

Matchings and Copeland’s Method

Telikepalli Kavitha and Rohit Vaish

Tata Institute of Fundamental Research, Mumbai, India
{kavitha,rohit.vaish}@tifr.res.in

Abstract. Given a graph $G = (V, E)$ where every vertex has a weak ranking over its neighbors, we consider the problem of computing an *optimal* matching as per agent preferences. The classical notion of optimality in this setting is *stability*. However stable matchings, and more generally, *popular* matchings need not exist when G is non-bipartite. Unlike popular matchings, *Copeland winners* always exist in any voting instance—we study the complexity of computing a matching that is a Copeland winner and show there is no polynomial-time algorithm for this problem unless $P = NP$.

We introduce a relaxation of both popular matchings and Copeland winners called *weak Copeland winners*. These are matchings with Copeland score at least $\mu/2$, where μ is the total number of matchings in G ; the maximum possible Copeland score is $(\mu - 1/2)$. We show a fully polynomial-time randomized approximation scheme to compute a matching with Copeland score at least $\mu/2 \cdot (1 - \varepsilon)$ for any $\varepsilon > 0$.

Keywords: Popular matching · Copeland score · FPRAS.

1 Introduction

Matching problems with preferences are of central importance in economics, computer science, and operations research [19,28,36]. Over the years, these problems have found several real-world applications such as in school choice [1,2], labor markets [34,35], and dormitory assignment [33]. We assume the input instance is a graph $G = (V, E)$ where the vertices correspond to *agents*, each with a weak ranking of its neighbors. The goal is to divide the agents into pairs, i.e., find a matching in G , while optimizing some criterion of agent satisfaction based on their preferences.

A classical criterion of agent satisfaction in the matching literature is *stability*, where no pair of agents mutually prefer each other over their prescribed matches [15]. Stability is an intuitively appealing notion, but it can be too demanding in the context of general (i.e., not necessarily bipartite) graphs, also known as *roommates instances*. Indeed, there exist simple roommates instances that do not admit any stable matching (Fig. 1a and Fig. 1b).

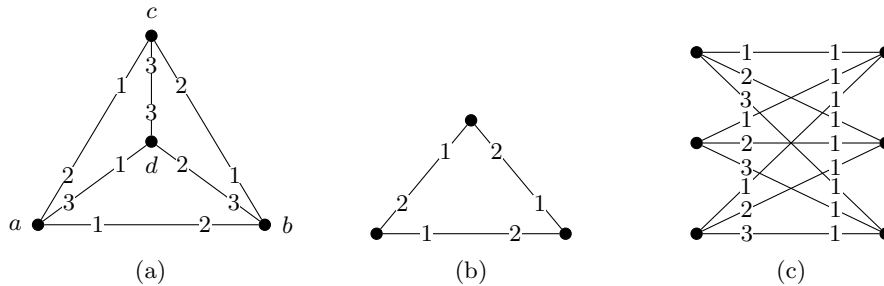


Fig. 1: (a) An instance with no stable matching but with two popular matchings $\{(a, d), (b, c)\}$ and $\{(a, c), (b, d)\}$. The numbers on the edges denote how the vertices rank each other, where a lower number denotes a more preferred neighbor. (b) A roommates instance without a popular matching. (c) A bipartite instance (with ties) without a popular matching.

Popularity is a natural relaxation of stability that captures welfare in a collective sense [16]. Intuitively, popularity asks for a matching that is not *defeated* by any matching in a head-to-head comparison. More

concretely, consider an election in which the matchings play the role of candidates and the agents/vertices act as voters. Given a pair of matchings M and N , each agent votes for the matching in $\{M, N\}$ that it prefers, i.e., where it gets a more preferred assignment, and abstains from voting if it is indifferent. Note that being left unmatched is the worst choice for any agent. In the M -vs- N election, let $\phi(M, N)$ be the number of votes for M and let $\phi(N, M)$ be the number of votes for N . Further, let $\Delta(M, N) := \phi(M, N) - \phi(N, M)$. We say that the matching N *defeats* (or is *more popular* than) the matching M if $\Delta(N, M) > 0$. A *popular* matching is one such that there is *no* “more popular” matching.

Definition 1 (Popular matching). *A matching M is popular if there is no matching that is more popular than M , i.e., $\Delta(M, N) \geq 0$ for all matchings N .*

Thus, a popular matching is a *weak Condorcet winner* in the underlying head-to-head election among matchings [9].¹ Note that under strict preferences, a stable matching is also popular [6], but a popular matching can exist in instances with no stable matching; e.g., there are two popular matchings $\{(a, d), (b, c)\}$ and $\{(a, c), (b, d)\}$ in Fig. 1a. Thus popularity is a more relaxed criterion than stability—it ensures “collective stability” as there is no matching that makes more agents better off than those who are worse off.

Unfortunately, the popularity criterion also suffers from similar limitations as stability and more. First, although popular matchings always exist in a bipartite graph with strict preferences [16], they could *fail to exist* with weak rankings, i.e., when preferences include ties (Fig. 1c) or when the graph is non-bipartite (Fig. 1b). Second, determining the existence of a popular matching is known to be NP-hard in roommates instances with strict preferences [13,18] and in bipartite instances with weak rankings [3,10].

Relaxations of popularity. A natural relaxation of popularity is *low unpopularity* and two of the well-known measures for quantifying the unpopularity of a matching are *unpopularity margin/factor* [29] that bound the additive/multiplicative gap of the worst pairwise defeat suffered by the matching. A popular matching has unpopularity margin 0 and unpopularity factor at most 1. It is easy to construct a bipartite instance on n vertices with weak rankings where every matching has unpopularity margin/factor $\Omega(n)$. Thus it can be the case that every matching suffers a heavy defeat against some other matching.

An intriguing alternative is to ask for a matching M such that $\Delta(M, N) \geq 0$ for a large number of matchings N . This is closely related to the notion of *semi-popularity* [26].

Definition 2 (Semi-popular matching). *A matching M is said to be semi-popular if $\Delta(M, N) \geq 0$ for at least half the matchings N in G .*

Being undefeated by a majority of matchings is a natural approximation of popularity. Unlike popular matchings, do semi-popular matchings always exist in any roommates instance with weak rankings? Is it easy to find one?

We answer these questions via a new class of matchings that we introduce here—this class is sandwiched between the set of popular matchings and the set of semi-popular matchings. To define these matchings, we draw inspiration from the voting interpretation of popular matchings.

Copeland’s method. The Copeland rule is a well-known *Condorcet-consistent* voting rule (i.e., it selects a Condorcet winner whenever one exists) that has a long history. Ramon Llull in the 13th century came up with a variant of this method in his treatise *Ars Electionis*. It was discussed by Nicholas of Cusa in the 15th century and by the Marquis de Condorcet in the 18th century. It is named after Arthur H. Copeland who advocated it independently in a 1951 lecture [7,20]. Variants of the Copeland rule are used in sports leagues around the world. Below we define this method in the setting of matchings.

For any matching M in the graph G , let $\text{wins}(M)$ (resp., $\text{ties}(M)$) be the number of matchings that are defeated by (resp., tie with) M in their head-to-head election. The *Copeland score* of a matching M is defined as $\text{score}(M) := \text{wins}(M) + \text{ties}(M)/2$. That is, the Copeland rule assigns one point for every win, half a point for every tie (this includes comparing the matching against itself), and none for a loss in a head-to-head comparison.

¹ A matching M is a *Condorcet winner* if $\Delta(M, N) > 0$ for every matching $N \neq M$ in G , and a *weak Condorcet winner* if $\Delta(M, N) \geq 0$ for every matching N in G .

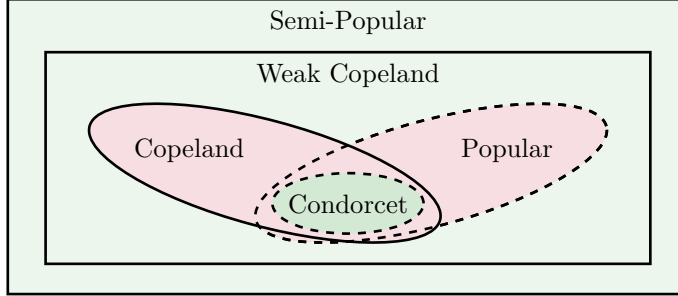


Fig. 2: Relationship among the various notions mentioned in this paper. A solid (resp., dashed) border indicates that the property is guaranteed to exist (resp., could fail to exist). Computational tractability (resp., intractability) is indicated via green (resp., red) color. Additionally, we use a lighter shade of green (for the two boxes) to denote tractability of the “almost” variant of the problem.

A matching with the maximum value of $\text{score}(\cdot)$ in the given instance will be called a *Copeland winner*. Social choice theory tells us that a Copeland winner satisfies many standard desirable properties [5,8,32]. A matching that is a Copeland winner is a natural candidate for optimizing the collective satisfaction of agents. Observe that for any matching M , we have $\text{score}(M) \leq \mu - 1/2$, where μ is the total number of matchings in G .² Interestingly, there is a polynomial time algorithm [4] to decide if there exists a matching M with $\text{score}(M) = \mu - 1/2$ (such an M has to be a Condorcet winner).³

What is the complexity of finding a Copeland winner in general? We consider this question here.

