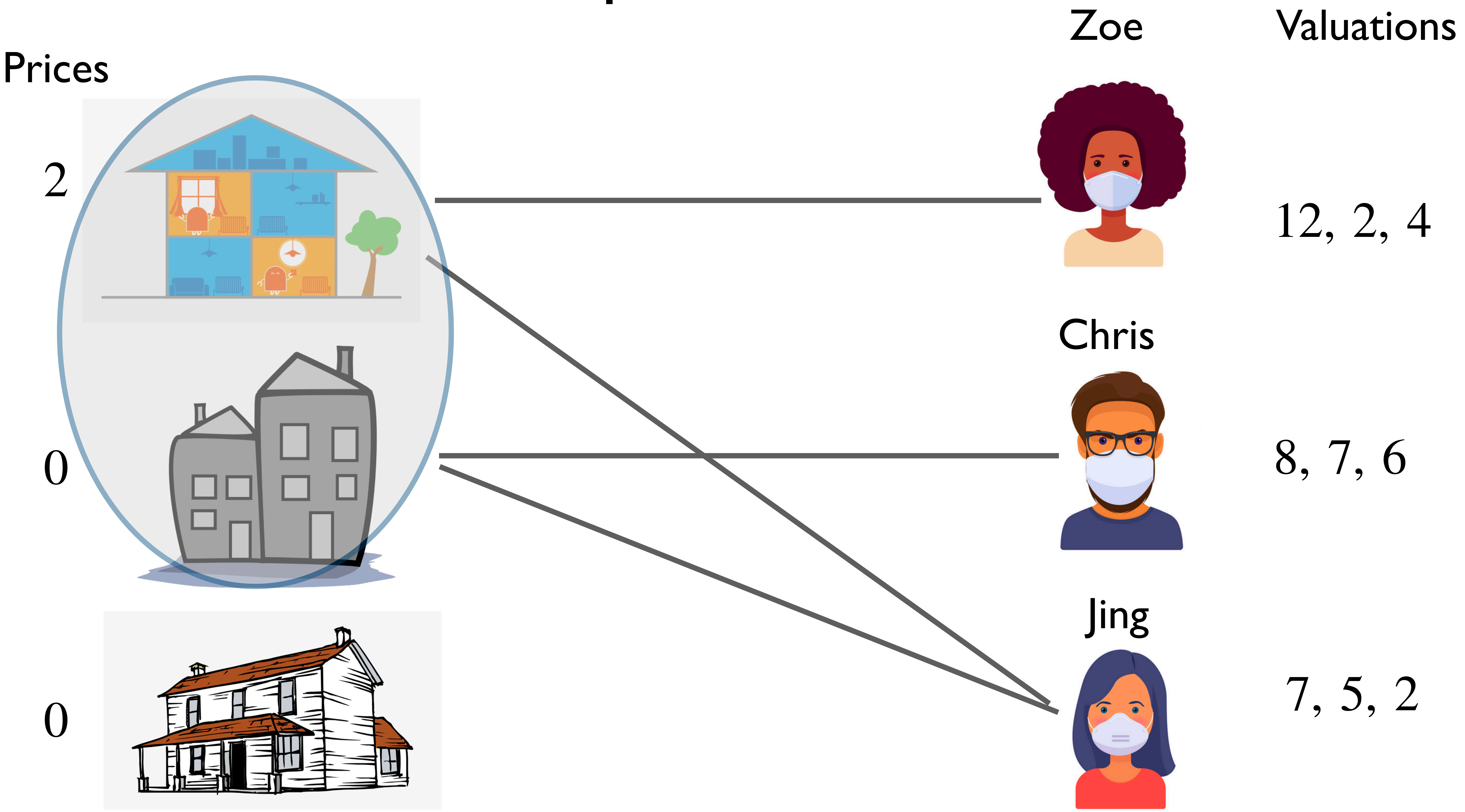
Preferred-Item Graph



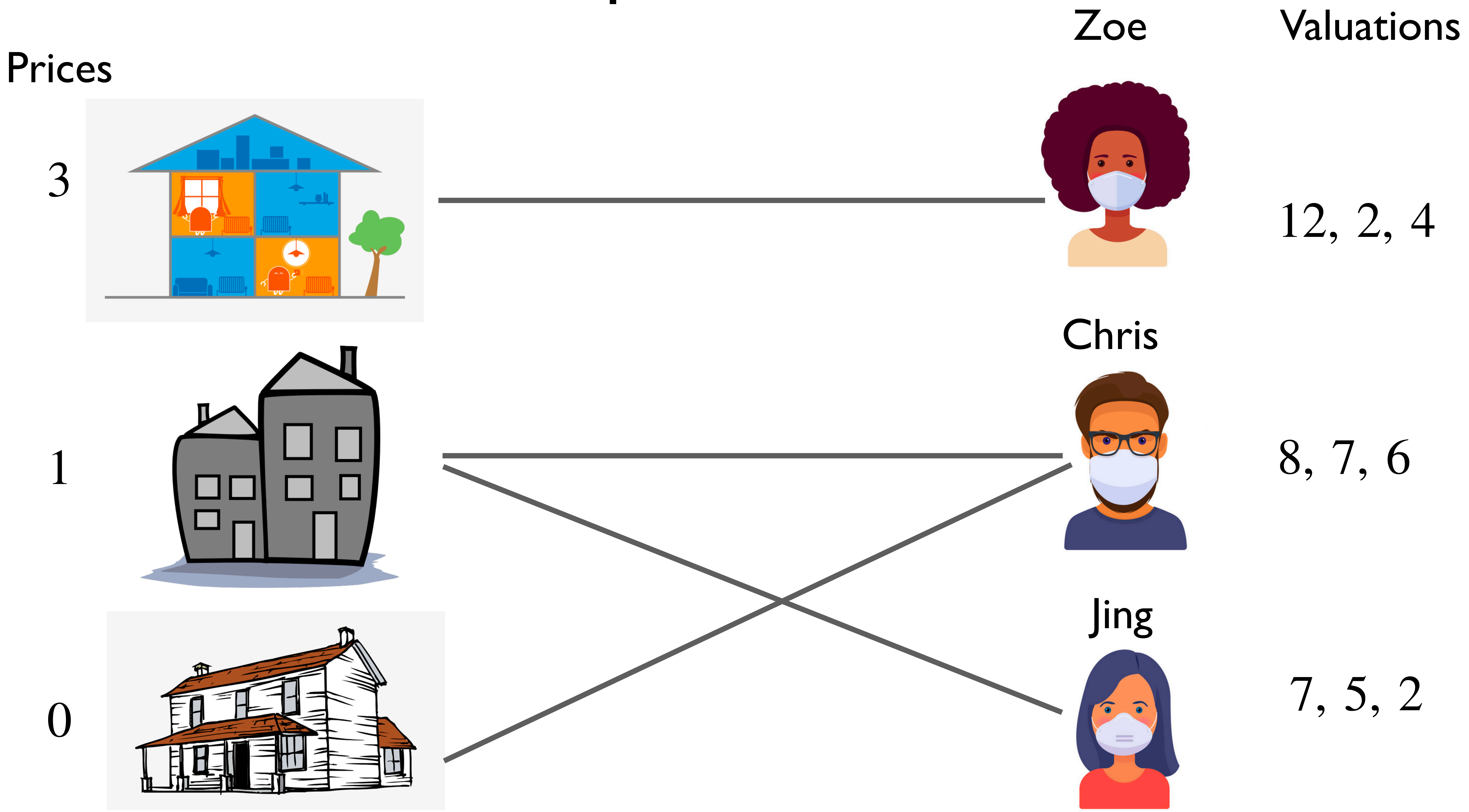
Preferred-Item Graph



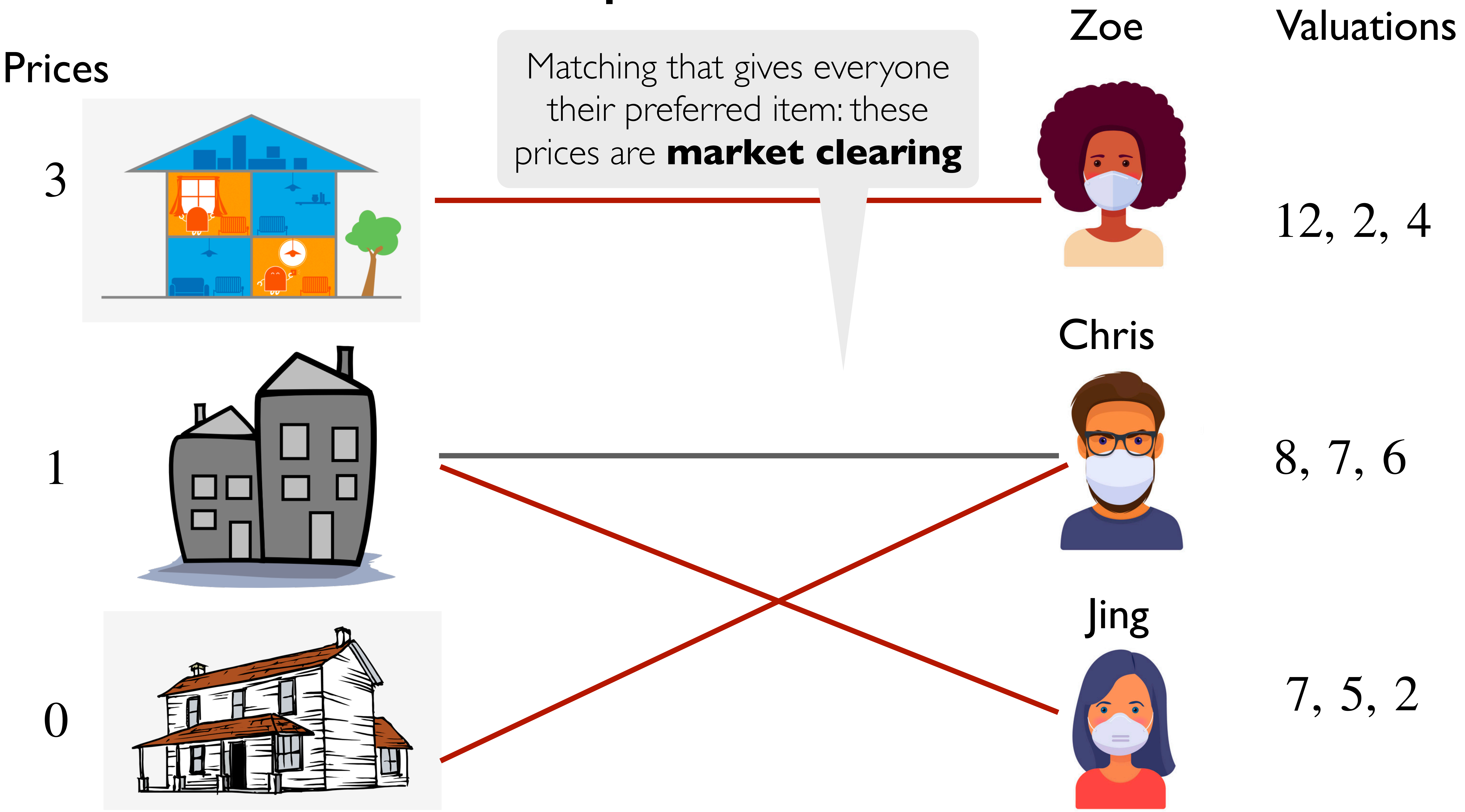
Preferred-Item Graph



Preferred-Item Graph



Preferred-Item Graph



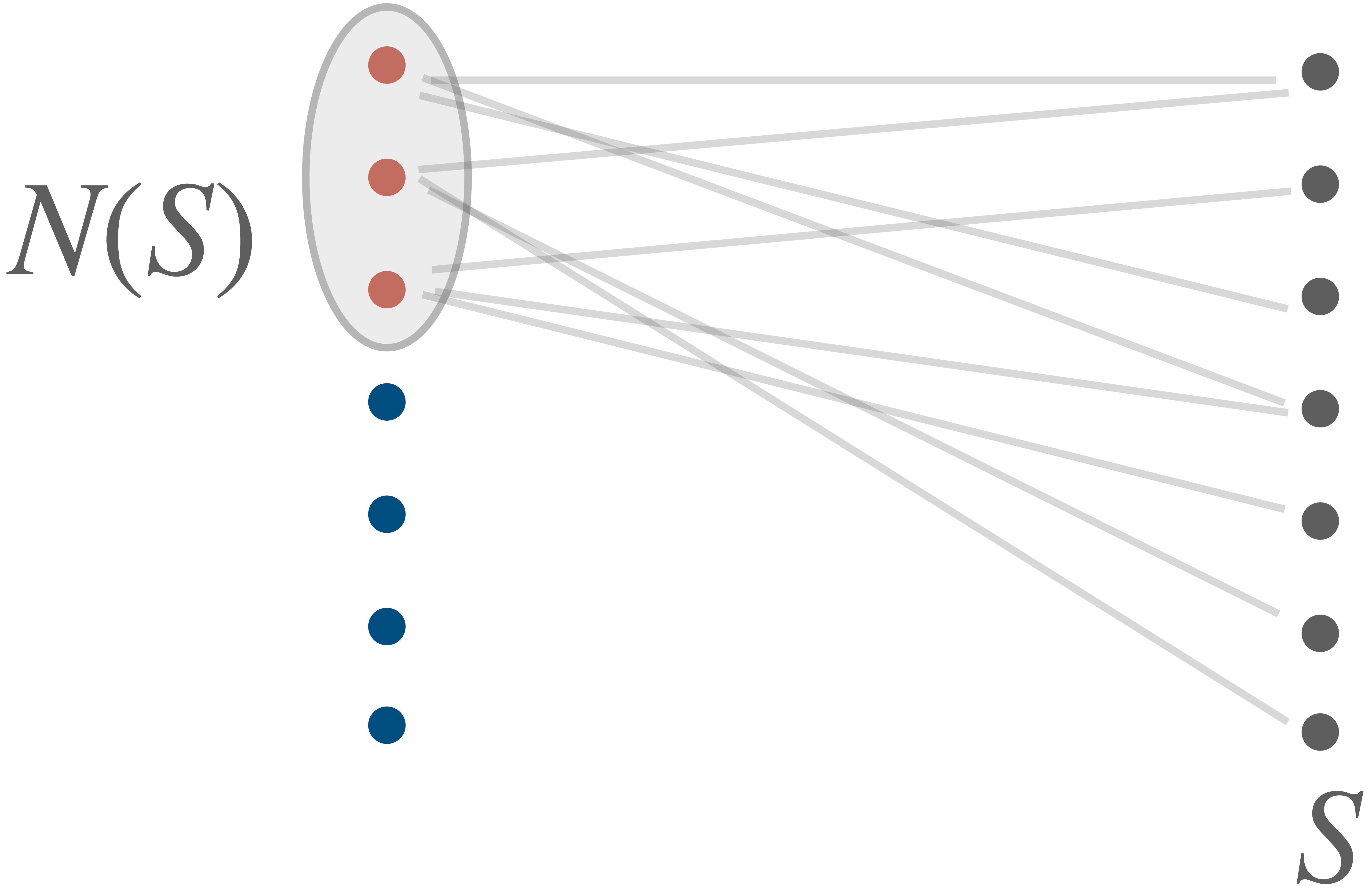
Ascending-Price Algorithm

- Start with prices of all items $p_j = 0$, assume all valuations $v_{ji} \in \mathbb{Z}$
- **Step 1.** Check if there is a **buyer-perfect matching** in preferred item graph
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1,\ldots,n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**
- **Invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$

