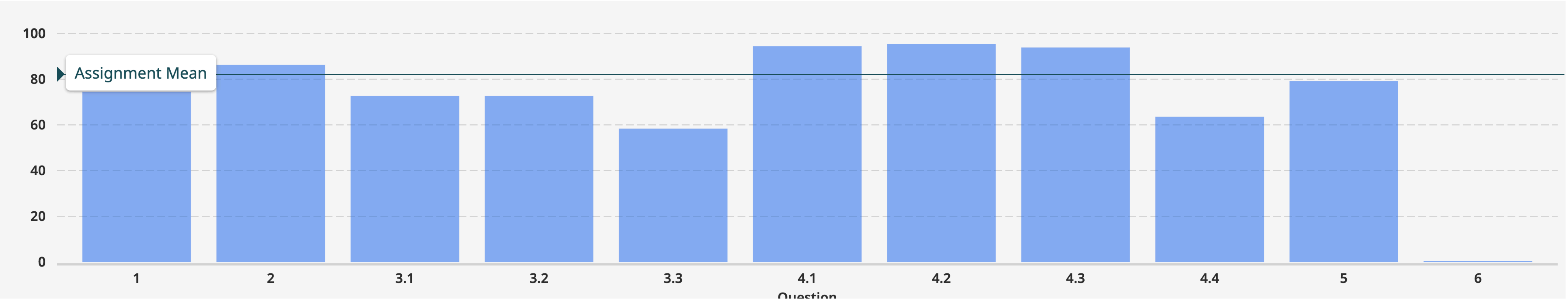
CS 357: Algorithmic Game Theory

Lecture 10: Stable Matchings

Shikha Singh

Announcements

- Pick up **HW 5** due Tuesday after break (April 8)
- Midterm graded feedback returned
  - Median: **87%**, Mean: **82%**
  - Bonus +2 for silly mistakes
  - Performance breakdown by question below
  - If anything in the feedback is unclear, please reach out
  - Only 15% of your final grade
  - Value growth, there is another exam on April 29



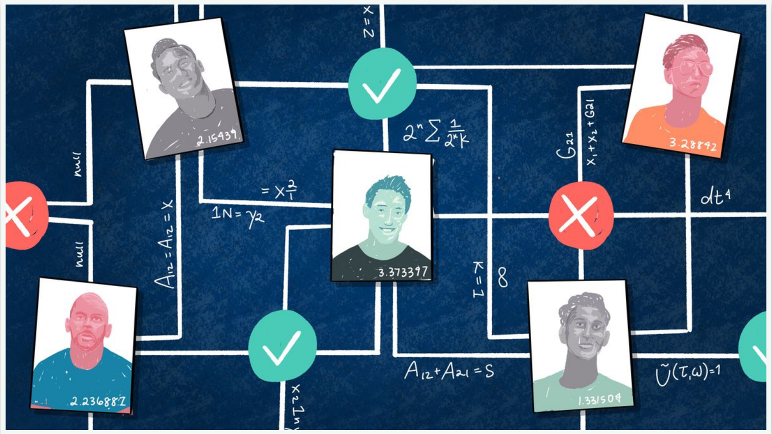
Paper Eval #2 on Friday

- Pick up **HW 5** due Tuesday after break (April 8)
- Mechanism Design of School Choice
  - New algorithm: **Top Trading Cycles**!
- Paper evaluation #2 due on Friday 1 pm
- What you need to do
  - **Part A.** Fill out google form (individual)
  - **Part B.** Answer both short questions & 1 proof (group)
    - Bring joint write up to class
    - Each group will be asked to present in class, for efficiency, please **prepare slides** this time!
    - You can submit a PDF/print out of the slides as your write up

Two-Sided Matching Markets

Two-Sided Markets

- Consider a two-sided market:
  - A set H of n hospitals, a set S of n students
  - Each hospital has a complete and strict preference ranking of students
  - Each student has a complete and strict preference ranking of hospitals
- **Goal.** A perfect matching M that is **stable** (has no blocking pairs)
  - A hospital h and student s form a **blocking pair** (h, s) in a matching M if h prefers s to its current match in M and s prefers h to its current match in M

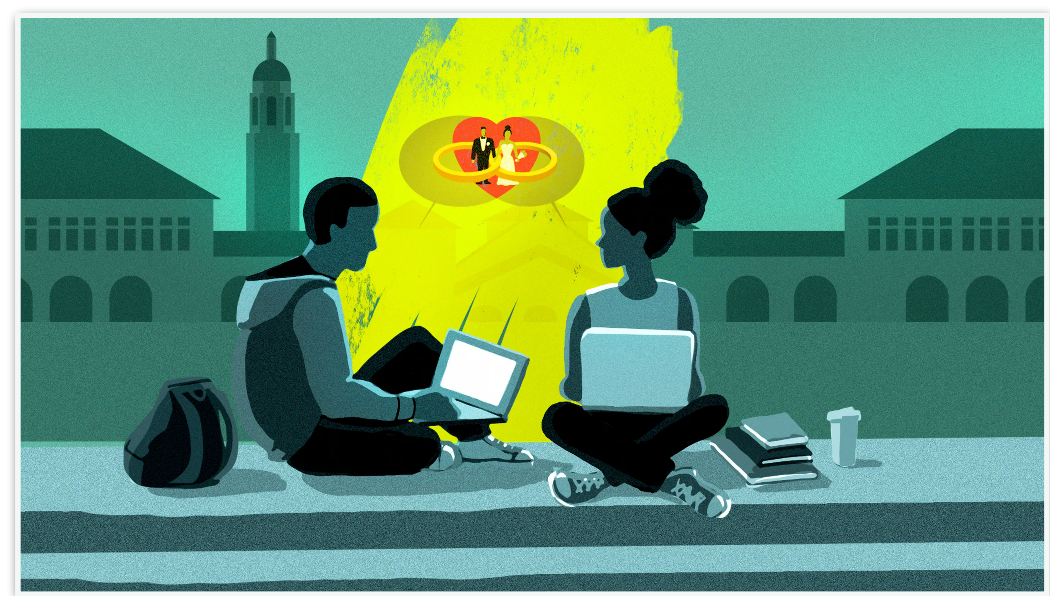


The Tinder algorithm explained: Vox

Stylized History:

the **"Stable Marriage"** Problem

The Dating Market: Medium



Dating apps are awful. But this algorithm offers just one match: your "backup plan." - Vox

Stylized Model of "Marriage" or "Dating"

1962, The American Mathematical Monthly

COLLEGE ADMISSIONS AND THE STABILITY OF MARRIAGE

D. GALE* AND L. S. SHAPLEY, Brown University and the RAND Corporation

3. Stable assignments and a marriage problem. In trying to settle the question of the existence of stable assignments we were led to look first at a special case, in which there are the same number of applicants as colleges and all quotas are unity. This situation is, of course, highly unnatural in the context of college admissions, but there is another "story" into which it fits quite readily.

1992

Stable Husbands

Donald E. Knuth, Rajeev Motwani, and Boris Pittel
Computer Science Department, Stanford University

2018

A Stable Marriage Requires Communication*

Yannai A. Gonczarowski† Noam Nisan‡ Rafail Ostrovsky§ Will Rosenbaum¶

2008

Sampling Stable Marriages: Why Spouse-Swapping Won't Work*

Nayantara Bhatnagar† Sam Greenberg‡ Dana Randall§

2003

Marriage, Honesty, and Stability

Nicole Immorlica* Mohammad Mahdian*

History of Stable Matching

- In 1900s matching medical residents to hospitals was decentralized
- Increasingly competitive
  - By the 1940s, appointments were often made as early as the beginning of the junior year of medical school

The market for law school graduate is also known for these problems. **Roth** in this article **"Who Gets What And Why"** quotes a law school student who in 2005, on a flight from her 1st interview to 2nd interview, got 3 voicemail messages: the 1st extending an offer from where she just interviewed; the 2nd to urge her to return the call soon; and the 3rd to rescind the offer. Her flight was only 35 mins long!

