CS 357: Algorithmic Game Theory **Spring 2025**

Assignment 4 (due 04/25/2025)

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LaTeX Template (Source): https://www.overleaf.com/read/qtzccrsxxxxx#690906

Instructions. This is a **partner assignment**. You must use the LaTeX solution template provided to write your answers and submit a joint PDF with your partner on Gradescope. Points will be awarded for *clarity, correctness and completeness* of the answers. Assignments are different from homeworks and **formal proofs** are expected for each question. This assignment is due Friday (04/25) at noon on Gradescope.

Problem 1. In this question, we consider a voting rule, *Schulze rule* that is a bit complicated to state, but satisfies most of the desirable criteria among preferential voting systems, e.g. Condorcet, Independence of clones, polynomial-time computability, etc.

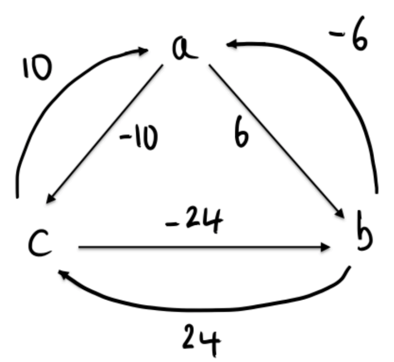
A *weighted-majority graph* is defined as follows: the candidates are the nodes, and there is a directed edge from a to b with weight w_{ab} = (no. of voters who prefer a to b) - (no. of voters who prefer b to a).

The *strength* of a path is defined as the weight of the **least-weight** edge on it. Let $S(x, y)$ be the maximum strength among all paths from x to y.

A candidate a **chain beats** a candidate b if $S(a, b) > S(b, a)$. The Schulze winner is a candidate that chain beats all others (such a winner is surprisingly guaranteed to exist).[1]

Consider an input with three candidates $\{a, b, c\}$, and sixty voters with the following breakdown of ranked orders (on the left), and the corresponding weighted-majority graph (on the right):

23	17	10	8	2
a	b	c	c	b
b	c	a	b	a
c	a	b	a	c



(a) Determine the Schulze winner in the above example by computing the $S(x, y)$ values.

(b) Prove that the Schulze rule is Condorcet consistent.

[1] Ties can be broken in a consistent way.

Problem 2. Consider the greedy strategy to solve the f-manipulation problem, when the social-choice rule f is the Borda rule:

Fix the ranked lists L_{-i} of all other voters. Compute the Borda score s_j of each alternative j under preference lists L_{-i}. Construct the misreport L'_i as follows: place i's favorite candidate a in the top position and rank the other alternatives in ascending order of their Borda scores s_j (that is highest-score candidate goes last).

We say the greedy algorithm is successful if it causes a to win whenever it is possible, given L_{-i}. Prove that the greedy algorithm successfully solves the f-manipulation for Borda rule. (*Hint.* Consider a list L_i^* such that a wins. Show that L_i^* can be transformed to L'_i through a series of swaps, such that a continues to win. This should is similar to how we prove greedy is optimal through exchange argument in CS256.)

Problem 3. In this problem, we look more closely at the two fairness criterion for fair division of divisible goods: envy-freeness and proportionality.

(a) Consider the cake cutting model from lecture. Formally prove the lemma discussed in class that the envy-free allocations are always proportional, for any number of players.

(b) Next, we show that the other direction is not true when there are three or more players.

Consider the moving-knife (Dubins Spanier) algorithm for dividing a unit-interval cake $[0, 1]$ proportionally between n players: a referee slowly moves a knife right from the start position 0. At any point if the knife is at a position $c_i \in [0, 1]$ in the cake such that $v_i([0, c_i]) = 1/n$, player i raises their hand and are given the slice $[0, c_i]$ of cake.[2] Then, the protocol continues between the remaining $n - 1$ players.

Recall that in lecture, we argued that this protocol creates a proportional division for any n players: that is, each player receives $\geq 1/n$th of the entire cake according to their own valuation, where $v_i([0, 1]) = 1$ for each player i. Show through an explicit example that even for $n = 3$ players, this protocol is not envy-free.

Problem 4. Consider a single-item market with $n \geq 2$ buyers. Each buyer i has a valuation v_i for the item. Add $n-1$ dummy items that all buyers value at 0. State and prove necessary and sufficient conditions on a market-clearing price vector $\mathbf{p} = (p_1, \ldots, p_n)$ of such a market (based on the valuations of the bidders).

Problem 5. Consider a matching market with n buyers and n items where each buyer only wants a single item. Let M^* be a maximum-weight matching in the graph, that is, a matching that maximizes the welfare: $\sum_{i=1}^{n} v_{iM^*(i)} \geq \sum_{i=1}^{n} v_{iM(i)}$ for any matching M. Let $\mathbf{p} = (p_1, \ldots, p_n)$ be *any* market clearing price of this market.

Prove that (M^*, p) is a competitive equilibrium of this market. Recall that $(M, \mathbf{p})$ is competitive-equilibrium iff $\mathbf{p}$ is a market-clearing price vector and M is a matching in the preferred-item graph defined by $\mathbf{p}$. Thus, to prove this statement, you must show that $M* \subseteq E_p$, where E_p are the edges in the preferred-item graph under $\mathbf{p}$.

[2] Ties are broken according to a predetermined tie-breaking rule