Theorem 3 (Hardness of Copeland winner). *Unless $P = NP$, there is no polynomial-time algorithm for finding a Copeland winner in a roommates instance with weak rankings.*

In spite of the above hardness result, Copeland’s method is useful to us to come up with the following relaxation of Copeland winners that includes all popular matchings. Observe that $\text{score}(M) \geq \mu/2$ for any popular matching M .

Definition 4 (Weak Copeland winner). *A matching M is a weak Copeland winner in G if $\text{score}(M) \geq \mu/2$.*

So, a weak Copeland winner defeats at least $\mu/2 - k$ matchings and ties with at least $2k$ matchings, for some $0 \leq k \leq \mu/2$. Note that the score of a weak Copeland winner is more than half of the *highest achievable* score of $\mu - 1/2$.

Observe that $\{\text{popular matchings}\} \subseteq \{\text{weak Copeland winners}\} \subseteq \{\text{semi-popular matchings}\}$. We show that weak Copeland winners, and thus semi-popular matchings, are *guaranteed to exist*. The universal existence of a weak Copeland winner implies, in particular, that the Copeland score of a Copeland winner is always at least $\mu/2$. See Fig. 2 for an illustration of relaxations among the various notions mentioned here.

Theorem 5 (Existence of a weak Copeland winner). *Every roommates instance admits a weak Copeland winner.*

The proof of Theorem 5 uses an averaging argument over the space of all matchings and is non-constructive. Thus, the above existence result for weak Copeland winners does not automatically provide an efficient algorithm for computing such outcomes. To address this, we provide a randomized algorithm that, with high probability, returns a matching that is *almost* weak Copeland.

Theorem 6 (FPRAS for a weak Copeland winner). *Given a roommates instance $G = (V, E)$ with weak rankings and any $\varepsilon > 0$, a matching M such that $\text{score}(M) > \mu/2 \cdot (1 - \varepsilon)$ with high probability can be computed in $\text{poly}(|V|, 1/\varepsilon)$ time.*

² This comprises of $\mu - 1$ points for winning against all other matchings and $1/2$ point for a tie against itself.

³ By contrast, it is NP-hard to decide if there is a matching M with $\text{wins}(M) + \text{ties}(M) = \mu$ (note that such an M has to be popular).

It is relevant to note that our algorithm works for *any* input graph G (not necessarily bipartite) and can also accommodate *weak rankings*, i.e., preferences with ties (and, more generally, partial order preferences). Thus, the weak Copeland winner notion satisfies universal existence (Theorem 5) and there is an efficient algorithm for computing an arbitrarily close approximation to it (Theorem 6) in the general roommates model.

Our fully polynomial-time randomized approximation scheme (FPRAS) result in Theorem 6 implies that relaxing Copeland winner to weak Copeland winner suffices for computational tractability. That is, though it is hard to find an exact Copeland winner in $G = (V, E)$, there is a randomized algorithm that finds a $(2 + \varepsilon)$ -approximation in $\text{poly}(|V|, 1/\varepsilon)$ time, for any $\varepsilon > 0$.

A weighted Copeland winner. A criticism of Copeland’s method is that it puts too much emphasis on the *number* of pairwise wins/defeats rather than their *magnitudes* [8]. Thus, one might ask whether it is possible to provide a meaningful bound on the total sum of the *margin* of wins and losses. To that end, we define the *weighted Copeland score* $\text{wt-score}(M) := (\sum_N \Delta(M, N))/\mu$, where the sum is over all matchings N and μ is the total number of matchings in G (note that a pairwise loss contributes negatively to the weighted score).

Let us call a matching M^* that maximizes $\text{wt-score}(\cdot)$ a *weighted Copeland winner*. A simple averaging argument shows that $\text{wt-score}(M^*) \geq 0$ [11]. Our final result provides an FPRAS for weighted Copeland winners.

Theorem 7 (FPRAS for a wt. Copeland winner). *Given a roommates instance $G = (V, E)$ with weak rankings and any $\varepsilon > 0$, a matching M such that $\text{wt-score}(M) \geq \text{wt-score}(M^*) - \varepsilon$ with high probability can be computed in $\text{poly}(|V|, 1/\varepsilon)$ time, where M^* is a weighted Copeland winner.*

Thus, if we imagine a uniform distribution over the set of all matchings, then Theorem 7 provides a randomized algorithm for computing a matching M with $\mathbb{E}[\Delta(M, N)] \geq \mathbb{E}[\Delta(M^*, N)] - \varepsilon$.

Background and related work. There are polynomial-time algorithms known for deciding if a roommates instance admits a stable matching [23] and if it admits a Condorcet winner (such a matching is also called *strongly popular*) [4]. Algorithmic aspects of popular matchings have been extensively studied in the last fifteen years within theoretical computer science and combinatorial optimization literature and we refer to [9] for a survey.

The special case of popular matchings in bipartite graphs with strict preferences has been of particular interest, where such matchings always exist. In this literature, the work that is closest to ours is [26], where semi-popular matchings were introduced to design an efficient bicriteria approximation algorithm for the min-cost popular matching problem. Prior to our work, it was not known if semi-popular matchings always exist in general graphs or in bipartite graphs with weak rankings.

Our techniques. We establish our main algorithmic result (Theorem 6) using a sampling-based procedure (Algorithm 1). Sampling matchings from a near-uniform distribution, while extensively studied in theoretical computer science [31], has not really been explored in the computational social choice literature. In this work, we use the sampling approach to search for an almost weak Copeland winner in the space of all matchings (which is exponentially large) in $G = (V, E)$. Specifically, we draw two independent samples, each containing $\Theta(\log |V|/\varepsilon^2)$ matchings, from a distribution that is $\varepsilon/4$ -close to the uniform distribution in total variation distance. By the seminal result of Jerrum and Sinclair [24], there is an algorithm with running time $\text{poly}(|V|, \log(1/\varepsilon))$ for generating a sample from such a distribution.

We then pit the two random samples against each other by evaluating all head-to-head elections between pairs of matchings, one matching from each sample, and pick the one with the highest Copeland score in these elections. It is easy to see that the chosen matching is a weak Copeland winner ‘on the sample’ (Lemma 8). By a standard concentration argument, we are able to show that this matching is an approximate weak Copeland winner with respect to *all* the matchings in the given instance with high probability (Lemma 10).

Our hardness result for Copeland winners (Theorem 3) uses a reduction from VERTEX COVER. At a high level, our construction is inspired by a construction in [12] which used a far simpler instance to

show that the extension complexity of the bipartite popular matching polytope is near-exponential. Our construction, on the other hand, is considerably more involved and we construct a non-bipartite instance G with weak rankings. What makes our construction particularly tricky is that *every* Copeland winner in G has to correspond to a minimum vertex cover in the input instance H . In general, we do not know how to characterize matchings that are Copeland winners.

While the notion of popularity treats *wins* and *ties* uniformly and it is easy to test a matching for popularity [3], a tie is worth only half a win for a Copeland winner and we do not know how to test if a given matching is a Copeland winner or not. This makes our reduction novel and challenging. We use the LP framework for popular matchings to analyze Copeland winners in G and this leads to our hardness proof. This proof is presented in Section 3. We prove Theorem 7 in Section 4.

2 Computing an Almost Weak Copeland Winner

Our input is a roommates instance $G = (V, E)$ where every vertex has a weak ranking over its neighbors. While it is easy to construct roommates instances that admit no popular matchings (see Fig. 1b), weak Copeland winners are always present in any instance G , as we show below.

Theorem 5 (Existence of a weak Copeland winner). *Every roommates instance admits a weak Copeland winner.*

Proof. Let μ be the total number of matchings in G . Consider the $\mu \times \mu$ table T defined below whose rows and columns are indexed by all matchings in G . For any $i, j \in \{1, \dots, \mu\}$, let us define

$$T[i, j] := \begin{cases} 1 & \text{if } M_i \text{ defeats } M_j; \\ \frac{1}{2} & \text{if the } M_i\text{-vs-}M_j \text{ election is a tie;} \\ 0 & \text{otherwise (i.e., if } M_j \text{ defeats } M_i). \end{cases}$$

The definition of the table T is such that for any matching M_i , the sum of entries in the i -th row of T is exactly $\text{score}(M_i)$. Observe that $T[i, j] + T[j, i] = 1$ for all i, j ; in particular, $T[i, i] = 1/2$ for all i . Thus the sum of all the entries in T is $\mu^2/2$. Hence, there exists a matching M_k such that the sum of entries in the k -th row of T is at least $\mu/2$. That is, $\text{score}(M_k) \geq \mu/2$; in other words, M_k is a weak Copeland winner. \square

Our algorithm. We will now show an FPRAS for computing a weak Copeland winner. In order to construct a matching M with $\text{score}(M) \geq \mu/2 \cdot (1 - \varepsilon)$ with high probability, as mentioned earlier, we will use the classical result from [24] that shows an algorithm with running time $\text{poly}(n, \log(1/\varepsilon))$ to sample matchings from a distribution ε -close to the uniform distribution in total variation distance (see [24, Corollary 4.3]).