"Who Gets What and Why" by A Roth

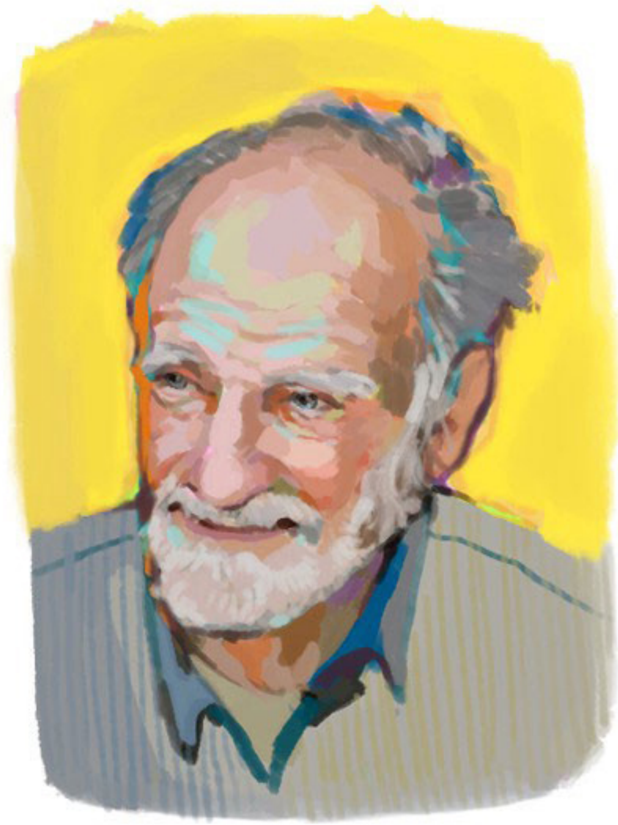
Why have Centralized Markets

- In 1900s matching medical residents to hospitals was decentralized
- Increasingly competitive
  - By the 1940s, appointments were often made as early as the beginning of the junior year of medical school
- In 1945, a variant of deferred acceptance implemented by **AAP (American Associated of Pediatrics)** and NRMP (National Resident Matching program) to match residents to hospitals
- This was the invention of **"the match"**

"The Origins, History, and Design of the Resident Match" by A Roth

Nobel Prize 2012: Shapley & Roth

David Gale
PROFESSOR, UC BERKELEY



Lloyd Shapley
PROFESSOR EMERITUS, UCLA

Stable matching: Theory, evidence, and practical design

*This year's Prize to **Lloyd Shapley** and **Alvin Roth** extends from abstract theory developed in the 1960s, over empirical work in the 1980s, to ongoing efforts to find practical solutions to real-world problems. Examples include the assignment of new doctors to hospitals, students to schools, and human organs for transplant to recipients. Lloyd Shapley made the early theoretical contributions, which were unexpectedly adopted two decades later when Alvin Roth investigated the market for U.S. doctors. His findings generated further analytical developments, as well as practical design of market institutions.*

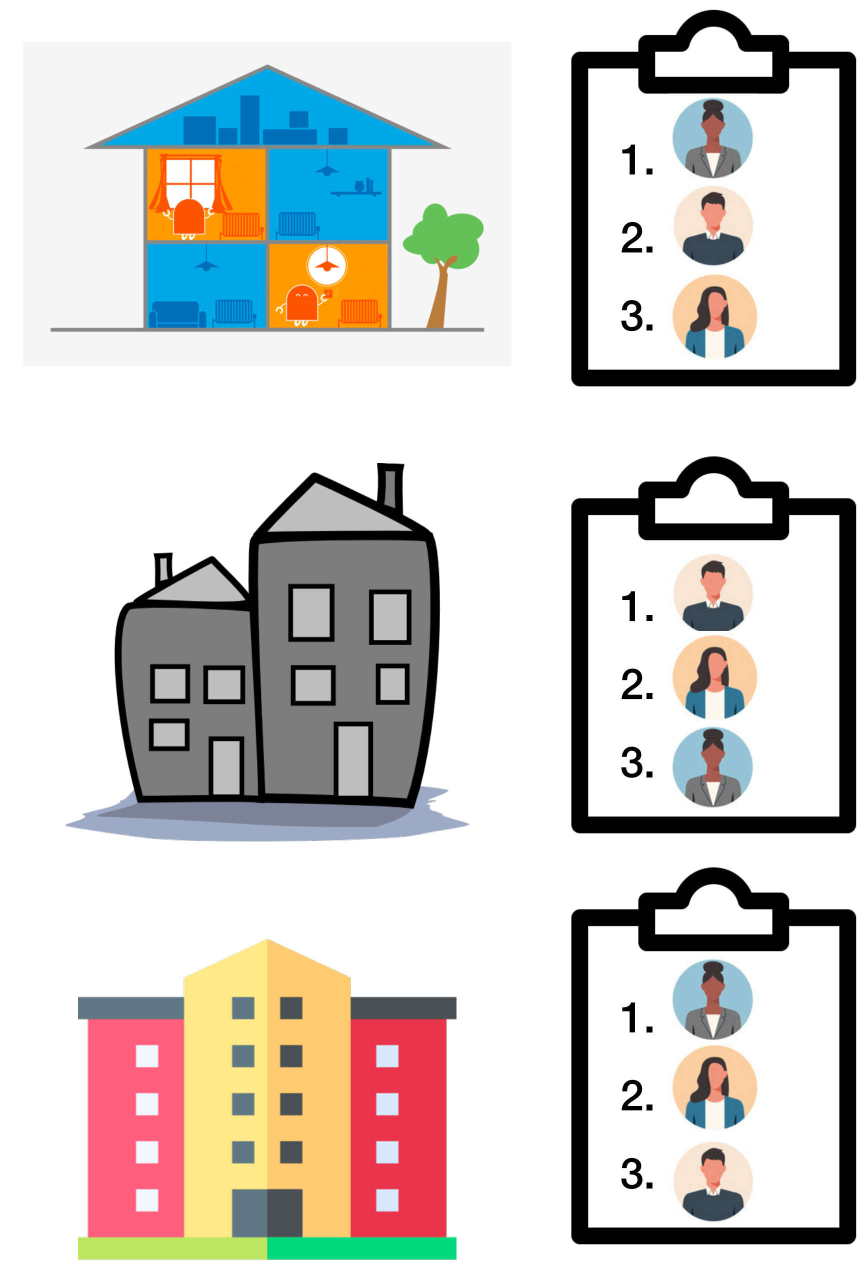
Why Stability: The Story of NRMP

- Empirical evidence in support
- In UK in the 60s, residency programs decided to move from a decentralized system to a centralized clearinghouse
- The details of the implementation were left to individual regions
- Roth looked at data from 7 regions
  - Two followed a stable implementation; they remain in use today
  - Five regions implemented unstable variants, 3 of which did not survive long (due to poor participation and negotiations outside the system)