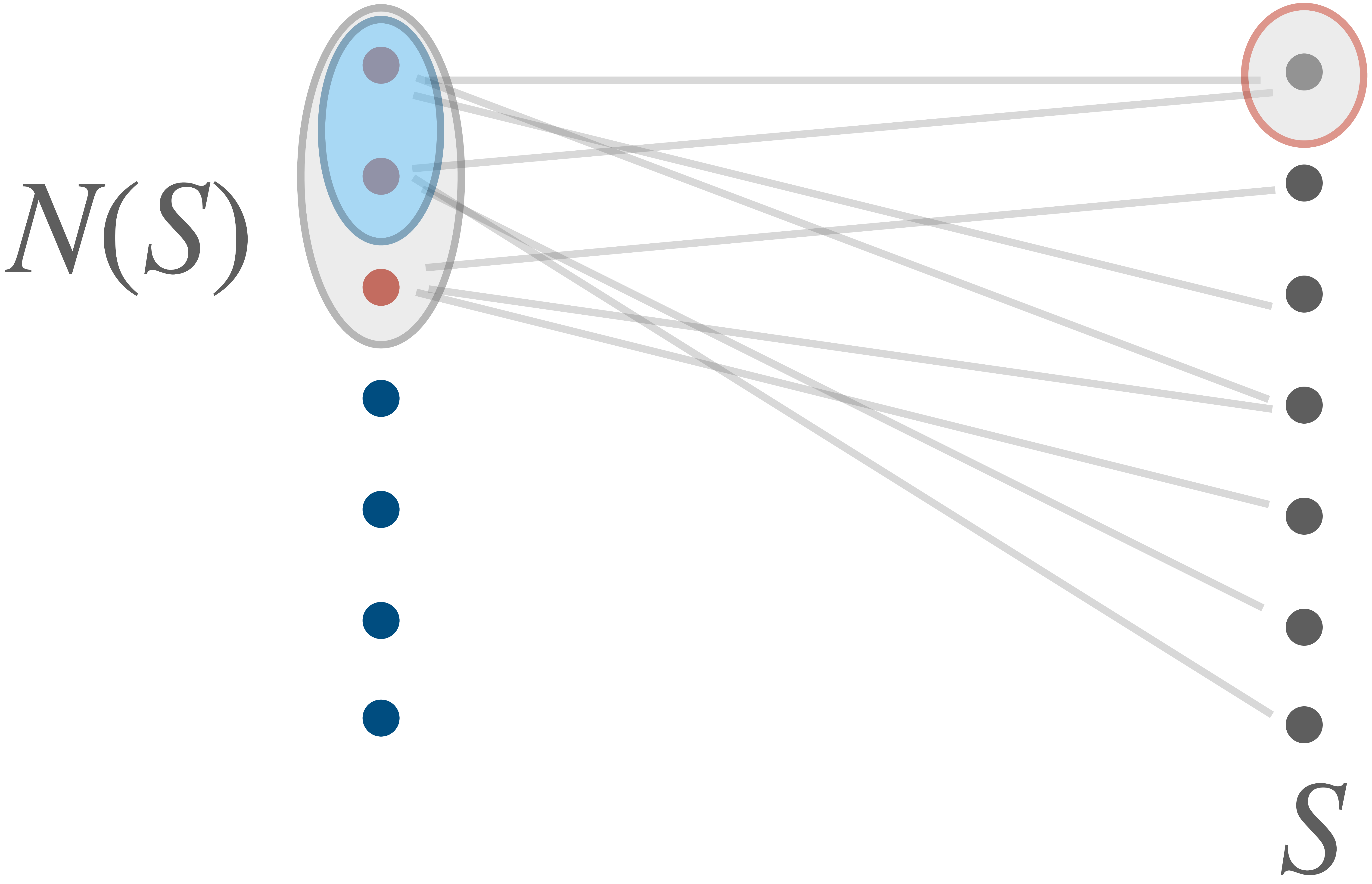
Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- At the new price, all edges between S to $N(S)$ still exist (buyers in S may have more edges to items outside that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, remove those edges from the matching)
    - Why is this matching possible?
    - We use Hall's theorem on items in $T = N(S)$