Our algorithm is presented as Algorithm 1. The input to our algorithm is a roommates instance $G = (V, E)$ on n vertices along with a parameter $\varepsilon > 0$. It computes two independent samples \mathcal{S}_0 and \mathcal{S}_1 of $k = \lceil (32 \ln n / \varepsilon^2) \rceil$ matchings—each from a distribution $\varepsilon/4$ -close to the uniform distribution (on all the matchings in G) in total variation distance.

For $M \in \mathcal{S}_0$ (resp., \mathcal{S}_1), let wins'_M be the number of matchings in \mathcal{S}_1 (resp., \mathcal{S}_0) that M wins against and let ties'_M be the number of matchings in \mathcal{S}_1 (resp., \mathcal{S}_0) that M ties with.

Our algorithm computes $\text{score}'_S = \text{wins}'_S + \text{ties}'_S/2$ for each $S \in \mathcal{S}_0 \cup \mathcal{S}_1$. It returns a matching in $\mathcal{S}_0 \cup \mathcal{S}_1$ with the maximum value of score' . We will now show that such a matching has a high Copeland score on the sample. Recall that $|\mathcal{S}_0| = |\mathcal{S}_1| = k$.

Lemma 8. *If S^* is the matching returned by our algorithm then $\text{score}'_{S^*} \geq k/2$.*

Proof. Consider the $k \times k$ table A whose rows are indexed by matchings in \mathcal{S}_0 and whose columns are indexed by matchings in \mathcal{S}_1 . For any $i, j \in \{1, \dots, k\}$, let us define

$$A[i, j] = \begin{cases} 1 & \text{if the } i\text{-th matching in } \mathcal{S}_0 \text{ defeats the } j\text{-th matching in } \mathcal{S}_1; \\ \frac{1}{2} & \text{if the } i\text{-th matching in } \mathcal{S}_0 \text{ ties with the } j\text{-th matching in } \mathcal{S}_1; \\ 0 & \text{otherwise.} \end{cases}$$

ALGORITHM 1: An FPRAS for Weak Copeland Winners.

Input: A graph $G = (V, E)$ on n vertices and a set of weak rankings for every vertex $v \in V$

Parameters: $\varepsilon > 0$

Output: A matching in G .

- 1 Produce two independent samples \mathcal{S}_0 and \mathcal{S}_1 of $k = \lceil (32 \ln n / \varepsilon^2) \rceil$ matchings where each matching is chosen from a distribution that is $\varepsilon/4$ -close to the uniform distribution (on all matchings in G) in total variation distance.
 - 2 **foreach** matching $M \in \mathcal{S}_0 \cup \mathcal{S}_1$ **do**
 - 3 Initialize $\text{wins}'_M = \text{ties}'_M = 0$.
 - 4 **foreach** matching $M \in \mathcal{S}_0$ **do**
 - 5 **foreach** matching $N \in \mathcal{S}_1$ **do**
 - 6 if $\Delta(M, N) > 0$ then $\text{wins}'_M = \text{wins}'_M + 1$
 - 7 if $\Delta(M, N) = 0$ then $\text{ties}'_M = \text{ties}'_M + 1$ and $\text{ties}'_N = \text{ties}'_N + 1$
 - 8 if $\Delta(M, N) < 0$ then $\text{wins}'_N = \text{wins}'_N + 1$.
 - 9 **return** a matching $S \in \mathcal{S}_0 \cup \mathcal{S}_1$ with the maximum value of $\text{wins}'_S + \text{ties}'_S/2$.
-

The definition of the table A is such that for the i -th matching $M_i \in \mathcal{S}_0$, the sum of entries in the i -th row of A is score'_{M_i} and for the j -th matching $N_j \in \mathcal{S}_1$, the sum of entries in the j -th column of A is $(k - \text{score}'_{N_j})$. So the sum of all the entries in A is $\sum_{M \in \mathcal{S}_0} \text{score}'_M = k^2 - \sum_{N \in \mathcal{S}_1} \text{score}'_N$.

- Suppose the sum of all the entries in A is at least $k^2/2$. Then $\sum_{M \in \mathcal{S}_0} \text{score}'_M \geq k^2/2$. Since $|\mathcal{S}_0| = k$, there has to be some $M \in \mathcal{S}_0$ with $\text{score}'_M \geq k/2$.
- Suppose the sum of all the entries in A is less than $k^2/2$. Then $\sum_{N \in \mathcal{S}_1} \text{score}'_N \geq k^2 - k^2/2 = k^2/2$. Since $|\mathcal{S}_1| = k$, there has to be some $N \in \mathcal{S}_1$ with $\text{score}'_N \geq k/2$.

Since S^* is the matching in $\mathcal{S}_0 \cup \mathcal{S}_1$ with the maximum value of score' , we have $\text{score}'_{S^*} \geq k/2$. □

We will show that the on-sample guarantee of Lemma 8 carries over, with high probability, to the entire set of matchings. This proof makes use of a tail bound for the random variable score'_S corresponding to the on-sample Copeland score of any fixed matching $S \in \mathcal{S}_0 \cup \mathcal{S}_1$.

Lemma 9. *Let $S \in \mathcal{S}_0 \cup \mathcal{S}_1$ be any fixed matching sampled by our algorithm. Then the probability that $\text{score}'_S \geq k \cdot (\text{score}(S)/\mu + \varepsilon/2)$ is at most $1/n$.*

Proof. Assume without loss of generality that $S \in \mathcal{S}_1$. If \mathcal{S}_0 was a set of k matchings chosen uniformly at random from the set of all matchings in G , then the probability that S defeats any matching in \mathcal{S}_0 is $\text{wins}(S)/\mu$ and the probability that it ties with any matching in \mathcal{S}_0 is $\text{ties}(S)/\mu$.

However, the matchings in \mathcal{S}_0 are sampled from a distribution $\varepsilon/4$ -close to the uniform distribution in total variation distance. So, the probability that S defeats any matching in \mathcal{S}_0 is at most $\text{wins}(S)/\mu + \varepsilon/4$ and the probability that it ties with any matching in \mathcal{S}_0 is at most $\text{ties}(S)/\mu + \varepsilon/4$.

Observe that $\text{score}'_S = X_1 + \dots + X_k$, where, for each i , the random variable $X_i \in \{0, \frac{1}{2}, 1\}$ denotes whether S loses/ties/wins against the i -th matching in \mathcal{S}_0 . Note that $\mathbb{E}[X_i] \leq \text{wins}(S)/\mu + \text{ties}(S)/2\mu + 3\varepsilon/8$ for each i . Since $\text{score}(S) = \text{wins}(S) + \text{ties}(S)/2$, linearity of expectation gives

$$\mathbb{E}[\text{score}'_S] \leq k \cdot \left(\frac{\text{score}(S)}{\mu} + \frac{3\varepsilon}{8} \right). \quad (1)$$

In light of Equation (1), it suffices to bound the probability of the event that

$$\text{score}'_S - \mathbb{E}[\text{score}'_S] \geq k \cdot \varepsilon/8, \text{ or, equivalently, } \text{score}'_S/k - \mathbb{E}[\text{score}'_S/k] \geq \varepsilon/8.$$

For this, we will use Hoeffding's inequality [21]. Recall that if X_1, \dots, X_k are bounded independent random variables such that $X_i \in [0, 1]$ for all $i \in [k]$ and $Y := (X_1 + \dots + X_k)/k$, then Hoeffding's inequality says that for any $t \geq 0$,

$$\Pr [Y - \mathbb{E}[Y] \geq t] \leq e^{-2kt^2}.$$

Applying the above inequality for $Y := \text{score}'_S/k$ and $t := \varepsilon/8$, we get that

$$\Pr [\text{score}'_S/k - \mathbb{E}[\text{score}'_S/k] \geq \varepsilon/8] \leq e^{-k\varepsilon^2/32}. \quad (2)$$

Substituting $k = \lceil (32 \ln n / \varepsilon^2) \rceil$ in Equation (2), we get that the right-hand side is at most $1/n$. Thus with probability at least $1 - 1/n$, we have $\text{score}'_S < \mathbb{E}[\text{score}'_S] + k \cdot \varepsilon/8$. Using the upper bound in Equation (1), it follows that $\text{score}'_S < k \cdot (\text{score}(S)/\mu + \varepsilon/2)$ with probability at least $1 - 1/n$. \square

By Lemmas 8 and 9, we have: $k/2 \leq \text{score}'_{S^*} < k \cdot (\text{score}(S^*)/\mu + \varepsilon/2)$ with high probability, where S^* is the matching returned by our algorithm.

Lemma 10. *If S^* is the matching returned by our algorithm then $\text{score}(S^*) > \mu/2 \cdot (1 - \varepsilon)$ with high probability.*

Lemma 10 follows from Lemmas 8 and 9. So our algorithm computes a matching whose Copeland score is more than $\mu/2 \cdot (1 - \varepsilon)$ with high probability. Its running time is polynomial in n and $1/\varepsilon$. Hence Theorem 6 follows.

Theorem 6 (FPRAS for a weak Copeland winner). *Given a roommates instance $G = (V, E)$ with weak rankings and any $\varepsilon > 0$, a matching M such that $\text{score}(M) > \mu/2 \cdot (1 - \varepsilon)$ with high probability can be computed in $\text{poly}(|V|, 1/\varepsilon)$ time.*

3 Finding a Copeland Winner: A Hardness Result

In this section we will prove Theorem 3, i.e., we will show that under standard complexity-theoretic assumptions, there is no polynomial-time algorithm for finding a Copeland winner.

