CSCI 357: HW 4 (due 03/11/2025) **Your Name:** ______________________

Applying Revenue Equivalence

A corollary of the Myerson's lemma for Bayes Nash equilibrium (BNE) is that that as long as two auctions have the same allocation at BNE, they have the same expected payments, and thus the same revenue that is generated at equilibrium.

One application of this corollary is that it lets us reason about the equilibrium of auctions that are harder to analyze by equating their expected payments at equilibrium to that of a strategically-simple dominant-strategyproof auction where bidders bid their value.

Steps to Use Revenue Equivalence for BNE. As an example of how revenue equivalence is useful to "guess" the BNE of an auction, let's use it to recreate the BNE for first-price auction.

- **Step 1.** Guess what the allocation might be in a Bayes-Nash equilibrium (usually a welfare-maximizing allocation). In this case, we guess that the highest-valued bidder wins in a first-price auction at equilibrium.

- **Step 2.** Write down the expression for the expected payment of a bidder as a function of their value in a strategically-simpler auction (usually the dominant-strategyproof version). In this case, consider bidder i, their expected payment in a second-price auction is

$$\begin{aligned} E[p_i] &= E[\text{second highest bid} \mid v_i \text{ wins}] \cdot \Pr[v_i \text{ wins}] + 0 \cdot \Pr[v_i \text{ loses}] \\ &= E[\text{second highest value} \mid v_i \text{ wins}] \cdot \Pr[v_i \text{ wins}] \\ &= E[X_{(1)} \text{ in } n-1 \text{ samples from } [0, v_i]] \cdot \Pr[v_i \text{ wins}] \\ &= \frac{n-1}{n} v_i \cdot \Pr[v_i \text{ wins}] \end{aligned}$$

  Here the last equality is because the expected value of the second-highest bidder given v_i is the highest bidder is the same as the expected value of the first-order statistic when $n-1$ samples are drawn from a uniform distribution on $[0, v_i]$. Let $X_1, \ldots, X_n'$ denote n' samples drawn i.i.d. from the uniform distribution on $[0, b]$. Let X_k be the kth largest value among them, then the last step follows from using $E[X_{(k)}] = \frac{n'-(k-1)}{n'+1} b$ with $n' = n-1$ and $k = 1$.

- **Step 3.** Write the expression for the expected payment in terms of the strategy of the bidder in the auction you are trying to solve for the BNE. In this case, a bidder i pays their bid in a first-price auction if they win and pay zero otherwise. Assuming a bidder i shades their bid down by a factor α of their value, their expected payment is

$$\begin{aligned} E[p_i] &= E[b_i \mid v_i \text{ wins}] \cdot \Pr[v_i \text{ wins}] + 0 \cdot \Pr[v_i \text{ loses}] \\ &= E[\alpha v_i \mid v_i \text{ wins}] \cdot \Pr[v_i \text{ wins}] = \alpha v_i \cdot \Pr[v_i \text{ wins}] \end{aligned}$$

In the last step, $E[\alpha v_i] = \alpha v_i$ because of linearity of expectation and because bidder i knows their own value (the expected utility is over their uncertainty of other bidders bids and values).

- **Step 4.** Solve for the BNE strategy by equating the expected payments between the two auctions. In this case, setting the expected payments equal in first and second price auction, assuming that the probability of winning is the same in both, we get $\alpha = \frac{n-1}{n}$, and thus revenue equivalence suggests that $s(v_i) = \frac{n-1}{n} v_i$ should be the symmetric Bayes Nash equilibrium of the first-price auction with n bidders.

- **Step 5.** Finally, verify that this strategy profile computed in Step 4 is actually a symmetric BNE. We did this step in lecture for the first-price auction.

Using Revenue Equivalence for kth-price auction. Consider a single-item auction with n bidders where the highest bidder wins and pays the kth-highest bid, where $2 \leq k < n$.

Problem 1. Using Steps 1-4 in the approach described above, solve for the symmetric Bayes' Nash equilibrium strategies of the kth-price auction when the agent values are drawn i.i.d. from $U(0, 1)$.

State your answer in the box below.

The symmetric BNE of the k-th price auction is when each bidder i bids as follows:

$s(v_i) =$

Remark. Notice that for $k = 2$ this reduces to truthful bidding (as expected) and for $k > 2$, surprisingly the bidders bid *above* their value at equilibrium!

You may show your steps below for partial credit.