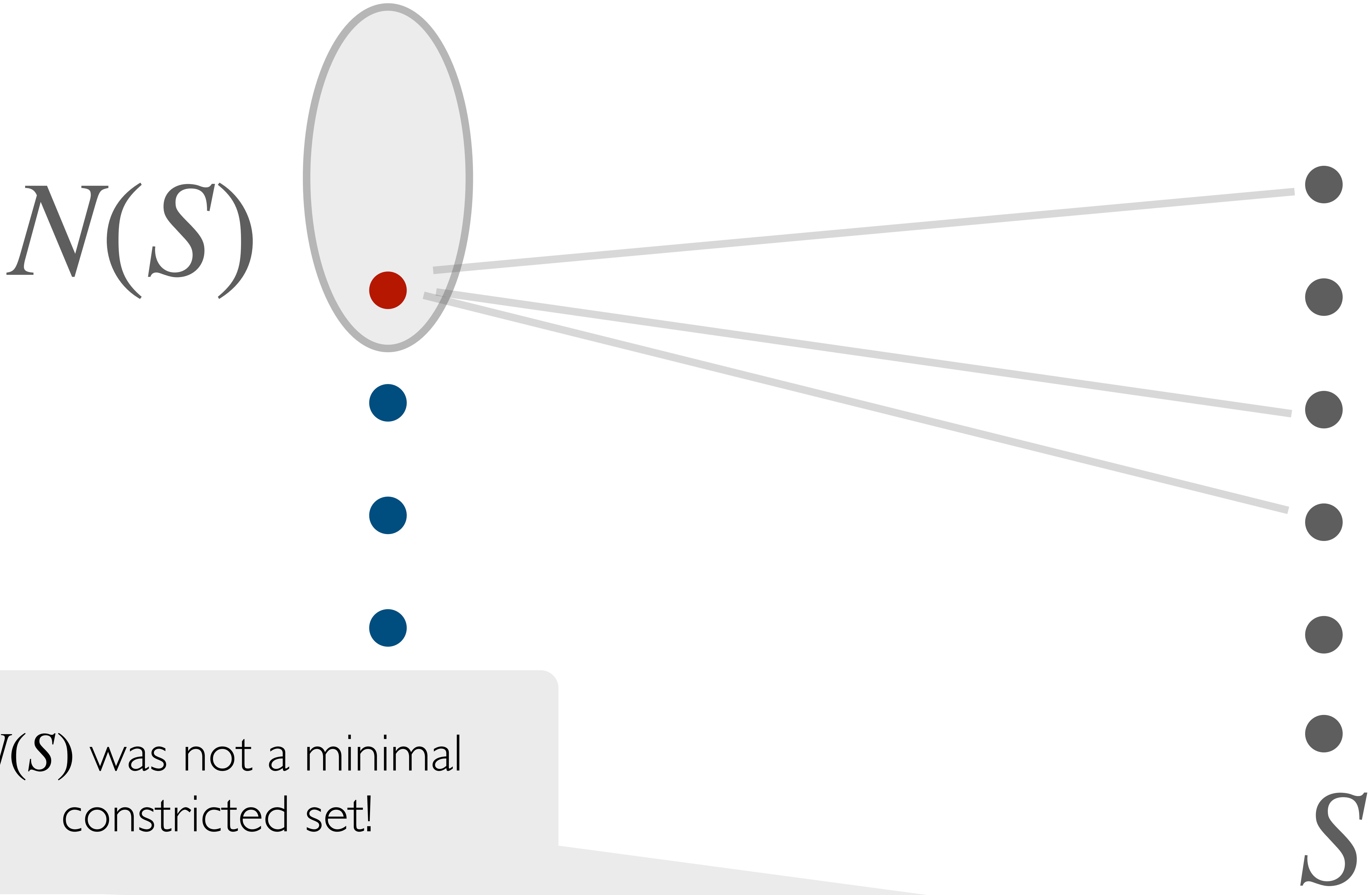
Why Such a Matching Exists



Why Such a Matching Exists



Why Such a Matching Exists



$N(S)$ was not a minimal constricted set!

Proving Our Algorithm Terminates

- **So far:** showed that if the algorithm ends, the prices are market clearing.
- **Theorem.** The ascending price auction terminates.
- **Proof.** Show that algorithm starts with a certain amount of **"potential energy"** which goes down by at least 1 in each iteration
- Let the potential of any round be defined as:

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_i^*$$

- where p_j is the price of item j in that round and u_i^* is the maximum utility i can obtain given prices $\mathbf{p}$ in that round

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_i^*$$

- **Theorem.** The ascending price auction terminates.
- **Proof.**
- At the the beginning, all prices are zero and $u_i^* = \max_j v_{ij}$
- Thus, before the auctions starts $E_0 = \sum_i \max_j v_{ij}$
- To wrap up proof, we show
  - Potential can never be negative $E \geq 0$
  - Potential at each step goes down by at least 1
- Thus, in E_0 steps the algorithm terminates. ∎

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

- **Lemma:** Potential energy E is always non-negative.
- **Proof.**
- If there is at least one item with price 0 then each $u_i^* \geq 0$
  - Why is this true? Use our invariant!
  - Every non-zero priced item is matched, thus when $n-1$ items are matched, no need to raise the price of nth item
- Since prices are always are always nonnegative $E \geq 0$

Proving Our Algorithm Terminates

$$E = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

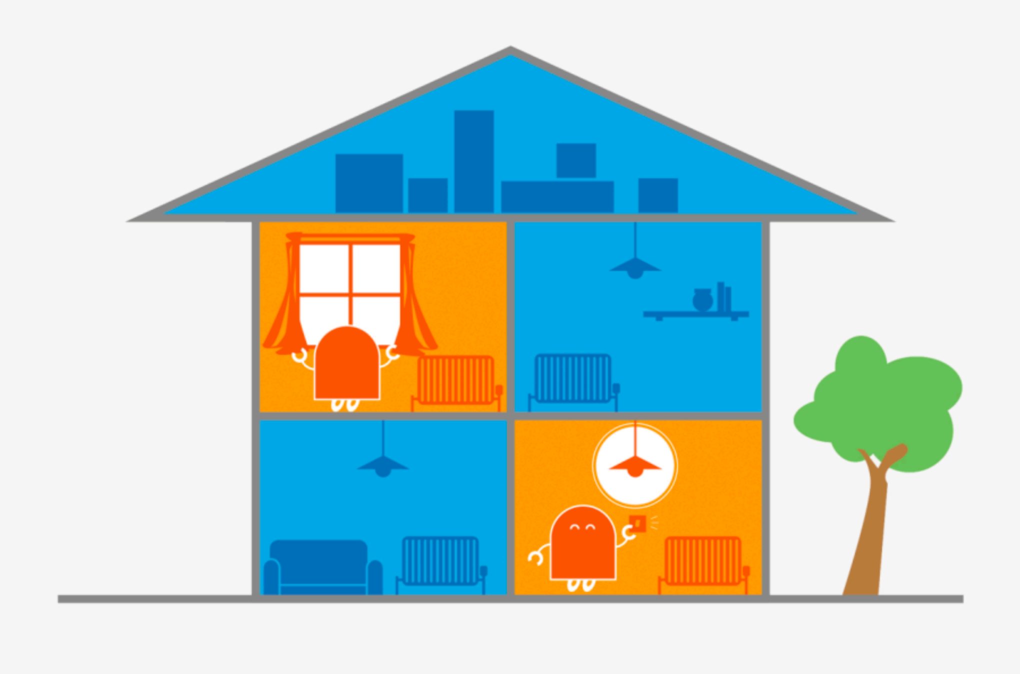
- **Claim.** Potential E goes down by at least one each step.
- **Proof**. At each step, we raise the price of all items in $N(S)$, how much does it increase the first term in E ?
  - $|N(S)|$
- However, the value of u_i^* goes down by one for each node in S, how much does this decrease the second term in E?
  - $|S|$
- Since $|N(S)| < |S|$, then potential decreases by at least 1
- Thus, the algorithm must terminate in E_0 steps ∎
- Our ascending auction terminates at market clearing prices!

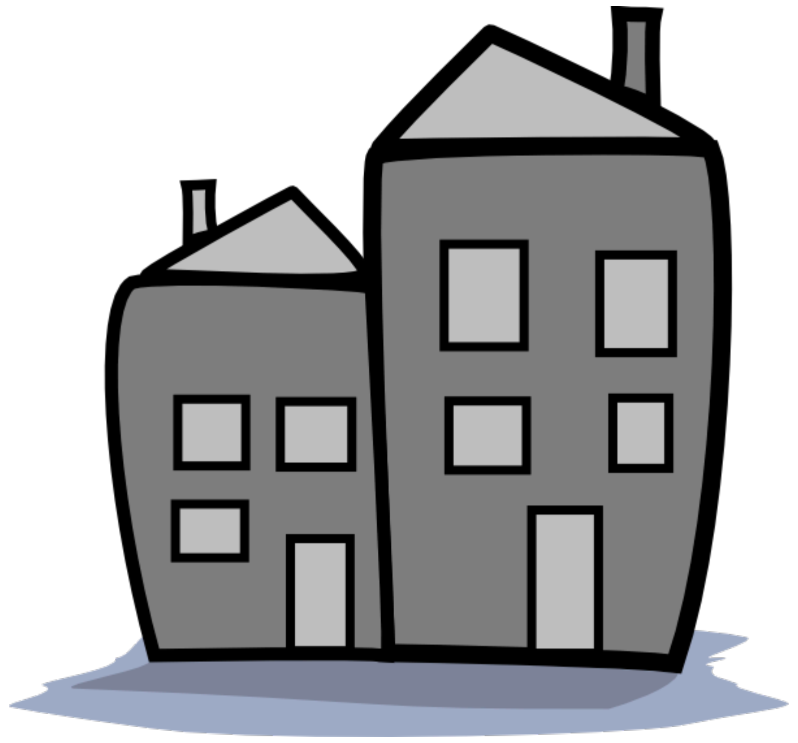
VCG Prices vs Market-Clearing

- VCG prices set **centrally**: ask each buyer to report their valuation and charge each buyer a **"personalized price"** for their allocation
- VCG prices are only set after a matching has been determined (the matching that maximizes total valuation of the buyers)
  - Not just about the item itself, but who gets the item
- Market-clearing prices are **"posted prices"** at which buyers are free to pick whatever item they like
  - Prices are chosen first and posted on the item
  - Prices cause certain buyers to select certain items leading to a matching