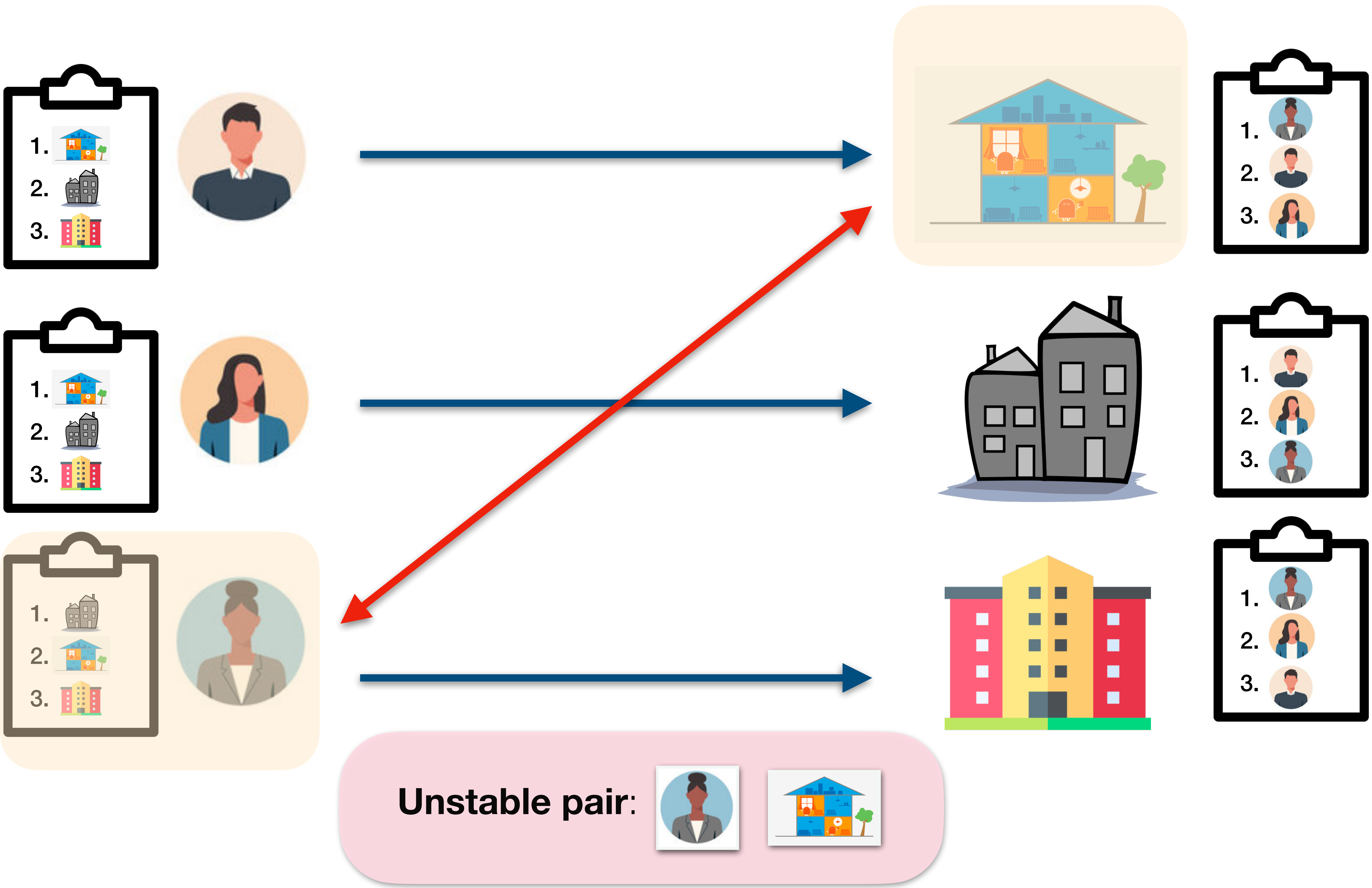
Classic Stable Matching Problem

- **Input**: n applicants and n jobs, complete preference lists

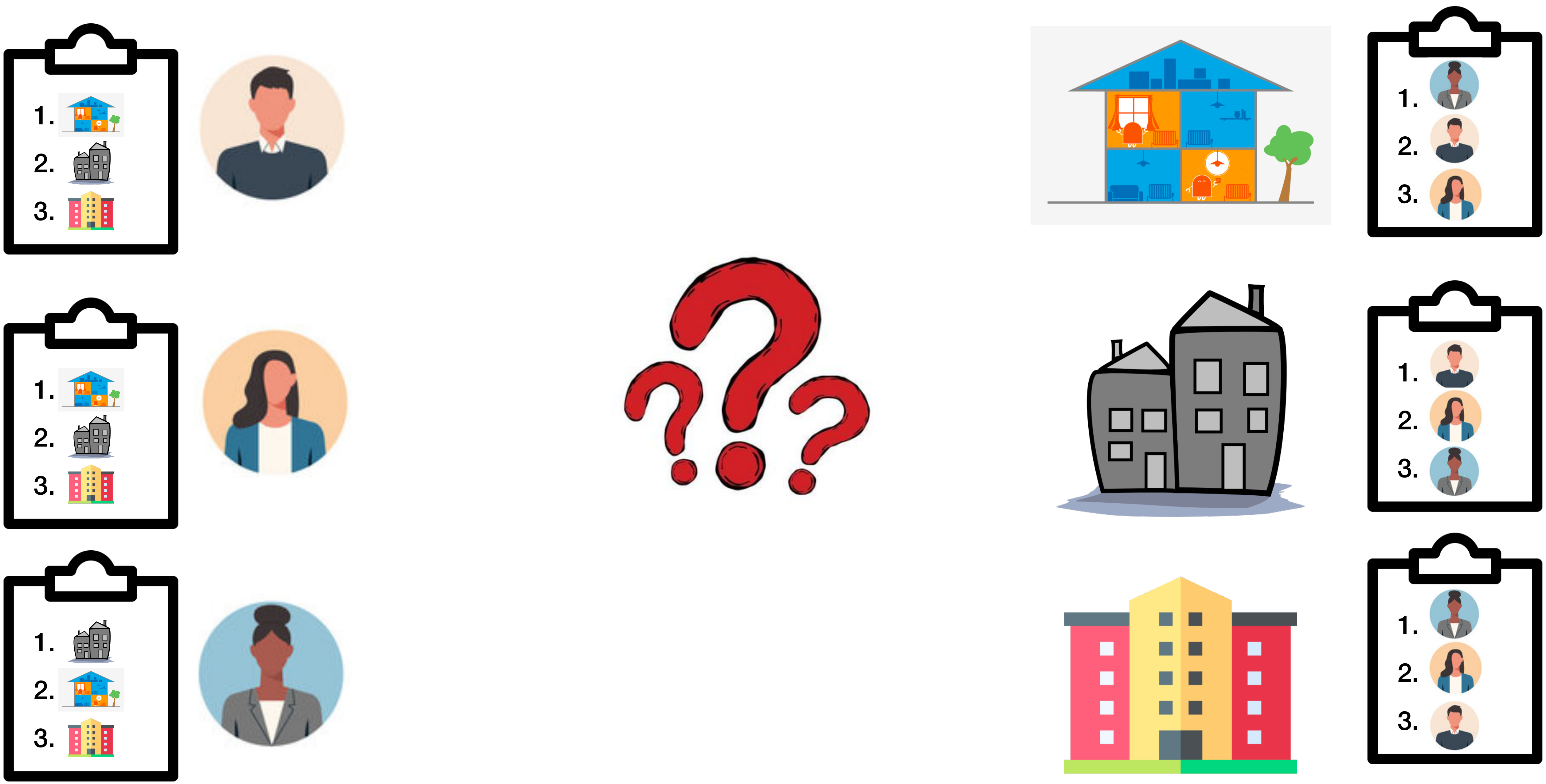
- **Output:** a **perfect matching** M that is **stable** (no applicant and job prefer each other to their match)