Theorem 3 (Hardness of Copeland winner). *Unless $P = NP$, there is no polynomial-time algorithm for finding a Copeland winner in a roommates instance with weak rankings.*

Given a VERTEX COVER instance $H = (V_H, E_H)$, we will construct a roommates instance $G = (V, E)$ such that any Copeland winner in G will correspond to a minimum vertex cover in H . We will assume that the vertices in the VERTEX COVER instance are indexed as $1, 2, \dots, n$, i.e., $V_H = \{1, \dots, n\}$. Specifically,

- for every vertex $i \in V_H$, there is a gadget Z_i in G on 4 *main* vertices a_i, b_i, a'_i, b'_i and 100 *auxiliary* vertices u_i^0, \dots, u_i^{99} , and
- for every edge $e \in E_H$, there is a gadget Y_e in G on 6 *main* vertices $s_e, t_e, s'_e, t'_e, s''_e, t''_e$ along with 8 *auxiliary* vertices $v_e, v'_e, w_e, w'_e, c_e, d_e, c'_e, d'_e$ (see Fig. 3).

The gadgets. The preferences of the vertices in the vertex gadget Z_i and the edge gadget Y_e are shown in Fig. 3. Observe that in the vertex gadget Z_i , all the vertices u_i^0, \dots, u_i^{99} are tied at the third position (which is the last acceptable position) in a_i 's preference order. Similarly, in the edge gadget Y_e , the vertices b_i and c_e are tied in d_e 's preference list and the vertices b_j and c'_e are tied in d'_e 's preference list.

Red state vs blue state. We will say a vertex gadget Z_i is in **red** state in M if $\{(a_i, b_i), (a'_i, b'_i)\} \subset M$ and in **blue** state if $\{(a_i, b'_i), (a'_i, b_i)\} \subset M$.

A high-level overview of our hardness reduction. Let M be any Copeland winner in G . We will show that M has the following properties.

- M does not use any *inter-gadget* edge (see Lemma 18).

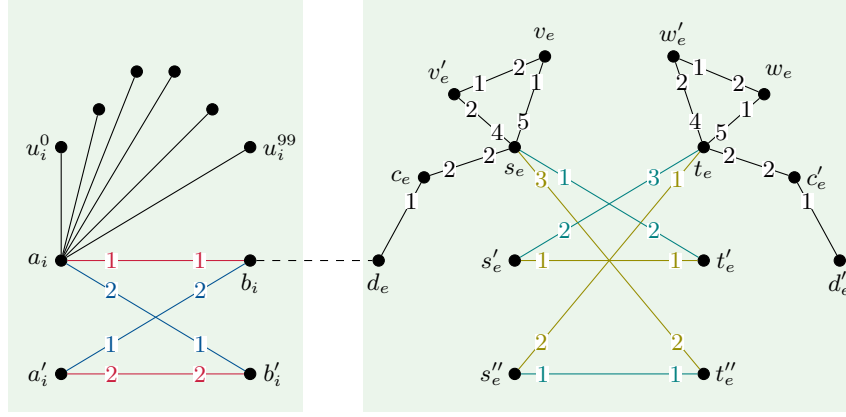


Fig. 3: (Left) The vertex gadget Z_i . The red colored edges within Z_i indicate a stable matching. (Right) The edge gadget Y_e on 14 vertices. All unnumbered edges incident to a vertex should be interpreted as “tied for the last acceptable position”. The edges $(c_e, d_e), (v_e, v'_e), (w_e, w'_e), (c'_e, d'_e)$ along with the olive-colored (resp., teal-colored) edges define the matching F_e (resp., L_e). The dashed edge (b_i, d_e) is an inter-gadget edge.

- For any vertex $i \in V_H$, its vertex gadget Z_i is either in red state or in blue state in M (see Lemma 13).
- For any edge $e = (i, j)$, at least one of Z_i, Z_j has to be in blue state in M (see Lemma 16).
- The vertices i whose vertex gadgets Z_i are in blue state in M form a *minimum vertex cover* in H (see Lemma 17).

Let us start with the proof of Lemma 13. In order to prove this result, we will find it useful to establish two other intermediate results in Lemmas 11 and 12. The first lemma discusses the number of matchings that tie with the matching F_e (or L_e) in the subgraph restricted to the edge gadget Y_e (see Fig. 3).

Two useful matchings. Let $e \in E$. In the edge gadget Y_e , we will find it convenient to define the matchings $F_e = \{(s_e, t''_e), (s'_e, t'_e), (s''_e, t_e), (v_e, v'_e), (w_e, w'_e), (c_e, d_e), (c'_e, d'_e)\}$ and $L_e = \{(s_e, t_e), (s'_e, t'_e), (s''_e, t''_e), (v_e, v'_e), (w_e, w'_e), (c_e, d_e), (c'_e, d'_e)\}$. These are highlighted with olive and teal colors in Fig. 3, respectively.

Lemma 11 (\star^4). *In the subgraph restricted to Y_e , there are exactly 10 matchings that are tied with F_e and no matching defeats F_e . Furthermore, an analogous implication holds for the matching L_e .*

Next, Lemma 12 shows that there is *no* matching within the subgraph restricted to Y_e that “does better” than F_e or L_e in terms of the number of matchings that defeat or tie with it.

Lemma 12 (\star). *For any matching T_e in the subgraph restricted to Y_e , there are at least 10 matchings within this subgraph that either defeat or tie with T_e .*

Our next result (Lemma 13) shows that a Copeland winner matching that does not use any inter-gadget edge must have each vertex gadget in either red or blue state.

Lemma 13 (\star). *Let M be a Copeland winner in G . If M does not use any inter-gadget edge, then in any vertex gadget Z_i , either $\{(a_i, b_i), (a'_i, b'_i)\} \subset M$ or $\{(a_i, b'_i), (a'_i, b_i)\} \subset M$.*

Observation 1 *Consider the subgraph induced on Z_i .*

- The red matching $R_i = \{(a_i, b_i), (a'_i, b'_i)\}$ is tied with 2 matchings in this subgraph. These are R_i itself and the blue matching $B_i = \{(a_i, b'_i), (a'_i, b_i)\}$.
- The blue matching $B_i = \{(a_i, b'_i), (a'_i, b_i)\}$ is tied with 3 matchings in this subgraph: these are B_i itself, the red matching R_i , and the matching $\{(a_i, b_i)\}$.

⁴ Lemmas marked by an asterisk (\star) are proved in the appendix.

Moreover, no matching in this subgraph defeats either the *red* matching R_i or the *blue* matching B_i .

For any gadget X , let $M \cap X$ denote the edges of matching M in the subgraph restricted to X .

Lemma 14 (\star). *Let $e = (i, j) \in E$. Let M be any matching in G such that both Z_i and Z_j are in *red* state in M . Then there are at least 100 matchings within $Y_e \cup Z_i \cup Z_j$ that defeat or tie with $M \cap (Y_e \cup Z_i \cup Z_j)$.*

Next, we will show in Lemma 16 that in a Copeland winner matching that does not use any inter-gadget edge, for any edge gadget, at least one of its adjacent vertex gadgets must be in *blue* state. The proof of Lemma 16 will make use of the following key technical lemma.

Lemma 15 (\star). *Let M^* be any matching in G that satisfies the following three conditions:*

1. *Every vertex gadget is either in *red* or *blue* state.*
2. *For every edge gadget, at least one of its adjacent vertex gadgets is in *blue* state.*
3. *For each edge $e = (i, j)$ where $i < j$, if the vertex gadget Z_i is in *blue* state, then $M^* \cap Y_e = F_e$ otherwise $M^* \cap Y_e = L_e$.*

Then (i) M^* is popular in G and (ii) any matching that contains an inter-gadget edge loses to M^* .

We will use LP duality to prove Lemma 15. The LP framework for popular matchings shows that a matching M^* is popular if and only if it can be realized as a max-weight perfect matching in the graph G augmented with self-loops and with appropriate edge weights—note that these edge weights are a function of the matching M^* . We will construct a *dual certificate* to prove M^* 's popularity in G and we will use complementary slackness to show property (ii).

Lemma 16. *Let M be a Copeland winner in G . If M does not use any inter-gadget edge, then, for every edge $e = (i, j)$, at least one of Z_i, Z_j has to be in *blue* state in M .*

Proof. Suppose, for contradiction, that both Z_i and Z_j are in *red* state in M for some edge $e = (i, j)$. Let \mathcal{X} denote the set of all vertex and edge gadgets in the matching instance. Consider a partitioning of \mathcal{X} into *single* gadgets and *auxiliary* gadgets. Each auxiliary gadget is a triple of an edge and two adjacent vertex gadgets (Y_e, Z_i, Z_j) where $e = (i, j)$ is an edge such that both Z_i and Z_j are in *red* state in M . While there is such an edge e with both its vertex gadgets in *red* state and these vertex gadgets are “unclaimed” by any other edge, the edge e claims both its vertex gadgets and makes an auxiliary gadget out of these three gadgets. All the remaining vertex and edge gadgets are classified as single gadgets, including those edges one or both of whose endpoints have been claimed by some other edge(s). Observe that

$$\text{loss}(M) + \text{ties}(M) \geq \sum_{X \in \mathcal{X}} (\text{loss}(M \cap X) + \text{ties}(M \cap X))$$

where X is any single or auxiliary gadget, and $\text{loss}(M \cap X)$ (resp., $\text{ties}(M \cap X)$) is the number of matchings that defeat (resp., tie with) $M \cap X$ within the gadget X . Also, $\text{loss}(M)$ (resp., $\text{ties}(M)$) is the number of matchings that defeat (resp., tie with) the matching M .