Applying VCG

Prices





VCG. Need to find surplus maximizing allocation first

Valuations

Zoe

12, 2, 4

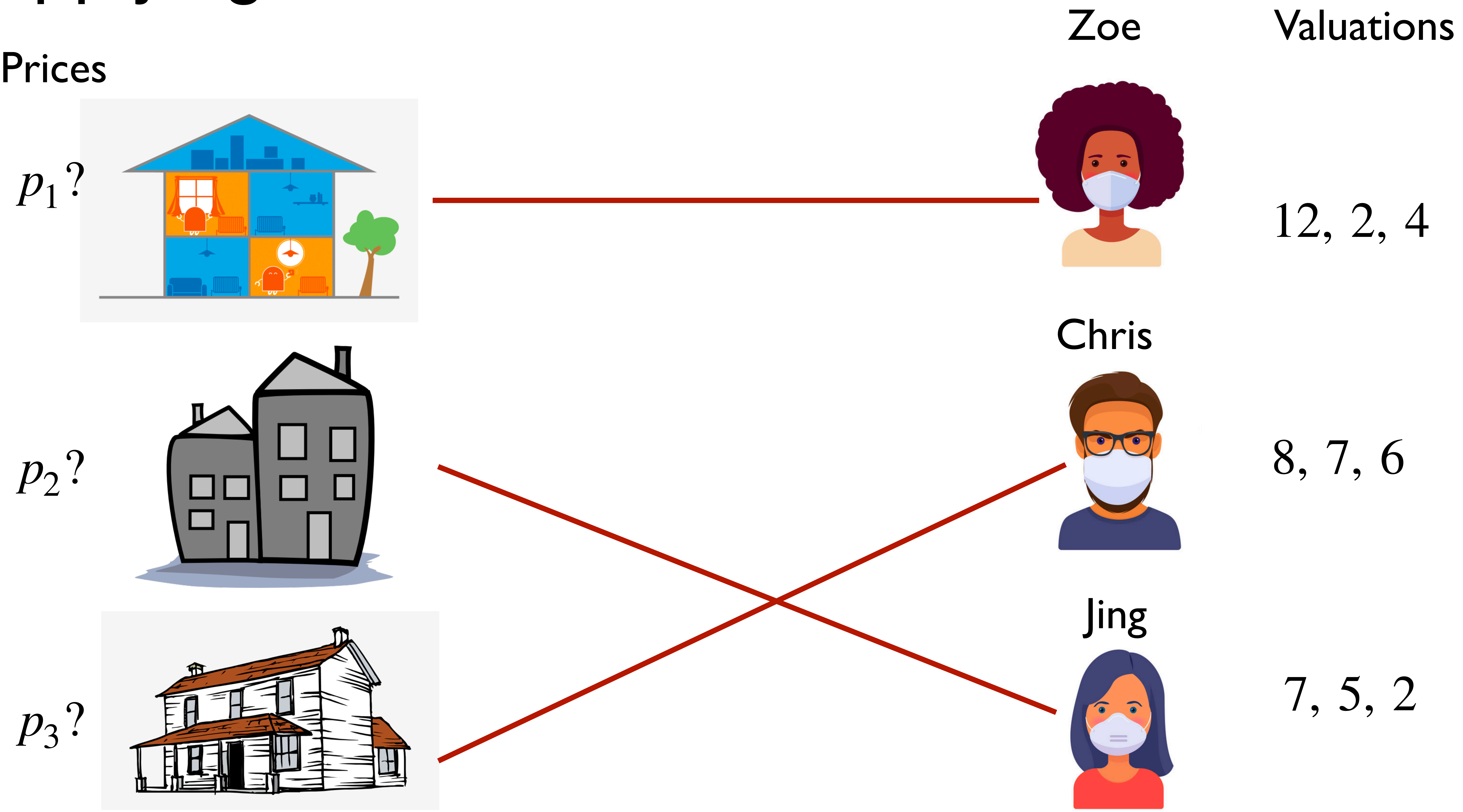
Chris

8, 7, 6

Jing

7, 5, 2

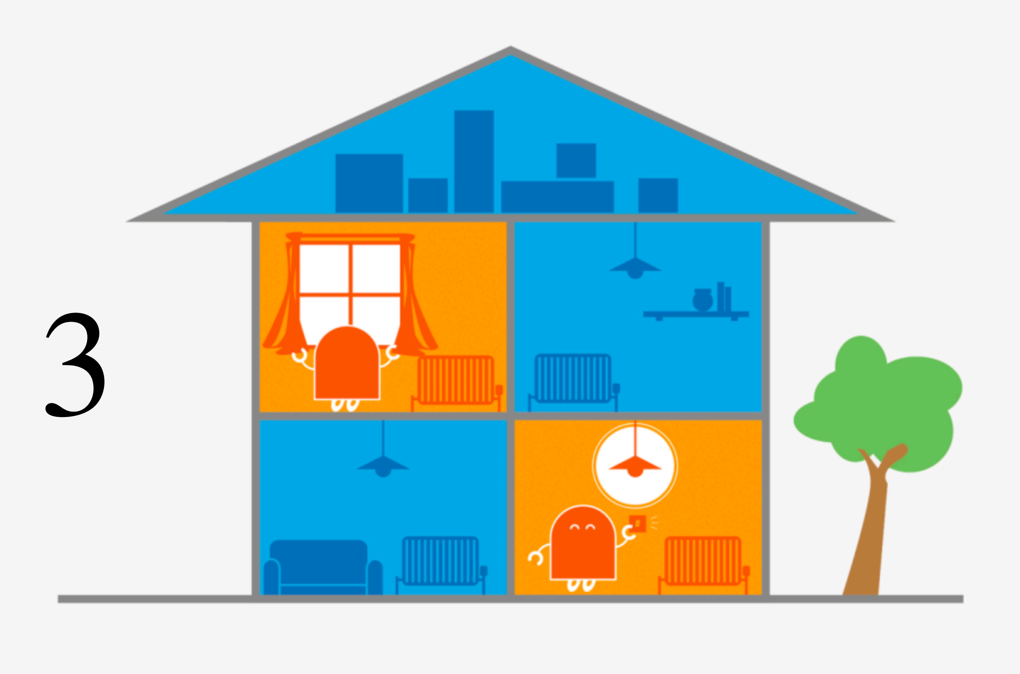
Applying VCG

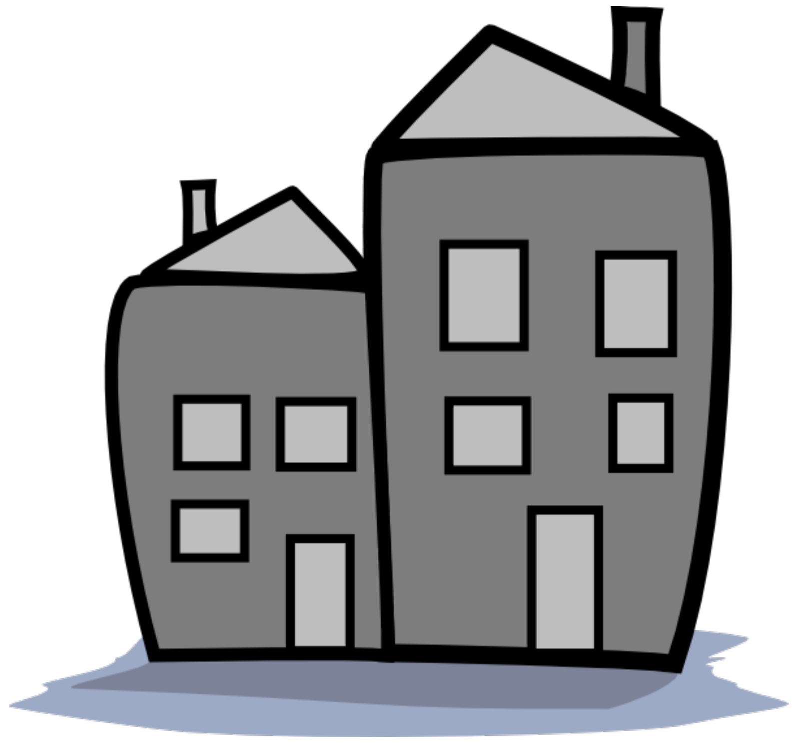


Applying VCG

Prices

$p_1 = 3$





Surplus without Zoe: **7+7 = 14**
Surplus by others when Zoe is present:
6 + 5 = 11