Classic Stable Matching Problem



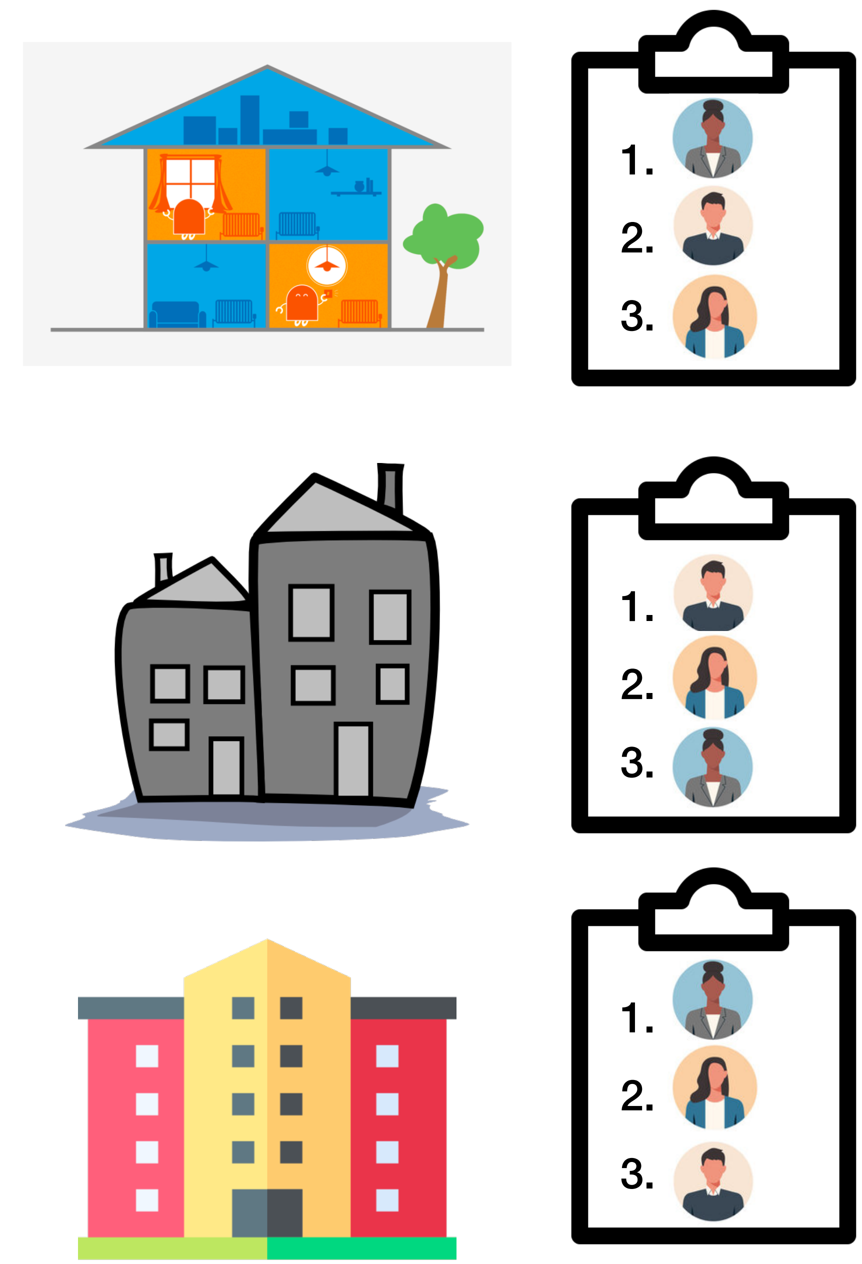
Classic Stable Matching Problem



[Gale Shapley 1952] A perfect stable matching **always exists**!

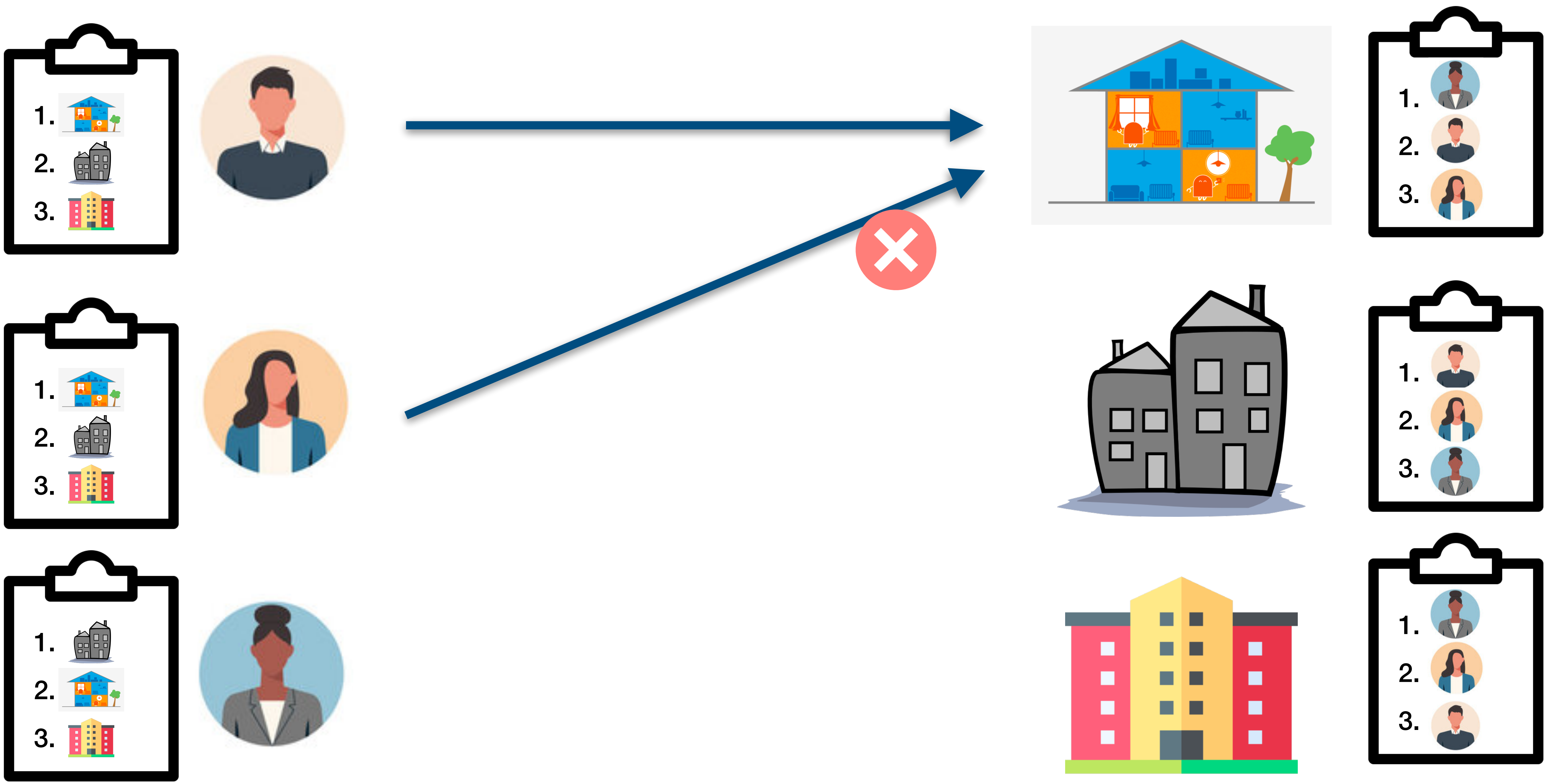
Deferred Acceptance (DA) Algorithm

- Proceeds in rounds
  - Each **unmatched** applicant **"proposes"** to their most preferred job
  - jobs retain the best proposal they have received & reject others
- Matching is finalized when each applicant is matched