Consider any auxiliary gadget $X = (Y_e, Z_i, Z_j)$. From Lemma 14, we know that $\text{loss}(M \cap X) + \text{ties}(M \cap X)$ is at least 100. Thus, we have

$$\text{loss}(M) + \text{ties}(M) \geq 2^{n'} \cdot 3^{n''} \cdot 10^{m'} \cdot 100^t, \quad (3)$$

where n' (resp., n'') is the number of vertices present as single gadgets in *red* (resp., *blue*) state, m' is the number of edges that are present as single gadgets, and t is the number of auxiliary gadgets in the aforementioned partition. Note that we used Lemma 12 here in bounding $\text{loss}(M \cap Y_e) + \text{ties}(M \cap Y_e)$ by 10 for every edge gadget Y_e that is present as a single gadget, and used Lemma 13 and Observation 1 in bounding $\text{loss}(M \cap Z_i) + \text{ties}(M \cap Z_i)$ by 2 (resp., 3) for every vertex gadget Z_i (acting as single gadget) that is in *red* (resp., *blue*) state.

We will now construct another matching with Copeland score higher than that of M to establish the desired contradiction. Let M^* be the matching obtained by converting both Z_i and Z_j in each auxiliary

gadget (Y_e, Z_i, Z_j) in our partition above into **blue** state and let $M^* \cap Y_e$ be either F_e or L_e (it does not matter which). For every edge $e' = (i', j')$ such that $Y_{e'}$ is present as a single gadget, observe that at least one of i', j' is either in **blue** state or in an auxiliary gadget. If it is $\min\{i', j'\}$ that is in **blue** state/auxiliary gadget, then let $M^* \cap Y_{e'} = F_{e'}$, else let $M^* \cap Y_{e'} = L_{e'}$. Any vertex gadget that is present as a single gadget remains in its original **red** or **blue** state.

Notice that M^* satisfies the conditions in Lemma 15. So M^* is popular in G , i.e., $\text{loss}(M^*) = 0$, and any matching that ties with M^* must not use any inter-gadget edge. Therefore, the number of matchings that tie with M^* across the entire graph G is simply the product of matchings that tie with it on individual single and auxiliary gadgets. Hence,

$$\text{loss}(M^*) + \text{ties}(M^*) = \text{ties}(M^*) = 2^{n'} \cdot 3^{n''} \cdot 10^{m'} \cdot 90^t, \quad (4)$$

where n', n'', m' , and t are defined as in (3). Note that the bound of 90 for an auxiliary gadget in (4) follows from taking the product of 3, 3, and 10, which is the number of matchings that tie with M^* in the two vertex gadgets and their common edge gadget (see Observation 1 and Lemma 11).

Recall that $\text{score}(N) = \mu - \text{loss}(N) - \text{ties}(N)/2$ for any matching N . Comparing (3) and (4) along with the fact that $\text{loss}(M) \geq 0 = \text{loss}(M^*)$, we have $\text{score}(M^*) > \text{score}(M)$ as long as there is even a single edge (i, j) such that both Z_i and Z_j are in **red** state in M . Indeed,

$$\begin{aligned} \text{loss}(M^*) + \text{ties}(M^*)/2 &= 2^{n'-1} \cdot 3^{n''} \cdot 10^{m'} \cdot 90^t, \text{ and} \\ \text{loss}(M) + \text{ties}(M)/2 &\geq \frac{1}{2} (\text{loss}(M) + \text{ties}(M)) \geq 2^{n'-1} \cdot 3^{n''} \cdot 10^{m'} \cdot 100^t. \end{aligned}$$

This contradicts the fact that M is a Copeland winner. Hence, for every edge $e = (i, j)$, at least one of Z_i, Z_j must be in **blue** state in M . This proves Lemma 16. \square

In Lemma 16, we showed that for any Copeland winner M , the vertices whose gadgets are in **blue** state in M constitute a vertex cover in H . The next result shows that the set of such vertices is, in fact, a *minimum* vertex cover. Thus Theorem 3 stated in Section 1 follows.

Lemma 17. *Let M be a Copeland winner in G . If M does not use any inter-gadget edge, then the vertices whose gadgets are in **blue** state in M constitute a minimum vertex cover in H .*

Proof. We know from Lemma 16 that $\text{loss}(M) + \text{ties}(M) \geq \Pi_S(\text{loss}(M \cap S) + \text{ties}(M \cap S))$ where the product is over all gadgets S . Further, from Lemma 13, Lemma 12, and Observation 1, we know that the right hand side in the above inequality is at least $2^{n-k} \cdot 3^k \cdot 10^m$, where k is the number of vertex gadgets in **blue** state and n (resp., m) is the number of vertices (resp., edges) in the VERTEX COVER instance H . Thus, $\text{score}(M) = \mu - \text{loss}(M) - \text{ties}(M)/2 \leq \mu - \frac{1}{2} \cdot 2^{n-k} \cdot 3^k \cdot 10^m$. Moreover, $k \geq |C|$, where C is a minimum vertex cover in H (by Lemma 16).

Let us construct a matching M_C where the vertex gadgets corresponding to the vertices in the minimum vertex cover C are in **blue** state, those corresponding to the remaining vertices are in **red** state, and for every edge $e = (i, j)$, if $\min\{i, j\}$ is in **blue** state then let $M_C \cap Y_e = F_e$, else $M_C \cap Y_e = L_e$. Then, the matching M_C satisfies the conditions of Lemma 15. Therefore, by a similar argument as in the proof of Lemma 16, we get that $\text{score}(M_C) = \mu - \frac{1}{2} \cdot 2^{n-c} \cdot 3^c \cdot 10^m$, where $|C| = c$. Thus $\text{score}(M_C) > \text{score}(M)$ if $c < k$. Since $\text{score}(M)$ has to be the highest among all matchings, it follows that $c = k$. In other words, the set of vertices whose gadgets are in **blue** state in M constitute a minimum vertex cover in H . \square

Finally, we will show that the assumption that a Copeland winner does not use an inter-gadget edge is without loss of generality.

Lemma 18 (\star). *If M is a Copeland winner in G , then M does not use any inter-gadget edge.*

Remark 19. By reducing from a restricted version of VERTEX COVER on 3-regular graphs, which is also known to be NP-hard [17], the intractability of Theorem 3 can be shown to hold even when there are only a *constant* number of neighbors per vertex.

4 Weighted Copeland Winner

For any matching M in G , recall that we defined $\text{wt-score}(M) = (\sum_N \Delta(M, N))/\mu$ in Section 1. In this section we will see a polynomial-time algorithm to compute a matching whose $\text{wt-score}(\cdot)$ is close to the maximum value. It will be useful to augment G with self-loops such that every vertex is its *bottom-most* acceptable choice, i.e., it finds being matched to itself more preferable than being unmatched. So we can restrict our attention to perfect matchings in the augmented graph. We show the following result here.

Lemma 20. *There is a weight function wt^* on the augmented edge set such that a matching M is a weighted Copeland winner in G if and only if $\tilde{M} = M \cup \{(u, u) : u \text{ is unmatched in } M\}$ is a max-weight perfect matching under this weight function wt^* in the augmented graph.*

Before we prove Lemma 20, it will be useful to recall some preliminaries on popularity. The notion of popularity can be extended to *mixed matchings* or convex combinations of integral matchings [27]. For any mixed matching \vec{p} in $G = (V, E)$, let \tilde{p} be the perfect mixed matching in the augmented graph (on the edge set $E \cup \{(u, u) : u \in V\}$) obtained by augmenting \vec{p} with self-loops so as to fully match all vertices.

We can compare a mixed matching \vec{p} with any integral matching M by asking every vertex u to cast a *fractional vote* for \tilde{M} versus \tilde{p} . This fractional vote is:

$$\text{vote}_u(M, \vec{p}) = \sum_{v \prec_u \tilde{M}(u)} \tilde{p}_{(u,v)} - \sum_{v \succ_u \tilde{M}(u)} \tilde{p}_{(u,v)}. \quad (5)$$

The first term is the sum of coordinates in \tilde{p} where u is matched to *worse* neighbors than its partner in \tilde{M} and the second term is the sum of coordinates in \tilde{p} where u is matched to *better* neighbors than its partner in \tilde{M} .