Zoe Valuations

12, 2, 4

Chris

8, 7, 6

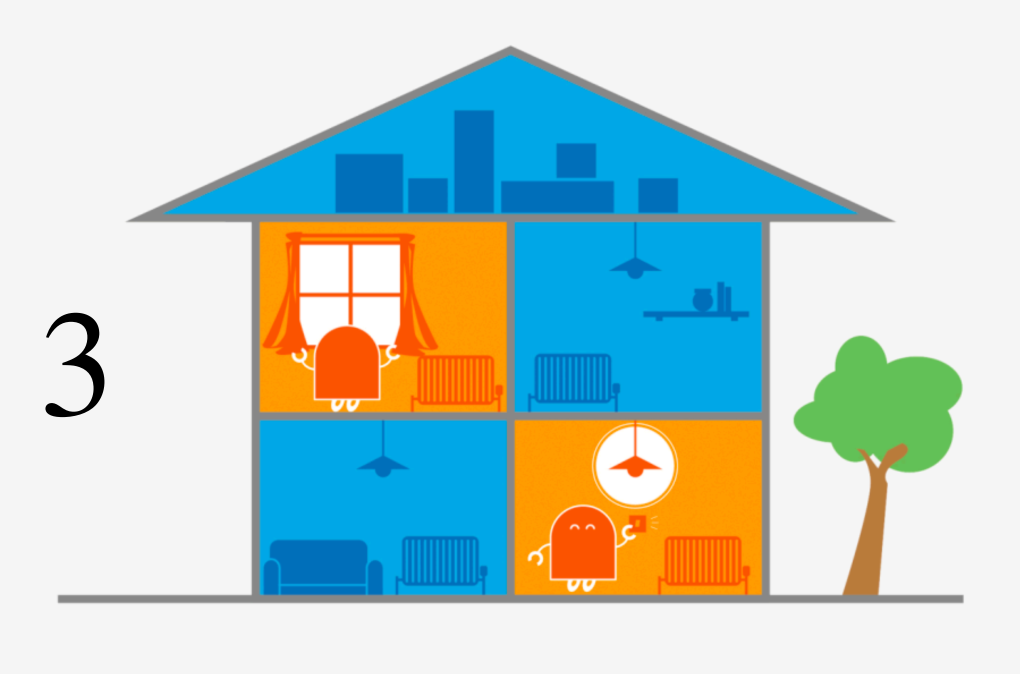
Jing

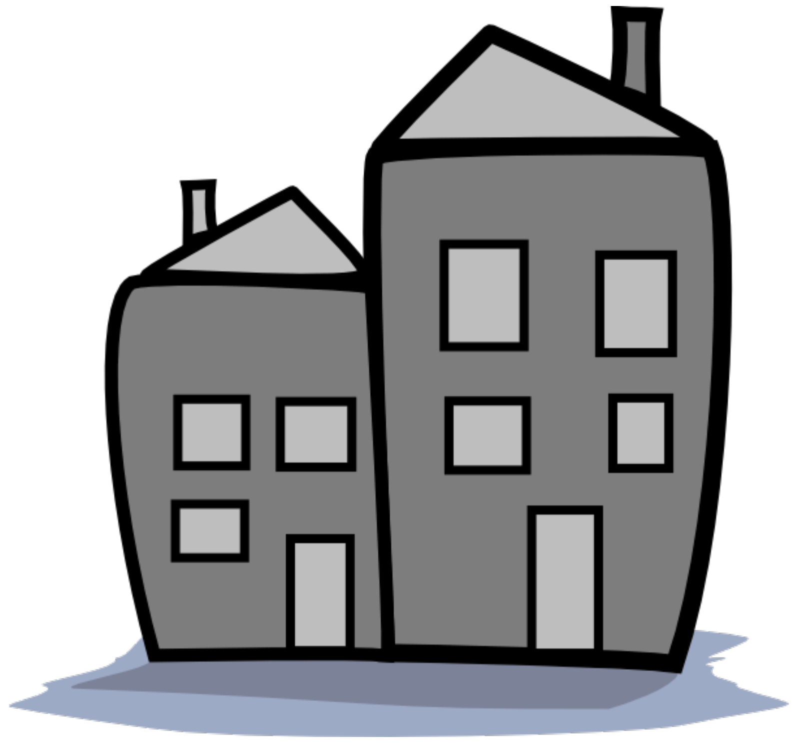
7, 5, 2

Applying VCG

Prices

$p_1 = 3$





$p_3 = 0$

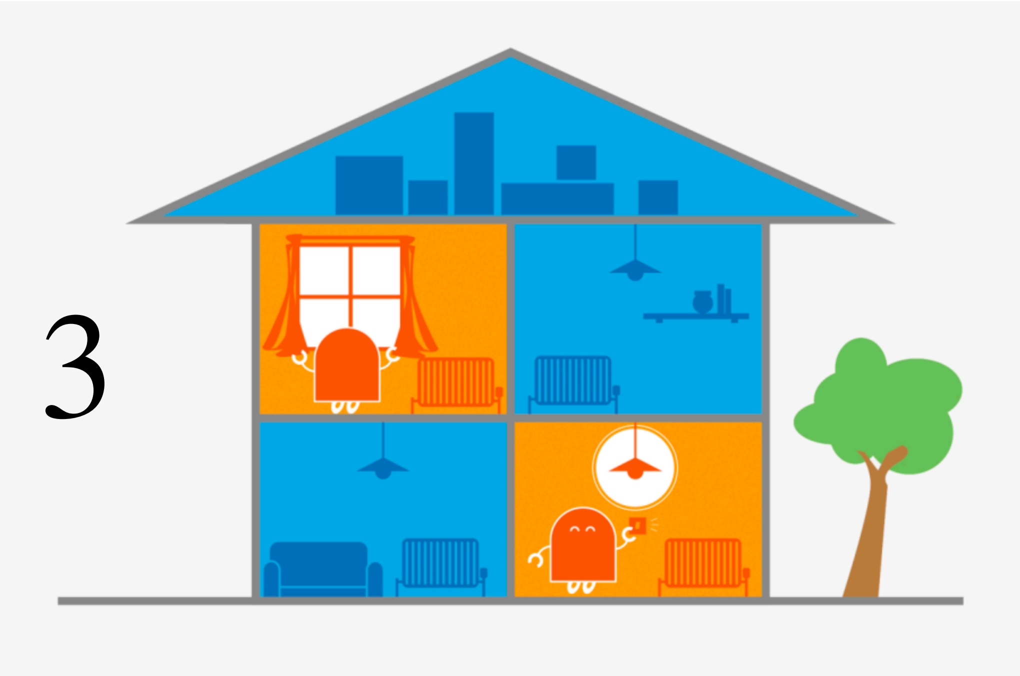
Surplus without Chris: **12+5 = 17**
Surplus by others when Chris is present: **12+5 = 17**

