CS 357: Algorithmic Game Theory **Spring 2025**

Assignment 2 (due 03/07/2025)

Instructor: Shikha Singh

LaTeX Template (Source): https://www.overleaf.com/read/ytncckzkjpyb#b48aad

Instructions. This is a **partner assignment**. You must use the LaTeX solution template provided to write your answers and submit a joint PDF with your partner on Gradescope. Points will be awarded for *clarity, correctness and completeness* of the answers. Assignments are different from homeworks and formal proofs as the justification is expected for each question. This assignment is due Friday (02/21) at noon on Gradescope.

Sequential Auctions

Many traditional auction-houses, as well as eBay, run multi-round auctions that allow bidders to respond to bids placed by others. Such auctions provide bidders with new kinds of strategically relevant information—they may be able to observe the bids of others and use it to revise their own.

Two common formats are: (a) *ascending clock* auctions (also known as English auctions), where the ask price increases in fixed increments over time, and (b) *descending clock* auctions (also known as Dutch auctions), where the ask price decreases in fixed increments.

Ascending-clock auction. Consider the following **ascending-clock auction**: the ask price starts low and increases continuously. Each bidder can drop out at any price and the auction closes when only one bidder remains. The bidder wins, and pays the ask price at which the last competitor dropped out.

This auction seems very similar to the sealed-bid second-price (SBSP) auction. However, how the ascending-clock auction is implemented—in particular, what information is visible to other bidders is crucial in formally comparing them. Recall that in a SBSP auction player values are their private information, and thus each player's strategy s_i is a function that only depends on their value v_i and maps it to an action (a bid b_i). On the other hand, in a sequential auction, their strategy can map any information available to them to their action.

We say that two auctions are **strategically equivalent** if, for any strategy profile in one auction, there exists a strategy profile in the other auction such that the outcomes (allocations and payments) are the same, for all value profiles (and vice versa). At a high level, strategic equivalence means that bidders participating in the two auctions have the same "expressive power" in terms of the strategies they can follow and its outcome (regardless of whether these strategies are *reasonable* or not).

Problem 1. Below, we compare the multi-round auctions to second-price auctions from the point of view of how bidders should participate in them.

(a) First, consider the ascending-clock auction where bidders can drop out privately (unobserved by other bidders). Show that the SPSB auction is strategically equivalent to the ascending-clock auction described above. (You need to prove both directions here.)

Solution. □

(b) Now consider an ascending-clock auction where a bidder can **observe** the drop-out points of other bidders (e.g., each bidder who is still in the game has their hand raised). Is the public-drop-out variant of the ascending-clock auction also strategically equivalent to a SBSP auction?

To help answer this, we will establish a property about **winner determination** in SBSP auctions. Fix a strategy profile s of the bidders. Let $\overline{v_1} = (v_i, v_{-i})$ and $\overline{v_2} = (v'_i, v_{-i})$ be two valuation profiles that only differ in bidder i's value ($v_i \neq v'_i$).[1]

Let w_1 be the winner of SBSP when bidders bid under $\overline{v_1}$ and w_2 when bidders bid under $\overline{v_2}$ (using the same strategy profile s). Show that in a SBSP auction, a change in a bidders value can only affect their own allocation. In particular, show that either the winner is unaffected: $w_1 = w_2$ or i is be the winner in one of them: $w_1 = i$ or $w_2 = i$.

Solution. □

(c) Using part (b), show that the ascending-clock auction where a bidder can observe the bids of others is **not strategically** equivalent to a SBSP auction. (*Hint.* Give a strategy profile for which the allocation cannot be achieved in any strategy in the SPSB auction.)

Solution. □

[1] As strategies map values to bids, this also changes bidder i's bids: $b_i = s_i(v_i)$ and $b'_i = s_i(v'_i)$.

Welfare Maximization and Externality Pricing

Problem 2. (Nisan, Roughgarden and Tardos)

(a) Consider an arbitrary single-parameter environment, with feasible allocation $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Given bids $\mathbf{b} = (b_1, \ldots, b_n)$, the welfare-maximizing allocation rule is $\mathbf{x}(\mathbf{b}) = \operatorname{argmax}_{(x_1,\ldots,x_n)\in X} \sum_{i=1}^{n} b_i x_i$. Prove that this allocation rule is monotone.[2] (*Hint.* Consider a player i, and fix $\mathbf{b}_{-i}$. Increase i's bid from b_i to b_i', where $b_i' = b_i + \delta$ and $\delta > 0$, and show that i's allocation cannot get worse.)

Solution. □

(b) Continuing part (a), we consider a welfare maximizing allocation rule $\mathbf{x}(\mathbf{b})$ and now restrict to feasible allocations X that contain only 0-1 vectors—that is, each bidder either wins or loses.

In this case, Myerson's payment rule can be written as:

$$p(b_i, \mathbf{b}_{-i}) = \begin{cases} 0 & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 0 \\ b_i^*(\mathbf{b}_{-i}) & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 1 \end{cases}$$

where $b_i^*(\mathbf{b}_{-i})$ is the bidder i's *critical bid*, that is, the lowest bid at which i gets a non-zero allocation.

Given feasible allocations containing 0-1 vectors, we can identify each feasible allocation with a "winning set" of bidders (the set of bidders i with $x_i = 1$ in that allocation).