For any pair of matchings M and N in $G = (V, E)$, observe that $\Delta(M, N) = \sum_{u \in V} \text{vote}_u(M, N)$ where $\text{vote}_u(M, N)$ is 1 if u prefers $\tilde{M}(u)$ to $\tilde{N}(u)$, it is -1 if u prefers $\tilde{N}(u)$ to $\tilde{M}(u)$, else it is 0. We would now like to generalize $\Delta(M, N)$ to $\Delta(M, \vec{p})$ where \vec{p} is a mixed matching in G . So

$$\Delta(M, \vec{p}) := \sum_{u \in V} \text{vote}_u(M, \vec{p}),$$

i.e., $\Delta(M, \vec{p})$ is the sum of fractional votes of all vertices for \tilde{M} versus \tilde{p} . Suppose $\vec{p} = \sum_i c_i \cdot I_{N_i}$ where $\sum_i c_i = 1$, $c_i \geq 0$ for all i , and for any matching N , $I_N \in \{0, 1\}^m$ is its edge incidence vector. Another natural definition for $\Delta(M, \vec{p})$ is $\sum_i c_i \cdot \Delta(M, N_i)$. It was shown in [27, Lemma 1] that the two are the same, i.e., $\sum_{u \in V} \text{vote}_u(M, \vec{p}) = \sum_i c_i \cdot \Delta(M, N_i)$.

Proof. (of Lemma 20) Let μ be the number of matchings in G . Let n_e be the number of matchings that contain the edge e . Similarly, let ℓ_u be the number of matchings that leave u unmatched. Define the mixed matching $\vec{q} := \sum_N I_N / \mu$ – note that this corresponds to the uniform distribution on all matchings in G . So $\vec{q} \in [0, 1]^{m+n}$ in the augmented graph \tilde{G} is as given below:

$$\vec{q} = \frac{\sum_{e \in E} \chi_e + \sum_{u \in V} \chi_u}{\mu},$$

where for each $e \in E$, χ_e is the vector in \mathbb{R}^{m+n} with n_e in its e -th coordinate and 0 in all other coordinates and for each $u \in V$, χ_u is the vector in \mathbb{R}^{m+n} with ℓ_u in its u -th coordinate and 0 in all other coordinates. For any matching M in G , observe that $\Delta(M, \vec{q}) = \text{wt-score}(M)$. This is because:

$$\Delta(M, \vec{q}) = \Delta\left(M, \frac{1}{\mu} \sum_N I_N\right) = \frac{1}{\mu} \cdot \sum_N \Delta(M, N) = \text{wt-score}(M).$$

For any mixed matching \vec{p} , it is known [27,29] that the function $\Delta(\cdot, \vec{p})$ corresponds to an appropriate edge weight function in the augmented graph. Consider the following edge weight function wt^* . For any edge $(u, v) \in E$, let

$$\text{wt}^*(u, v) = \left(\sum_{v' \prec_u v} \tilde{q}_{(u, v')} - \sum_{v' \succ_u v} \tilde{q}_{(u, v')} \right) + \left(\sum_{u' \prec_v u} \tilde{q}_{(u', v)} - \sum_{u' \succ_v u} \tilde{q}_{(u', v)} \right).$$

The first term on the right is $\text{vote}_u(v, \vec{q})$, i.e., u 's fractional vote for v versus its assignment in \vec{q} , and the second term on the right is $\text{vote}_v(u, \vec{q})$, i.e., v 's fractional vote for u versus its assignment in \vec{q} . For a self-loop (u, u) , let $\text{wt}^*(u, u) = \tilde{q}_{(u, u)} - 1$.

The definition of $\text{wt}^*(e)$ for each edge/self-loop e is such that $\text{wt}^*(\tilde{M}) = \sum_u \text{vote}_u(M, \vec{q}) = \Delta(M, \vec{q}) = \text{wt-score}(M)$ for any matching M in G . So a matching M has the maximum value of $\text{wt-score}(\cdot)$ if and only if \tilde{M} is a max-weight perfect matching under the weight function wt^* in the augmented graph. \square

Consider the following linear program. **LP1** is optimizing over the perfect matching polytope of the augmented graph – so it admits an optimal solution that is integral. We know from Lemma 20 that any such solution is a weighted Copeland winner in G .

$$\text{maximize } \sum_{e \in \tilde{E}} \text{wt}^*(e) \cdot x_e \tag{LP1}$$

subject to

$$\begin{aligned} \sum_{e \in \tilde{\delta}(u)} x_e &= 1 \quad \forall u \in V \\ \sum_{e \in E[S]} x_e &\leq \lfloor |S|/2 \rfloor \quad \forall S \in \Omega \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}. \end{aligned}$$

Here $\tilde{E} = E \cup \{(u, u) : u \in V\}$ and $\tilde{\delta}(u) = \delta(u) \cup \{(u, u)\}$, where $\delta(u) \subseteq E$ is the set of edges incident to vertex u . Also Ω is the set of all odd subsets of V and $E[S] \subseteq E$ stands for the set of edges with both endpoints in S . If we knew the function wt^* then solving **LP1** gives us a weighted Copeland winner. However we do not know how to efficiently compute this weight function wt^* since it is $\#\text{P}$ -hard to compute μ [38].

We will use the fully polynomial-time randomized approximation scheme (FPRAS) in [24] to approximately count with high probability (i) the total number μ of matchings in G , (ii) for each $e \in E$, the number n_e of matchings in G that contain the edge e , and (iii) for each $u \in V$, the number ℓ_u of matchings in G that leave the vertex u unmatched. This will give us an efficient approximation of wt^* .

Theorem 7 (FPRAS for a wt. Copeland winner). *Given a roommates instance $G = (V, E)$ with weak rankings and any $\varepsilon > 0$, a matching M such that $\text{wt-score}(M) \geq \text{wt-score}(M^*) - \varepsilon$ with high probability can be computed in $\text{poly}(|V|, 1/\varepsilon)$ time, where M^* is a weighted Copeland winner.*

Proof. Let $|V| = n$. Given $\varepsilon > 0$, let us define $\varepsilon' = \varepsilon/8n$. Using the FPRAS in [24], we can compute in time polynomial in $n, 1/\varepsilon$, and $\log(1/\delta)$ (for a suitable $\delta > 0$ that will be fixed later):

1. a value μ' such that $\Pr[(1 - \varepsilon')\mu \leq \mu' \leq (1 + \varepsilon')\mu] \geq 1 - \delta$.
2. for each $e \in E$: a value n'_e such that $\Pr[(1 - \varepsilon')n_e \leq n'_e \leq (1 + \varepsilon')n_e] \geq 1 - \delta$.
3. for each $u \in V$: a value ℓ'_u such that $\Pr[(1 - \varepsilon')\ell_u \leq \ell'_u \leq (1 + \varepsilon')\ell_u] \geq 1 - \delta$.

So the vector $\tilde{r} \in [0, 1]^{m+n}$ that we have with us is the following:

$$\tilde{r} = \frac{\sum_{e \in E} \chi'_e + \sum_{u \in V} \chi'_u}{\mu'}$$

where for each $e \in E$, χ'_e is the vector in \mathbb{R}^{m+n} with n'_e in its e -th coordinate and 0 in all other coordinates and for each $u \in V$, χ'_u is the vector in \mathbb{R}^{m+n} with ℓ'_u in its u -th coordinate and 0 in all other coordinates. It follows from items 1-3 above that with probability at least $1 - (m+n+1)\delta$, every coordinate in the vector \tilde{r} is within $(1 \pm 2\varepsilon')$ of the corresponding coordinate in $\tilde{q} \in [0, 1]^{m+n}$ (see the proof of Lemma 20 for more details on \tilde{q}).

Let $\delta = 1/(m+n+1)n$. So with probability at least $1 - 1/n$, we have $\tilde{r}_c \in [(1 - 2\varepsilon')\tilde{q}_c, (1 + 2\varepsilon')\tilde{q}_c]$ for every coordinate c ; note that every coordinate in \tilde{r} and in \tilde{q} is either an edge or a vertex.

We will use the following edge weight function wt_r . For any edge $(u, v) \in E$,

$$\text{wt}_r(u, v) = \left(\sum_{v' \prec_u v} \tilde{r}_{(u, v')} - \sum_{v' \succ_u v} \tilde{r}_{(u, v')} \right) + \left(\sum_{u' \prec_v u} \tilde{r}_{(u', v)} - \sum_{u' \succ_v u} \tilde{r}_{(u', v)} \right). \quad (6)$$

For a self-loop (u, u) , let $\text{wt}_r(u, u) = \tilde{r}_{(u, u)} - 1$. Recall the edge weight function wt^* defined in the proof of Lemma 20. The definition of $\text{wt}^*(u, v)$ is analogous to (6) with \tilde{q}_c replacing \tilde{r}_c for every coordinate c .

The fact that with high probability $\tilde{r}_c \in [(1 - 2\varepsilon')\tilde{q}_c, (1 + 2\varepsilon')\tilde{q}_c]$ for all c , $\sum_{v' \in \delta(u)} \tilde{q}_{(u, v')} = 1$ for every vertex u , and $\tilde{q}_c \geq 0$ for all c implies that with high probability for every edge/self-loop e , we have $\text{wt}^*(e) - 4\varepsilon' \leq \text{wt}_r(e) \leq \text{wt}^*(e) + 4\varepsilon'$. Hence for any matching \tilde{N} in G , with high probability:

$$\text{wt}^*(\tilde{N}) - 4\varepsilon'|\tilde{N}| \leq \text{wt}_r(\tilde{N}) \leq \text{wt}^*(\tilde{N}) + 4\varepsilon'|\tilde{N}|.$$

Since $|\tilde{N}| \leq n$ and $\varepsilon' = \varepsilon/8n$, this implies that $\text{wt}^*(\tilde{N}) - \varepsilon/2 \leq \text{wt}_r(\tilde{N}) \leq \text{wt}^*(\tilde{N}) + \varepsilon/2$ with high probability.