	Valuations
Zoe 	12, 2, 4
Chris 	8, 7, 6
Jing 	7, 5, 2

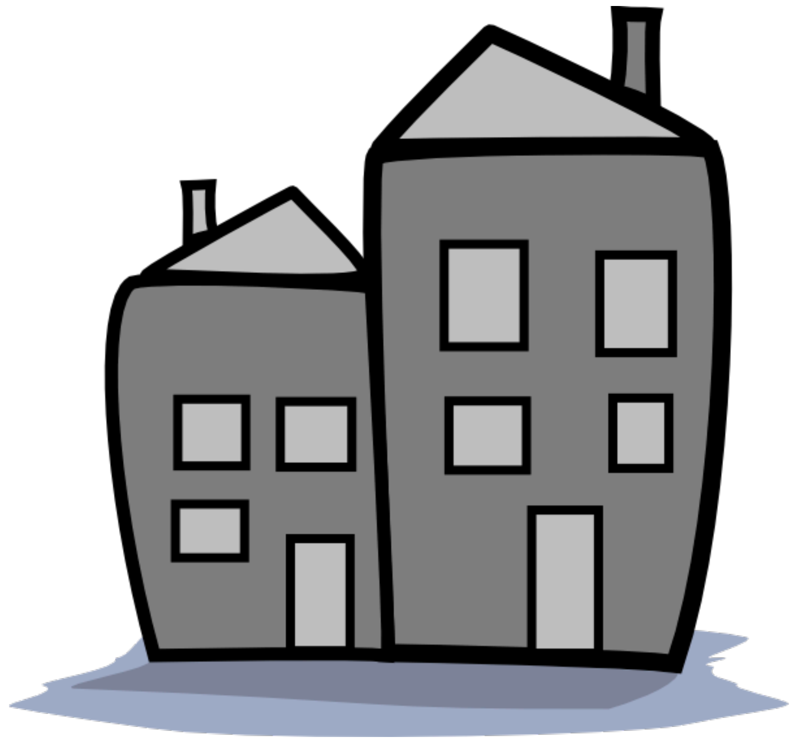
Applying VCG

Prices

$p_1 = 3$



$p_2 = 1$



$p_3 = 0$

Surplus without Jing: **12+7 = 19**
Surplus by others when Jing is present:
12+6 = 18

Zoe | Valuations

12, 2, 4

Chris

8, 7, 6

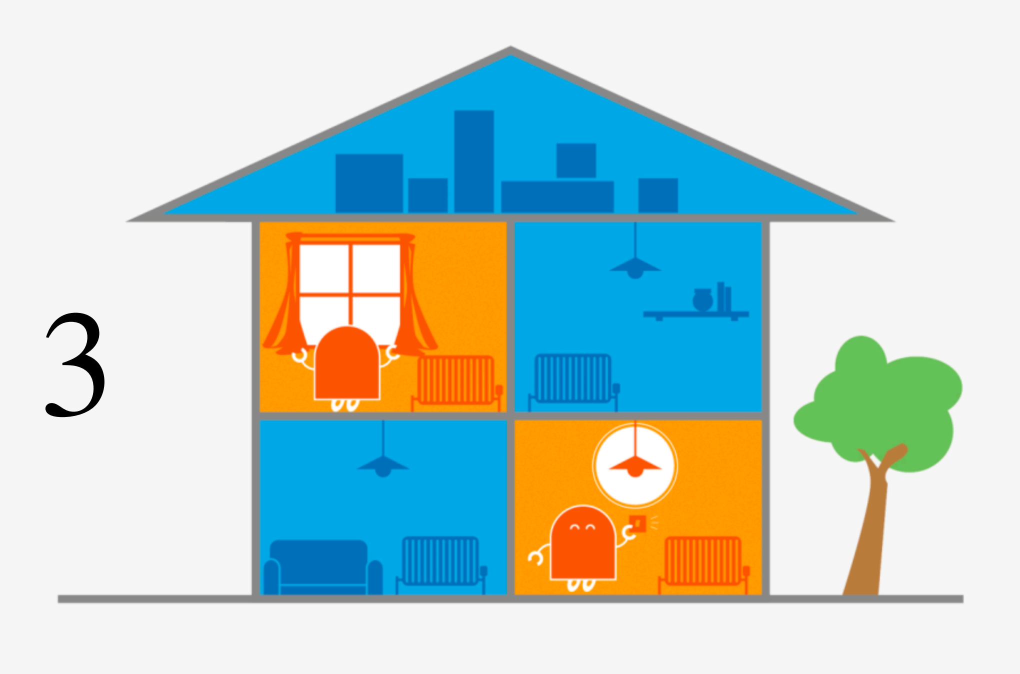
Jing

7, 5, 2

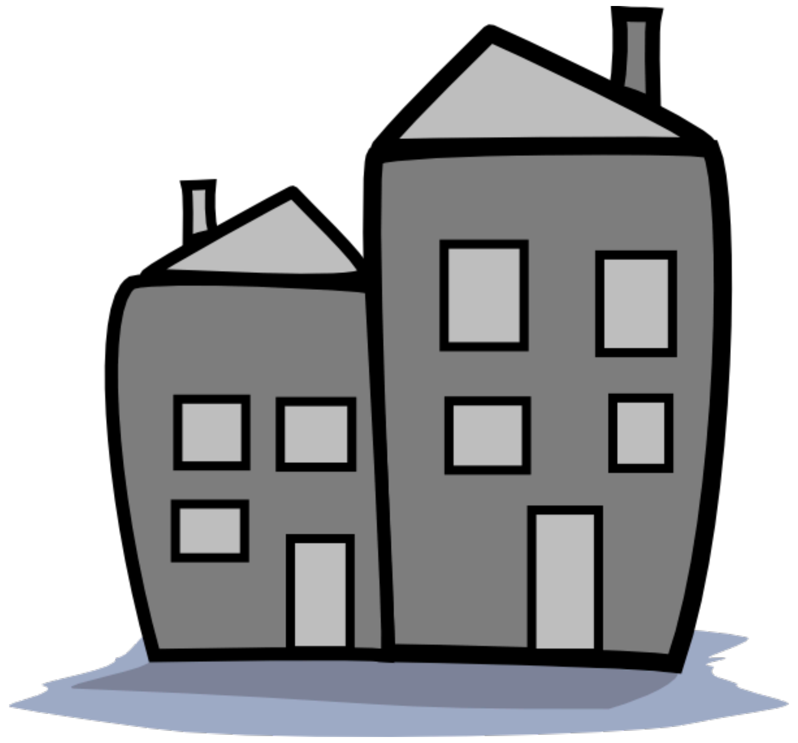
Applying VCG

Prices

$p_1 = 3$



$p_2 = 1$



$p_3 = 0$

We got the same **prices** & **matching** as our **competitive equilibrium**

Zoe

Chris

Jing

Valuations

12, 2, 4

8, 7, 6

7, 5, 2

VCG Prices are Market Clearing

- Despite their definition as personalized prices, VCG prices are always market clearing (for the case when each buyer wants a single item)
- Suppose we computed VCG prices for a given matching market
- Then, instead of assigning the VCG allocation and charging the price, we post the prices publicly
  - Without requiring buyers to follow the VCG match
- Despite this freedom, each buyer will in fact achieve the highest utility by selecting the item that was allocated by the VCG mechanism!
- **Theorem.** In any matching market (where each buyer can receive a single item) the VCG prices form the unique set of **market clearing prices of minimum total sum**.