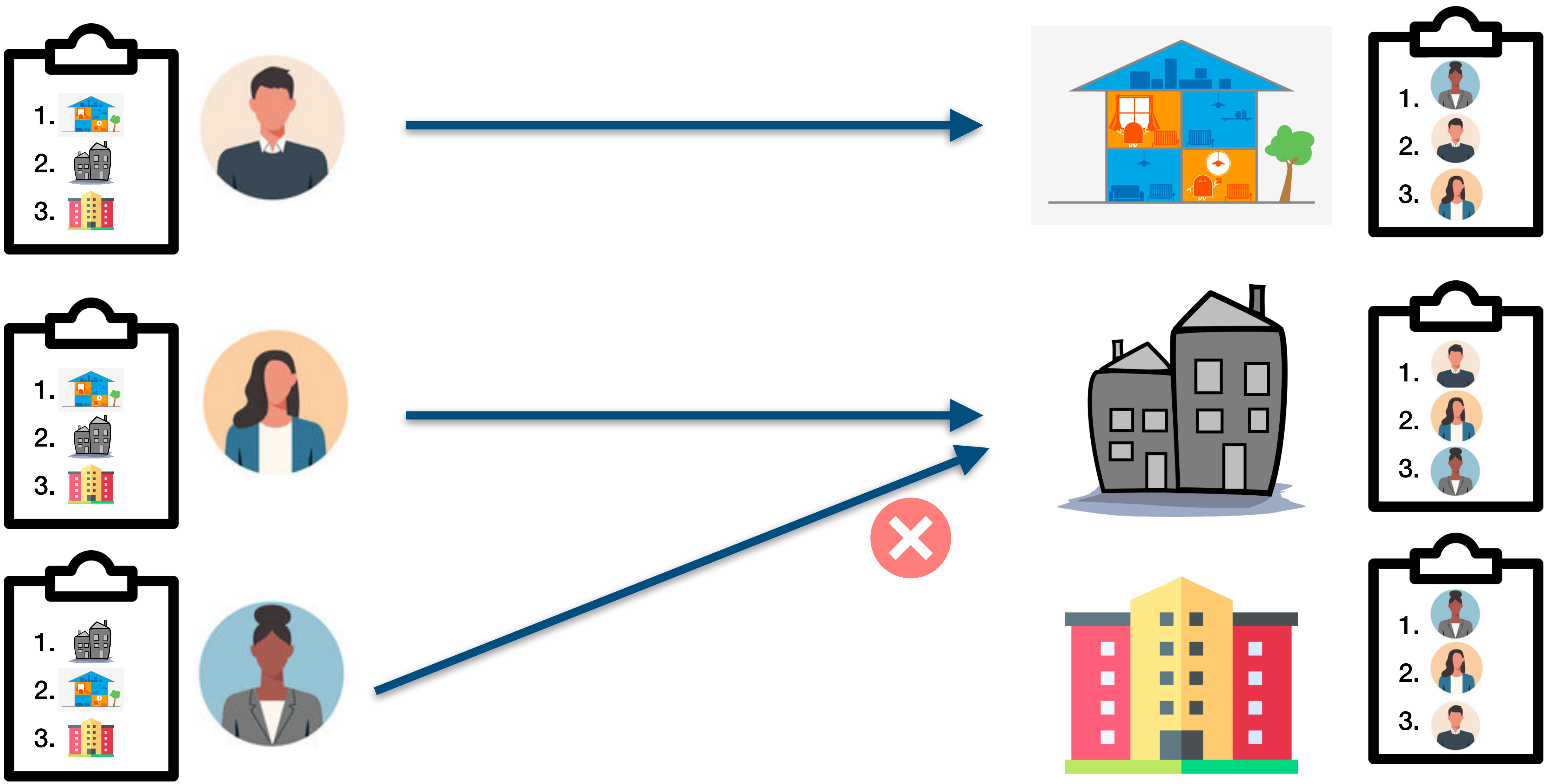


[Gale Shapley 1952] A perfect stable matching always exists!