We are interested in solving the following linear program LP2:

$$\text{maximize } \sum_{e \in \tilde{E}} \text{wt}_r(e) \cdot x_e \quad (\text{LP2})$$

subject to

$$\begin{aligned} \sum_{e \in \delta(u)} x_e &= 1 \quad \forall u \in V \\ \sum_{e \in E[S]} x_e &\leq \lfloor |S|/2 \rfloor \quad \forall S \in \Omega \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}. \end{aligned}$$

Let \tilde{M} be an optimal solution to LP2. So $\text{wt}_r(\tilde{M}) \geq \text{wt}_r(\tilde{M}^*)$ where \tilde{M}^* is a weighted Copeland winner. It follows from our discussion above that $\text{wt}_r(\tilde{M}^*) \geq \text{wt}^*(\tilde{M}^*) - \varepsilon/2$ with high probability.

So $\text{wt}_r(\tilde{M}) \geq \text{wt}^*(\tilde{M}^*) - \varepsilon/2$ with high probability. Moreover, $\text{wt}_r(\tilde{M}) \leq \text{wt}^*(\tilde{M}) + \varepsilon/2$ with high probability. Thus with high probability: $\text{wt}^*(\tilde{M}) \geq \text{wt}^*(\tilde{M}^*) - \varepsilon$. Since $\text{wt}^*(\tilde{M}) = \text{wt-score}(\tilde{M})$, it follows that $\text{wt-score}(\tilde{M}) \geq \text{wt-score}(\tilde{M}^*) - \varepsilon$ with high probability, as desired. \square

5 Open Questions

An open problem is to resolve the computational complexity of finding an *exact* weak Copeland winner and more generally, a semi-popular matching. Similarly, the complexity of finding an *exact* weighted Copeland winner is open. Is there an efficient algorithm for *testing* whether a given matching is a Copeland or a weak Copeland winner? Notice that a positive answer would place the relevant search problem in the complexity class TFNP [30].

The class of Copeland $^\alpha$ rules generalizes the Copeland rule where wins/ties/losses are weighted as $1/\alpha/0$ for some $\alpha \in [0, 1]$ [14,37]. Our intractability result for Copeland winners (Theorem 3) extends to all Copeland $^\alpha$ rules for $\alpha \in [0, 1)$. Whether there always exists a weak Copeland winner in all scenarios and whether it can be efficiently (approximately) computed is an interesting question for future work.

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Appendix: Missing Proofs from Section 3

Proof of Lemma 11

Lemma 11. *In the subgraph restricted to Y_e , there are exactly 10 matchings that are tied with F_e and no matching defeats F_e . Furthermore, an analogous implication holds for the matching L_e .*

Proof. We prove this lemma for the matching F_e . The proof for the matching L_e is totally analogous. Let us first fix the edges $(c_e, d_e), (c'_e, d'_e), (v_e, v'_e), (w_e, w'_e)$ and see what matchings that contain these 4 edges tie with F_e . There are 4 such matchings and the remaining edges in these matchings are:

1. $(s_e, t''_e), (s'_e, t'_e), (s''_e, t_e)$ and 2. $(s_e, t'_e), (s'_e, t_e), (s''_e, t''_e)$
3. $(s'_e, t'_e), (s''_e, t''_e)$ and 4. $(s_e, t'_e), (s''_e, t''_e)$

There are other matchings that contain the pair of edges $(c_e, d_e), (c'_e, d'_e)$ and are tied with F_e . There are again 4 such matchings and the remaining edges in these matchings are:

5. $(s_e, t'_e), (s''_e, t''_e), (v_e, v'_e), (t_e, w_e)$ and 6. $(s'_e, t'_e), (s''_e, t''_e), (v_e, v'_e), (t_e, w_e)$
7. $(s'_e, t'_e), (s''_e, t''_e), (s_e, v_e), (w_e, w'_e)$ and 8. $(s'_e, t'_e), (s''_e, t''_e), (s_e, v_e), (t_e, w_e)$

Finally, the following 2 matchings are also tied with F_e . Note that both these matchings leave the vertex d_e unmatched.

9. $\{(s'_e, t'_e), (s''_e, t''_e), (s_e, c_e), (v_e, v'_e), (w_e, w'_e), (c'_e, d'_e)\}$
10. $\{(s'_e, t'_e), (s''_e, t''_e), (s_e, c_e), (v_e, v'_e), (t_e, w_e), (c'_e, d'_e)\}$

Thus there are 10 such matchings that tie with F_e (note that F_e is also included here). It can be verified that other than these 10 matchings, no other matching in this subgraph is tied with F_e .

We now need to show that within the subgraph induced on Y_e , there is no matching that *defeats* F_e . The matching F_e matches all vertices in this subgraph and there is exactly one blocking edge wrt F_e in this subgraph: this is the edge (s''_e, t''_e) . It follows from the characterization of popular matchings in [22] that a matching N defeats F_e if and only if $N \oplus F_e$ has an alternating cycle C with the blocking edge (s''_e, t''_e) ; moreover, no edge in C should be *negative*⁵.

It is easy to check there is no such alternating cycle wrt F_e . The absence of such a cycle implies that F_e is popular in this subgraph. In other words, no matching in this subgraph defeats F_e . \square

⁵ An edge (x, y) is negative with respect to matching M if both x and y prefer their partners in M to each other.

Proof of Lemma 12

Lemma 12. *For any matching T_e in the subgraph restricted to Y_e , there are at least 10 matchings within this subgraph that either defeat or tie with T_e .*

Proof. Observe that it is enough to show this lemma for *Pareto optimal* matchings.⁶ We have already seen that F_e (similarly, L_e) loses to *no* matching in the subgraph induced on Y_e . Hence for any matching $T_e \notin \{F_e, L_e\}$ in this subgraph, the 3 matchings F_e, L_e , and T_e itself either defeat or tie with T_e . We need to show at least 7 more such matchings. We will divide this proof into three parts.

(1) *Suppose neither s_e nor t_e is matched in T_e to any of its top 3 neighbors.* So each of s_e, t_e is either matched to one of its neighbors in the *triangle* that it is a part of or it is left *unmatched*.

Observe that due to Pareto optimality of T_e , the edges (s'_e, t'_e) and (s''_e, t''_e) are in T_e . We will crucially use the triangles $\langle s_e, v_e, v'_e \rangle$ and $\langle t_e, w_e, w'_e \rangle$ here. Let us use the phrase “rotate the matching edge in a triangle” to denote that we replace the edge of T_e in this triangle with another edge that makes two of these vertices happy and one unhappy.

- We immediately get 3 matchings that defeat T_e by rotating the matching edge in (i) $\langle s_e, v_e, v'_e \rangle$, (ii) $\langle t_e, w_e, w'_e \rangle$, and (iii) both these triangles.
- Rotate the matching edge in $\langle s_e, v_e, v'_e \rangle$ and *free* the vertex t_e from $\langle t_e, w_e, w'_e \rangle$, i.e., if t_e is matched in T_e then replace $(t_e, *)$ with (w_e, w'_e) . We now get 3 matchings that tie with T_e by each of these 3 choices: (i) replacing the edge (s'_e, t'_e) with (s'_e, t_e) , (ii) replacing the edge (s''_e, t''_e) with (s''_e, t_e) , and (iii) replacing the edge (c'_e, d'_e) with (c'_e, t_e) .
Each of the operations (i)-(iii) makes two vertices unhappy and one vertex t_e happy. Since rotating the matching edge made 2 vertices among s_e, v_e, v'_e happy and one unhappy, everything summed together, it follows that each of these 3 matchings is tied with T_e .
- We do a mirror image of what we did in the earlier step, i.e., we rotate the matching edge in $\langle t_e, w_e, w'_e \rangle$ and *free* the vertex s_e from $\langle s_e, v_e, v'_e \rangle$ and promote s_e to one of its top 3 neighbors (at the cost of making 2 vertices unhappy).

(2) *Suppose both s_e and t_e are matched in T_e to one of their top 3 neighbors.* The Pareto optimal matching that results when s_e (resp., t_e) is matched to its third choice neighbor is F_e (resp., L_e). So we only have to consider the case when both s_e and t_e are matched to one of their top 2 neighbors. Observe that there has to be a blocking edge incident to the partner of s_e (similarly, the partner of t_e).

- “Unblock” the blocking edge incident to s_e ’s partner, i.e., if $(s_e, t'_e) \in M$ then replace this with (s'_e, t'_e) . This creates a more popular matching where s_e is unmatched. To obtain another such matching, rotate the matching edge in $\langle s_e, v_e, v'_e \rangle$, i.e., replace (v_e, v'_e) with (s_e, v_e) . Thus we get 2 matchings that are more popular than T_e .
- Run a “mirror image” of the above step with t_e replacing s_e . This gives us 2 more matchings that are more popular than T_e .
- Unblock the blocking edge incident to s_e ’s partner *and* the blocking edge incident to t_e ’s partner. This gives us one matching more popular than M . Now rotate the matching edge in $\langle s_e, v_e, v'_e \rangle$ – this gives us another matching more popular than M . Instead, rotate the matching edge in $\langle t_e, w_e, w'_e \rangle$. Thus we can get 3 more matchings that are more popular than M .