This is a generalization of the VCG/GSP result (where valuations are constrained). The general proof is beyond the scope of this course.

General Demand

- Market clearing prices **may not exist in combinatorial markets**
- **Example**, suppose our market has two items $\{L, R\}$
- Two buyers Alice and Maya
- Alice wants both $v_a(\{L, R\}) = 5$, $v_a(\{L\}) = v_s(\{R\}) = 0$
- Maya wants either, $v_p(\{L\}) = v_p(\{R\}) = v_p(\{L, R\}) = 3$
- What's the welfare-maximizing allocation?
    - Give both to Alice
- What must the price of each be so that Maya doesn't want it?
    - $p(\{L\}) \geq 3, p(\{R\}) \geq 3$
- At a price of ≥ 6 does Alice want it?

Summary

- Ascending price auction is also called Hungarian algorithm in matching literature
- Hungarian algorithm is used to find max-weight bipartite matching
  - Prices are just a conceptual interpretation of "dual" variables
- Caveats:
  - No sales occur until prices have settled at their equilibrium point
  - Coordination required for tie breaks
  - Running time to convergence can be very slow

Competitive Equilibrium Research

- [Left] 2016 Article argues that competitive equilibrium's tie breaking requirement can be fairly strong
  - Use **learning theory** to predict buyer's behavior and demand and show convergence under such some mild assumptions
- [Right 2021]. Algorithms with predictions paper predicts "prices" for faster runtime

Do Prices Coordinate Markets?

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Faster Matchings via Learned Duals

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Abstract

A recent line of research investigates how algorithms can be augmented with machine-learned predictions to overcome worst case lower bounds. This area has revealed interesting algorithmic insights into problems, with particular success in the design of competitive online algorithms. However, the question of improving algorithm running times with predictions has largely been unexplored.

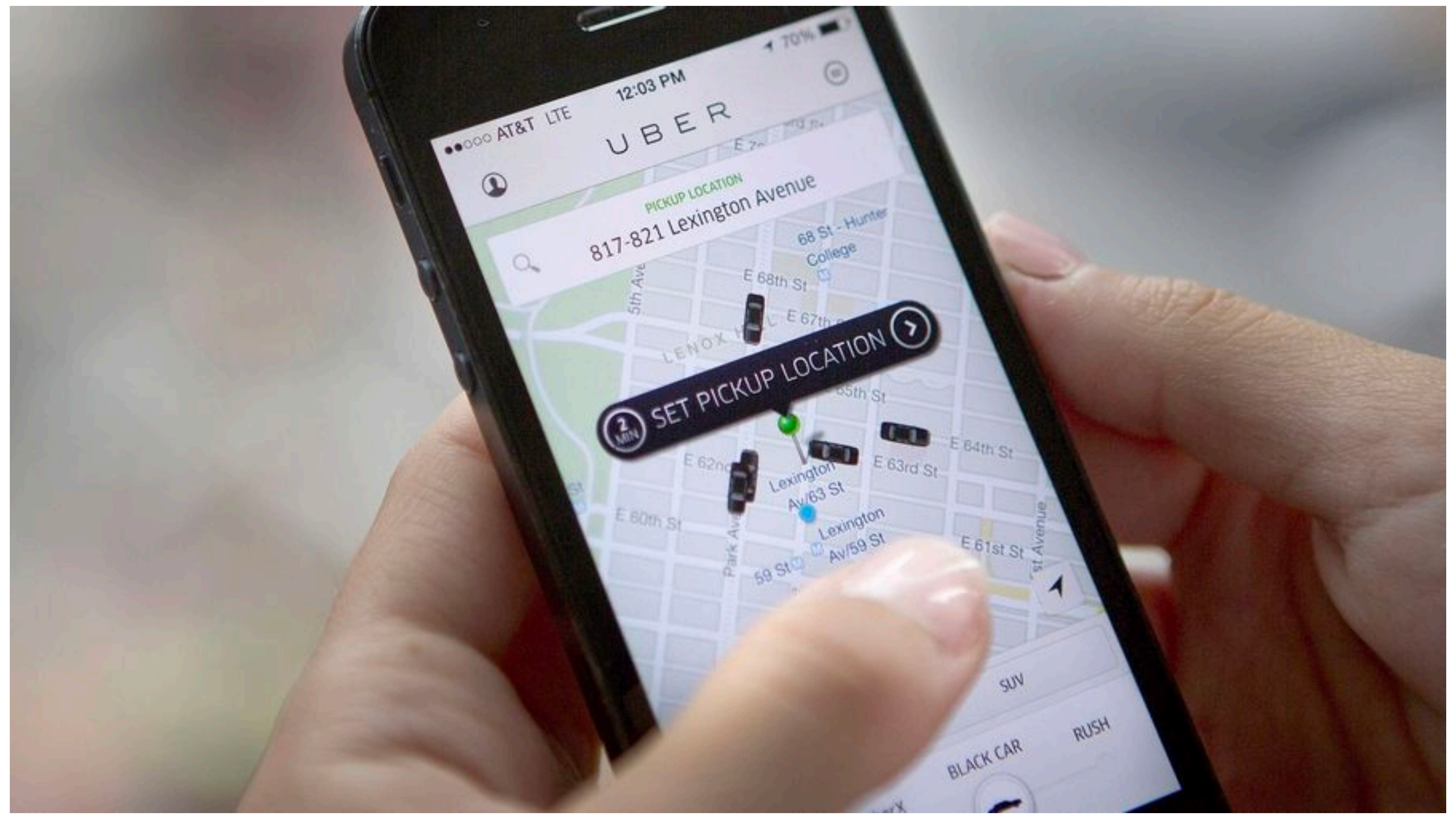
We take a first step in this direction by combining the idea of machine-learned predictions with the idea of "warm-starting" primal-dual algorithms. We consider one of the most important primitives in combinatorial optimization: weighted bipartite matching and its generalization to b-matching. We identify three key challenges when using learned dual variables in a primal-dual algorithm. First, predicted duals may be infeasible, so we give an algorithm that efficiently maps predicted infeasible duals to nearby feasible solutions. Second, once the duals are feasible, they may not be optimal, so we show that they can be used to quickly find an optimal solution. Finally, such predictions are useful only if they can be learned, so we show that the problem of learning duals for matching has low sample complexity. We validate our theoretical findings through experiments on both real and synthetic data. As a result we give a rigorous, practical, and empirically effective method to compute bipartite matchings.

The Myth of the Invisible Auctioneer

- One fundamental assumption when we executed the ascending price mechanism to compute market-clearing prices is:
  - The market does not actually clear until prices have settled at their equilibrium point
- As if an invisible auctioneer is coordinating the prices and lets the market know when the prices have converged and trade can actually take place
- In practice, one might imagine that sales are actually happening concurrently with price adjustment

Fluctuations in Practice: Research

- In practice, one might imagine that sales are actually happening concurrently with price adjustment
- It turns out, the way buyers and sellers respond to prices in the short-run can dramatically influence prices
- **Example**. Surge pricing on ride-sharing platforms can be viewed as an attempt to find market-clearing prices
- However, if passengers and drivers respond to prices myopically, the resulting behavior can be erratic
- Recent research in AGT studies **dynamic (online) resource allocation problems** that take these factors into account



Decentralized Markets without Money