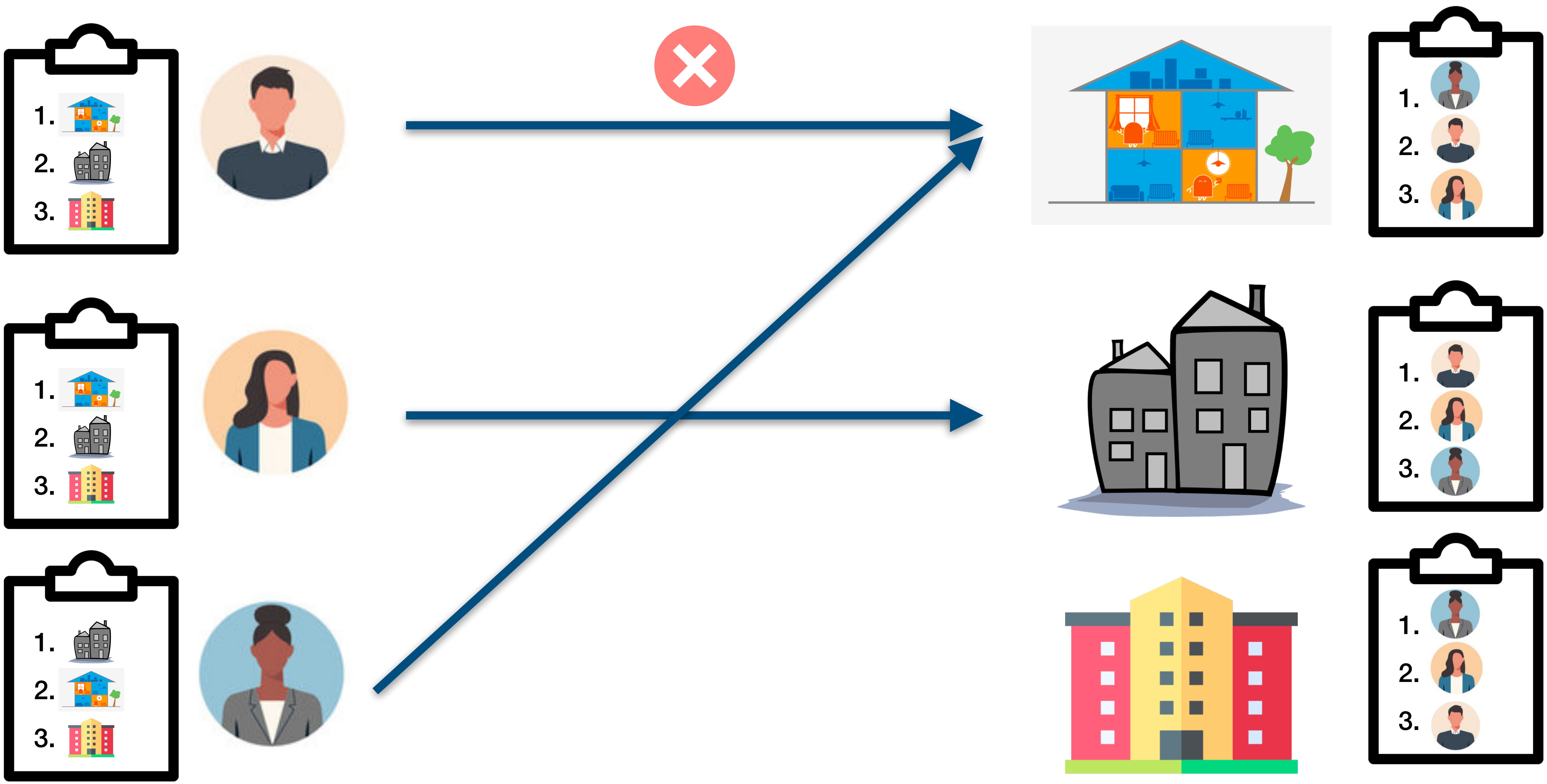
Classic Stable Matching Problem



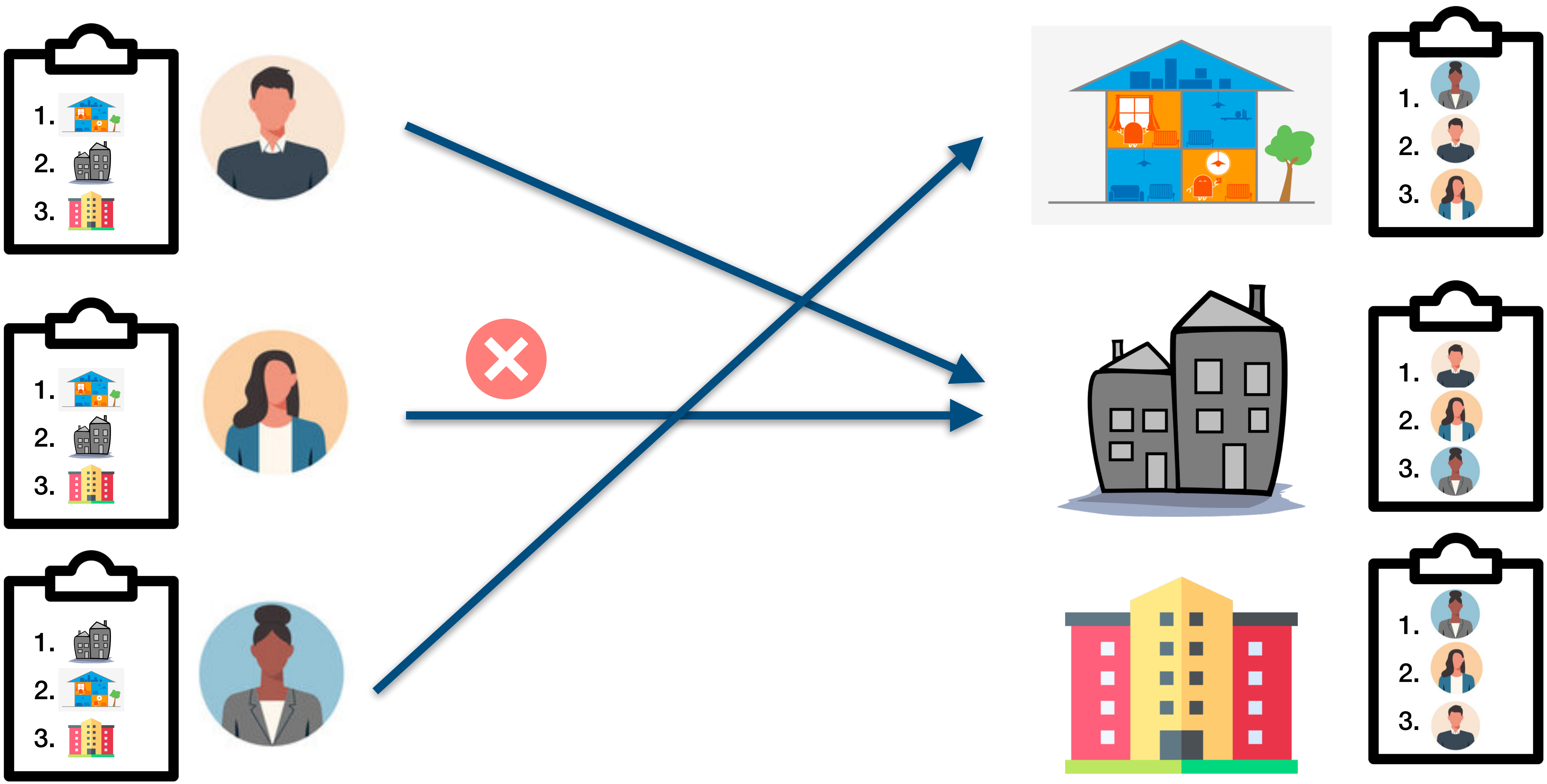
Classic Stable Matching Problem



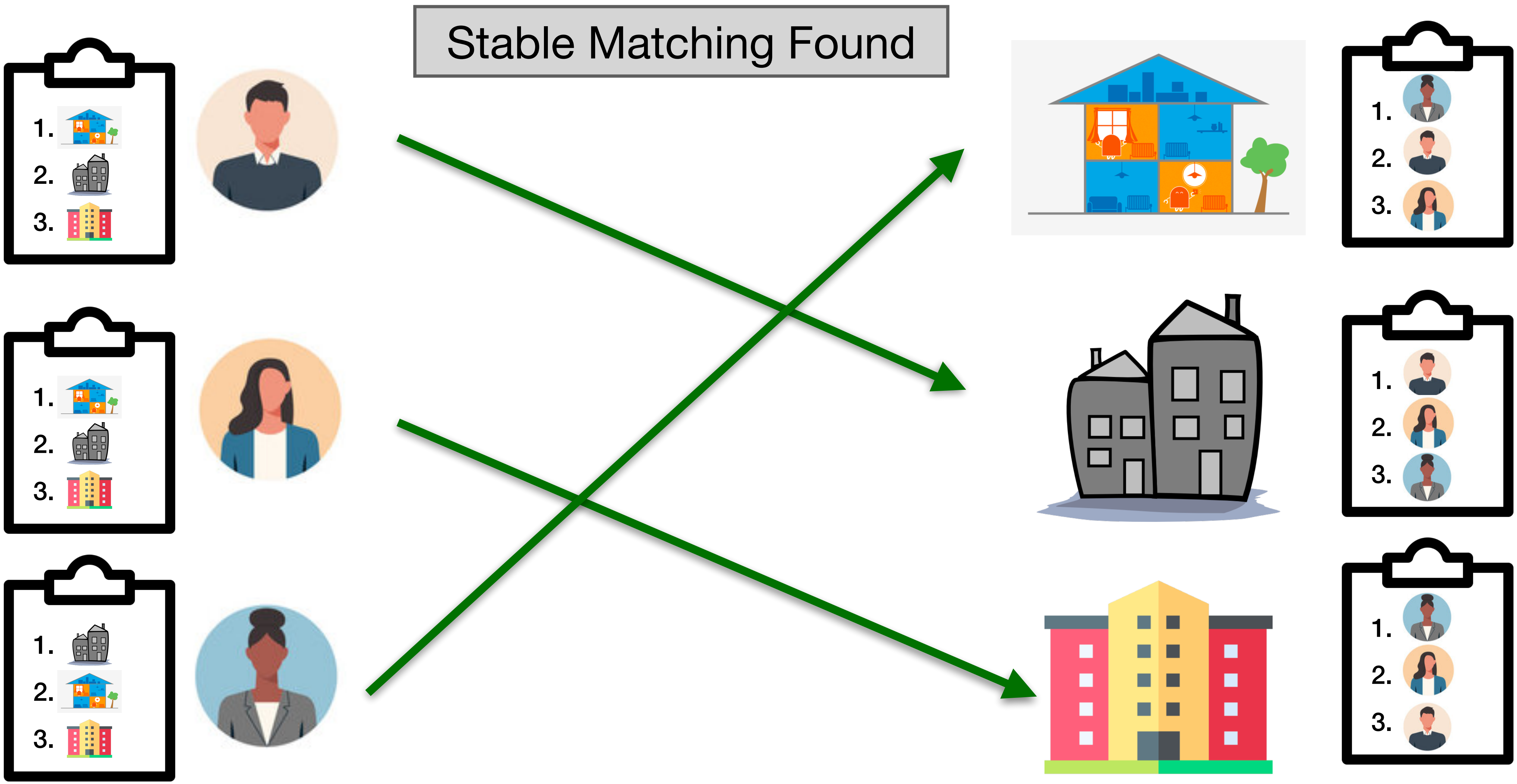
Classic Stable Matching Problem



Classic Stable Matching Problem



Classic Stable Matching Problem



- Output matching is **applicant optimal** and **job pessimal**

Switch to hospital-proposing-to-students DA

GALE–SHAPLEY (*preference lists for hospitals and students*)

INITIALIZE M to empty matching.

WHILE (some hospital h is unmatched and hasn't proposed to every student)

  $s \leftarrow$ first student on h's list to whom h has not yet proposed.

  IF (s is unmatched)

    Add h–s to matching M.

  ELSE IF (s prefers h to current partner h')

    Replace h'–s with h–s in matching M.

  ELSE

    s rejects h.

RETURN stable matching M.

Deferred Acceptance Properties

Lemma 1. DA algorithm always produces a stable matching.

Proof. (By contradiction) Let M be the resulting matching. Suppose $\exists (h, s)$ such that $(h, s'), (h', s) \in M$ and

- h prefers s over s' and s prefers h over h'

Thus h must have offered to s before s'

- Either s broke the match to h at some point for some h'', or s already had a match h'' that s preferred over h

But students always trade up, so s must prefer final match h' over h'', which they prefer over h. $(\Rightarrow\Leftarrow)$ ∎

Deferred Acceptance Properties

- The deferred-acceptance algorithm does not specify the order in which the hospitals should make offers
- Do all orders produce the same unique matching?
- Given an input instance, there may be several stable matchings.
- **Question.** Does Gale-Shapely produce the "best matching" for hospitals or students?
- Turns out hospital-proposing algorithm produces a unique matching that is hospital optimal and student pessimal
    - Matches hospital to "best achievable" student and student to "worst-achievable" hospital among all stable matchings

Best Achievable Partner

- **Lemma.** $M^* = \{(h, \text{best}(h)) \mid h \in H\}$ is the unique output of the hospital-proposing deferred-acceptance algorithm.
- **Proof (By Contradiction).** Suppose k is the first round where a hospital h is rejected by $s^* = \text{best}(h)$
  - s^* instead holds on to offer from h'
- **Claim.** $s^* = \text{best}(h')$
  - Suppose not, suppose $s' = \text{best}(h')$, then since h' proposed to s^* by round k, h' must have proposed to s' before round k and already been rejected
  - This contradicts the k is the first round where a hospital h is the first hospital to be rejected by $\text{best}(h)$ ∎

Best Achievable Partner

- **Lemma.** $M^* = \{(h, \text{best}(h)) \,|\, h \in H\}$ is the unique output of the hospital-proposing deferred-acceptance algorithm.
- **Proof (By Contradiction).** Suppose k is the first round where a hospital h is rejected by $s^* = \text{best}(h)$
    - s^* instead holds on to offer from h'
- **Claim.** $s^* = \text{best}(h')$
- Let M be a stable matching s.t. $(h, s^*) \in M$
- **Claim.** (h', s^*) is a blocking pair for matching M, why?
    - s^* prefers h' to h because they rejected h in M^* for h', and h' prefers s^* to all other stable partners, including one in M ($\Rightarrow\!\Leftarrow$) $\blacksquare$