Prove that, when S^* is the set of winning bidders and $i \in S^*$, then i's critical bid $b_i^*(\mathbf{b}_i)$ equals the difference between

(i) the maximum welfare of a feasible allocation that excludes i[3]—that is, the maximum welfare that can be generated if i was not present

(ii) the welfare $\sum_{j\in S^*\setminus\{i\}} b_j$ generated by the winners (other than i) in the chosen outcome S^*—that is, the welfare that is generated (by others) given i wins

Hint. Write down the welfare-maximizing allocation for the two cases when (a) $x_i = 1$ ($i \in S^*$) and (b) when $x_i = 0$ ($i \notin S^*$). The optimal allocation will choose whether or not to allocate to i (that is, when to switch from (b) to (a)) at a bid b_i where the welfare generated by case (a) is at least as good as the welfare generated by case (b).

Remark. In other words, a winning bidder pays their "externality"—the welfare loss they impose on others.

Solution. □

[2] Assume that ties are broken in a deterministic and consistent way, such as lexicographically.
[3] You should assume that there is at least one such feasible allocation.

Approximation Algorithms for NP Hard Auctions

Problem 3. (Roughgarden) Knapsack auctions are another widely applicable example of single-parameter mechanisms.

In a knapsack auction, each bidder i has a **publicly known** size w_i and a private valuation v_i. The seller has a capacity W. A feasible allocation X is defined as the 0-1 vectors $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} w_i \cdot x_i \leq W$. (As usual, $x_i = 1$ indicates that i is a winning bidder.) The goal is to design allocation and payment rules as to (a) maximize welfare $\sum_{i=1}^{n} x_i v_i$ subject to the capacity constraints, and (b) elicit truthful bids (dominant strategyproof).

Knapsack auctions come up whenever there is a shared resource with limited capacity. For example, each bidder's size could represent the duration of an advertisement, the valuation their willingness-to-pay for its ad being shown during the Super Bowl, and the seller capacity the length of a commercial break.

Unfortunately, problem of finding an allocation that maximizes welfare (the well-known Knapsack problem) is NP-hard. Thus, we resort to approximation algorithms.

Consider the following greedy approximation algorithm from lecture:

- Sort and relabel bidders such that:
$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \ldots \geq \frac{b_n}{w_n}$$
- In the above order, greedily try to fit as many bidders in the Knapsack as possible—let Q be the set of jobs that are chosen in this way. Let $S = \sum_{i \in Q} b_i$ denote the welfare generated by these bidders, assuming truthful bids.
- Let $i_{\max}$ be the single job with the highest bid $b_{\max}$. If $b_{\max} > S$, then allocate only to $i_{\max}$. Otherwise, allocate to bidders in Q.

(a) Assuming truthful bidding, show that this greedy algorithm is a 2-approximation: that is, if OPT is the maximum welfare possible by any feasible allocation, then this greedy algorithm generates welfare that is at least $1/2 \cdot \text{OPT}$. (*Hint.* Consider a Knapack with a larger capacity $W' \geq W$, then $\text{OPT}(W') \geq \text{OPT}(W)$. Use this and show that $S + b_{\max} \geq \text{OPT}$.)

Solution.

□

(b) We would like to use the 2-approximation algorithm to design a dominant strategyproof mechanism. Myerson's lemma tells us that a allocation can be made dominant strategyproof iff it is monotone. Argue that the above greedy 2-approximation scheme is monotone.

Solution. □

Problem 4. Consider a set of M distinct items and n bidders. Each bidder i wants a **publicly known** subset $S_i \subseteq M$ of items and has a **private value** v_i for obtaining it. If bidder i is given a subset T_i at a total price p, her utility is $v_i - p$ if $S_i \subseteq T_i$, and 0 otherwise: that is, the bidders want **all** the items in their desired subset or nothing at all. Each item can only be awarded to one bidder. Thus, a subset B of bidders can all receive their desired subsets S_i simultaneously if and only if $S_i \cap S_j = \emptyset$ for each distinct $i, j \in B$.[4]

Turns out that the problem of computing a welfare-maximizing feasible allocation is NP-hard.[5] Consider the following greedy heuristic for this problem:

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GreedyAllocation((b_1, ..., b_n), (S_1, ..., S_n)):

Initialize W = ∅ and X = M
Sort and relabel the bidders such that b_1 ≥ b_2 ≥ ... b_n
for  i = 1, 2 ..., n do
        if i's subset is still available, that is, S_i ⊆ X then
                remove S_i from X and add i to W
return the winning bidders W
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(a) Does this algorithm define a monotone allocation rule? Just state yes or no and convince yourself that if you had to provide a proof, you could. There are already two monotonicity proofs in this assignment so we will skip a formal proof here.

(b) Assuming truthful bidding, does the greedy allocation algorithm maximize the social welfare? Prove it or give an explicit counterexample.

(c) Using Myerson's lemma, give an expression for the payment that a winning bidder $i \in W$ should be charged, in terms of the subsets S_j and bids b_j of the bidders.

Solution. □

[4] This is a single-parameter setting with 0/1 allocations as a bidder either wins her desired subset or not.

[5] A reduction from a known NP hard problem such as maximum independent set works to show this.

Citation Sources

Using question-specific prompts on the Internet is a violation of the honor code. If you referred to lecture notes or other resources on the topic and you find that information helpful towards these questions, you must cite them here.