Pareto Efficiency

- Are stable matchings Pareto optimal?
  - Not among all matchings, that is, an unstable matching may Pareto dominate a stable matching
  - Example 1 in the School Choice paper
- **Lemma.** Let M^* be the output of the hospital-proposing deferred-acceptance algorithm on input I, then M^* is not Pareto dominated by any other stable matching on I.
  - Ideas on why this holds, how to prove it?

Strategyproofness and Stability

- **Question.** Is truthful reporting a dominant strategy for hospitals in a hospital-proposing DA?
  - Yes, while intuitive, this can be surprisingly annoying to prove
  - See Theorem 10.6.18 in http://www.masfoundations.org/mas.pdf
  - Challenge: stability is wrt to reported preferences
  - Proof is simpler if you allow "short lists" (agents to cut off their preference lists)
  - We will develop this proof on the next assignment

Strategyproofness and Stability

- **Question.** Is truthful reporting a dominant strategy for students in a hospital-proposing DA?
    - No, let's do a counter example as an exercise

Class Exercise

- Consider the following truthful preference profile
- Does there exist a student such that if they reported a different preference profile, they would get a better match (all else fixed)?

	1st	**2nd**	**3rd**
MA	Beth	Aamir	Chris
NH	Aamir	Chris	Beth
OH	Aamir	Beth	Chris

	1st	**2nd**	**3rd**
Aamir	MA	OH	NH
Beth	OH	MA	NH
Chris	MA	OH	NH

Class Exercise

- Stable matching under **truthful preferences**:
  - (MA, Beth), (NH, Chris), (OH, Aamir)

	1st	**2nd**	**3rd**
MA	Beth	Aamir	Chris
NH	Aamir	Chris	Beth
OH	Aamir	Beth	Chris

	1st	**2nd**	**3rd**
Aamir	MA	OH	NH
Beth	OH	MA	NH
Chris	MA	OH	NH

Class Exercise

- Suppose Amir misreports: swaps NH and OH
- New stable matching?
    - (MA, Aamir), (NH, Chris), (OH, Beth)

DA is not strategyproof (the receiving side can misreport and achieve a better match)

	1st	2nd	3rd
MA	Beth	**Aamir**	Chris
NH	Aamir	**Chris**	Beth
OH	Aamir	**Beth**	Chris

	1st	2nd	3rd
Aamir	**MA**	NH	OH
Beth	**OH**	MA	NH
Chris	MA	OH	**NH**

Can't Have Both

- Can there be a mechanism that is both strategy proof and stable?
  - Unfortunately, no
- **Theorem.** No mechanism for two-sided matching is both stable and strategyproof.
  - Proof partly developed in Assignment 3
- Many interesting questions:
  - How much information is needed to find a useful manipulation?
  - What is the optimal manipulation cheating strategy
- Empirically manipulations do not play a large role
  - If not many stable partners, can't gain much

Evolution of the Match

- NRMP Revisited. The original 1952 implementation of the DA algorithm was the hospital-optimal version
- Students protested that the match was favoring hospitals

MONDAY, OCTOBER 22, 1951. The New York Times MONDAY, OCTOBER 22, 1951. 25 L+

MEDICAL SENIORS HIT INTERNE PLAN

Delegates of 44 Schools Meet Here to Protest Selection by 'Matching Machine'

TEACHERS PRAISE SYSTEM

They Argue That It Bars Unfair Recruiting—Students Insist on Choosing Their Hospitals

Delegates representing seniors in nearly all of the country's leading medical schools met here yesterday to express overwhelming opposition to a proposed mathematically contrived plan to place medical students in hospitals as internes.

They indicated that a great majority of their classmates preferred the present system whereby the country's hospitals, which have 10,000 internships, scramble for the best of a year's 6,000 medical graduates.

The meeting was held at Bard Hall of the College of Physicians and Surgeons of Columbia University, but it was made clear that the university was not its sponsor.

Seventy students attended the meeting at which forty-four colleges were represented either by delegates or through communications giving the opinion of the medical school's seniors on "the matching plan for internship" organized by the National Interassociation Committee on Internship about two years ago.

A prospectus of the plan of leading hospitals calls it "the accepted procedure for 1951-52" in determining which medical graduate shall go to what hospital to complete his medical education. Medical men said yesterday that its success or failure would have much to do with the chances of a medical student properly to complete his education during a wartime emergency.

By Winston Churchill: *The Second World War*

Volume V—Closing the Ring

INSTALLMENT 15—TEHERAN: CONCLUSIONS

Book II—Teheran to Rome

NEED FOR TEACHERS EXPECTED TO GROW

1,200 More a Year Required in State, Board Officials Say —Triple Sessions Feared

LACK IS WORST IN GRADES

Special Subjects Also Suffer— Syracuse Parley Cites High Birth Rate, Low Salaries

ROOSEVELT, STALIN, CHURCHILL AGREE ON PLANS FOR WAR ON GERMANY IN TALKS AT TEHERAN; 1,500 MORE TONS OF BOMBS DROPPED ON BERLIN

AGREEMENT: The news on the morning of Dec. 4, 1943.

Evolution of the Match

- A new algorithm was adopted in 1997
  - Primary motivated was to give couples the option to get placed in geographically nearby programs
  - But in addition was made student-proposing
- Changes incentives for hospitals, but did it make a difference?
- Empirically, at least for the datasets arising in NRMP, less than 1% of the hospitals could have benefited by misreporting

Stable Matching Summary

- Hospital-proposing DA is hospital-optimal and student pessimal, among all stable matchings (regardless of the order of proposals)
- Stability matchings are not Pareto optimal overall, but are Pareto optimal among the set of all stable matchings
- Stable matchings are only strategyproof for the proposing side and cannot be strategyproof for both sides
- Lots of generalizations:
    - Incomplete preferences and ties
    - Stable "roommates" problem
    - Many-to-one stable matchings
    - Approximately stable matchings

Active Area of Research

Mar 2024

Structural Complexities of Matching Mechanisms*

Yannai A. Gonczarowski[†] Clayton Thomas[‡]

March 30, 2024

Abstract

We study various novel complexity measures for two-sided matching mechanisms, applied to the two canonical strategyproof matching mechanisms, Deferred Acceptance (DA) and Top Trading Cycles (TTC). Our metrics are designed to capture the complexity of various structural (rather than computational) concerns, in particular ones of recent interest within economics. We consider a unified, flexible approach to formalizing our questions: Define a protocol or data structure performing some task, and bound the number of bits that it requires. Our main results apply this approach to four questions of general interest; for mechanisms matching applicants to institutions, our questions are:

(1) How can one applicant affect the outcome matching?

(2) How can one applicant affect another applicant's set of options?

(3) How can the outcome matching be represented / communicated?

(4) How can the outcome matching be verified?

2021

Deferred Acceptance with Compensation Chains

PIOTR DWORCZAK, Stanford University, Graduate School of Business

I introduce a class of algorithms called Deferred Acceptance with Compensation Chains (DACC). DACC algorithms generalize the DA algorithms by Gale and Shapley [1962] by allowing both sides of the market to make offers. The main result is a characterization of the set of stable matchings: a matching is stable if and only if it is the outcome of a DACC algorithm.

Nov 2024

Stable Matching with Ties: Approximation Ratios and Learning

Shiyun Lin * Simon Mauras † Nadav Merlis ‡ Vianney Perchet §

November, 2024

Abstract

We study the problem of matching markets with ties, where one side of the market does not necessarily have strict preferences over members at its other side. For example, workers do not always have strict preferences over jobs, students can give the same ranking for different schools and more. In particular, assume w.l.o.g. that workers' preferences are determined by their utility from being matched to each job, which might admit ties. Notably, in contrast to classical two-sided markets with strict preferences, there is no longer a single stable matching that simultaneously maximizes the utility for all workers.

We aim to guarantee each worker the largest possible share from the utility in her best possible stable matching. We call the ratio between the worker's best possible stable utility and its assigned utility the *Optimal Stable Share* (OSS)-ratio. We first prove that distributions over stable matchings cannot guarantee an OSS-ratio that is sublinear in the number of workers. Instead, randomizing over possibly non-stable matchings, we show how to achieve a tight logarithmic OSS-ratio. Then, we analyze the case where the real utility is not necessarily known and can only be approximated. In particular, we provide an algorithm that guarantees a similar fraction of the utility compared to the best possible utility. Finally, we move to a bandit setting, where we select a matching at each round and only observe the utilities for matches we perform. We show how to utilize our results for approximate utilities to gracefully interpolate between problems without ties and problems with statistical ties (small suboptimality gaps).

Jan 2025

UNBALANCED RANDOM MATCHING MARKETS WITH PARTIAL PREFERENCES

ADITYA POTUKUCHI AND SHIKHA SINGH

ABSTRACT. Properties of stable matchings in the popular random-matching-market model have been studied for over 50 years. In a random matching market, each agent has complete preferences drawn uniformly and independently at random. Wilson (1972), Knuth (1976) and Pittel (1989) proved that in balanced random matching markets, the proposers are matched to their $\ln n$th choice on average. In this paper, we consider markets where agents have partial (truncated) preferences, that is, the proposers only rank their top d partners. Despite the long history of the problem, the following fundamental question remained unanswered: *what is the smallest value of d that results in a perfect stable matching with high probability?* In this paper, we answer this question exactly—we prove that a degree of $\ln^2 n$ is necessary and sufficient. That is, we show that if $d < (1-\varepsilon)\ln^2 n$ then no stable matching is perfect and if $d > (1+\varepsilon)\ln^2 n$, then every stable matching is perfect with high probability. This settles a recent conjecture by Kanoria, Min and Qian (2021).

We generalize this threshold for unbalanced markets: we consider a matching market with n agents on the shorter side and $n(\alpha+1)$ agents on the longer side. We show that for markets with $\alpha = o(1)$, the sharp threshold characterizing the existence of perfect stable matching occurs when d is $\ln n \cdot \ln\left(\frac{1+\alpha}{\alpha+(1/n(\alpha+1))}\right)$.

Finally, we extend the line of work studying the effect of imbalance on the expected rank of the proposers (termed the "stark effect of competition"). We establish the regime in unbalanced markets that forces this stark effect to take shape in markets with partial preferences.