(3) *The remaining case.* So exactly one of s_e, t_e is matched in T_e to one of its top 3 neighbors. Without loss of generality, assume that it is s_e that is matched to one of its top 3 neighbors. To begin with, let us assume that (s_e, t'_e) and (t_e, w_e) are in T_e .

- Rotate the matching edge in $\langle t_e, w_e, w'_e \rangle$, i.e., replace (t_e, w_e) with (t_e, w'_e) . This gives us a matching more popular than T_e .

⁶ A matching M is Pareto optimal if there is no matching N such that at least one vertex is *better off* in N than in M and no vertex is *worse off* in N .

- Unblock the blocking edge incident to s_e 's partner, i.e., replace (s_e, t'_e) with (s'_e, t'_e) . This gives us a second matching more popular than T_e . Also replace (t_e, w_e) with (t_e, w'_e) . This gives us yet another matching more popular than T_e .
- Replace (s_e, t'_e) with (s'_e, t'_e) and rotate the matching edge in $\langle s_e, v_e, v'_e \rangle$, i.e., replace (v_e, v'_e) with (s_e, v_e) . This gives us a 4th matching more popular than T_e . Now also rotate the matching edge in $\langle t_e, w_e, w'_e \rangle$. This gives us a 5th matching more popular than T_e .
- Replace the edge (w_e, t_e) with (s'_e, t_e) . This gives us a 6th matching more popular than T_e (this is not the same as L_e since the edge $(w_e, w'_e) \in L_e$ while w_e is unmatched here).
- Replace (s_e, t'_e) with (s'_e, t'_e) and replace (t_e, w_e) with (w_e, w'_e) . This gives us a 7th matching that is tied with T_e since s'_e, t'_e , and w'_e prefer this matching while s_e, t_e, w_e prefer T_e .

Suppose (s_e, c_e) and (t_e, w_e) are in T_e . Then it is easy to see that corresponding to the first 5 matchings listed above, we again have 5 matchings that defeat/tie with M . We can obtain two more matchings by unblocking the blocking edge incident to c_e , i.e., (s_e, c_e) is replaced with (c_e, d_e) , and (i) replace (s'_e, t'_e) and (t_e, w_e) with (s'_e, t_e) and (w_e, w'_e) (ii) replace (s''_e, t''_e) and (t_e, w_e) with (s''_e, t_e) and (w_e, w'_e) . Observe that both these matchings are tied with T_e .

When (s_e, t''_e) and (t_e, w_e) are in T_e , we can similarly show 7 matchings (other than F_e, L_e, T_e) that defeat/tie with T_e . When $(t_e, w'_e) \in T_e$ or when t_e is unmatched in T_e , the above cases are analogous. This finishes the proof of Lemma 12. \square

Proof of Lemma 13

Lemma 13. *Let M be a Copeland winner in G . If M does not use any inter-gadget edge, then in any vertex gadget Z_i , either $\{(a_i, b_i), (a'_i, b'_i)\} \subset M$ or $\{(a_i, b'_i), (a'_i, b_i)\} \subset M$.*

The function vote will be useful to us here. For any vertex x and any pair of matchings S and T in G , the vote of x for S versus T can be defined as follows.

$$\text{Let } \text{vote}_x(S, T) = \begin{cases} 1 & \text{if } x \text{ prefers } S \text{ to } T; \\ -1 & \text{if } x \text{ prefers } T \text{ to } S; \\ 0 & \text{otherwise, i.e., } x \text{ is indifferent between } S \text{ and } T. \end{cases}$$

For any two matchings S and T , observe that $\Delta(S, T) = \sum_x \text{vote}_x(S, T)$, where the sum is over all vertices x in G .

Proof. (of Lemma 13) We know that the matching M uses only intra-gadget edges. Let us start by showing that $(a_i, u_i^k) \notin M$ for any $k \in \{0, \dots, 99\}$. Suppose not, i.e., let $(a_i, u_i^k) \in M$ for some $k \in \{0, \dots, 99\}$. Then this means either b_i or b'_i is left unmatched in M .

1. If b_i is unmatched in M then let $N = M \cup \{(a_i, b_i)\} \setminus \{(a_i, u_i^k)\}$.
2. If b'_i is unmatched in M then let $N = M \cup \{(a_i, b'_i)\} \setminus \{(a_i, u_i^k)\}$.

It is easy to check that for any matching T , we have $\Delta(N, T) \geq \Delta(M, T)$. In more detail, either $\text{vote}_{b_i}(N, T) - \text{vote}_{b_i}(M, T) \geq 1$ or $\text{vote}_{b'_i}(N, T) - \text{vote}_{b'_i}(M, T) \geq 1$ since N matches either b_i or b'_i to its top choice neighbor a_i while M leaves that vertex unmatched. We also know that $\text{vote}_{u_i^k}(N, T) - \text{vote}_{u_i^k}(M, T) = -1$. For any other vertex x , we have $\text{vote}_x(N, T) - \text{vote}_x(M, T) \geq 0$.

Thus we have $\Delta(N, T) \geq \Delta(M, T)$ for all matchings T in G and so $\text{score}(N) \geq \text{score}(M)$. Moreover, $\Delta(N, M) > 0$ since two vertices prefer N to M (these are a_i and one of b_i, b'_i) while only u_i^k prefers M to N . Thus $\text{score}(N) > \text{score}(M)$, a contradiction to M being a Copeland winner.

Hence the matching M has to use only the remaining 4 edges in Z_i . Note that a Copeland winner has to be Pareto optimal (see footnote 6). This implies there are only two choices for M within the gadget Z_i : either $\{(a_i, b_i), (a'_i, b'_i)\} \subset M$ or $\{(a_i, b'_i), (a'_i, b_i)\} \subset M$. \square

Proof of Lemma 14

Lemma 14. *Let $e = (i, j) \in E$. Let M be any matching in G such that both Z_i and Z_j are in **red** state in M . Then there are at least 100 matchings within $Y_e \cup Z_i \cup Z_j$ that defeat or tie with $M \cap (Y_e \cup Z_i \cup Z_j)$.*

Proof. We will consider three cases here.

(1) *Either (s_e, c_e) or (t_e, c'_e) is in M .* In this case it is easy to see that there are at least 100 matchings that tie with $M \cap (Y_e \cup Z_i \cup Z_j)$. Assume wlog that $(s_e, c_e) \in M$; so d_e is left unmatched. Consider the matchings obtained by replacing the edge (a_i, b_i) with the pair of edges (d_e, b_i) and (a_i, u_i^k) . Since $k \in \{0, \dots, 99\}$, there are 100 such matchings and each is tied with $M \cap (Y_e \cup Z_i \cup Z_j)$.

(2) *Both s_e and t_e are matched to their top choice neighbors in M .* So (s_e, t'_e) and (s''_e, t_e) are in M . Consider the resulting matching obtained by replacing these two edges with (s'_e, t'_e) and (s''_e, t''_e) and replacing the edges (c_e, d_e) and (a_i, b_i) with (s_e, c_e) , (d_e, b_i) , and (a_i, u_i^k) . It is easy to check that this matching is tied with M since s'_e, t'_e, s''_e, t''_e , and u_i^k prefer this matching to M while a_i, b_i, c_e, s_e , and t_e prefer M and all the other vertices are indifferent. Since $k \in \{0, \dots, 99\}$, there are at least 100 such matchings that are tied with M .

(3) *The remaining case.* So either (i) s_e is matched to a neighbor worse than c_e or (ii) t_e is matched to a neighbor worse than c'_e . Assume wlog that it is s_e that is matched to a neighbor worse than c_e (this includes the case that s_e is unmatched). We can assume $(t_e, c'_e) \notin M$ (see case (1) above).

- Suppose (s'_e, t_e) or (s''_e, t_e) is in M . Consider the resulting matching obtained by replacing this edge with either (s'_e, t'_e) or (s''_e, t''_e) appropriately (and leaving t_e unmatched) and replacing the edges (c_e, d_e) and (a_i, b_i) with (s_e, c_e) , (d_e, b_i) , and (a_i, u_i^k) . In case s_e is matched in M to v_e or v'_e , we replace this edge with (v_e, v'_e) . For the sake of simplicity, we will assume that $(v_e, v'_e) \in M$. It is easy to check that this assumption is without loss of generality.

The matching constructed above is tied with M since s_e, u_i^k , and either s'_e, t'_e or s''_e, t''_e prefer this matching to M while a_i, b_i, c_e , and t_e prefer M and all the other vertices are indifferent. Since $k \in \{0, \dots, 99\}$, there are at least 100 matchings that are tied with M .

- If either t_e was left unmatched in M or matched to one of w_e, w'_e , then we can rotate the matching edge in the triangle $\langle t_e, w_e, w'_e \rangle$ to make two vertices happy and one unhappy (notice that the preferences within the triangle are cyclic). Again we assume wlog that $(v_e, v'_e) \in M$.

Consider the resulting matching obtained by rotating the edge of M in $\langle t_e, w_e, w'_e \rangle$ and replacing the edges (c_e, d_e) and (a_i, b_i) with (s_e, c_e) , (d_e, b_i) , and (a_i, u_i^k) . This matching is tied with M since s_e, u_i^k , and two among t_e, w_e, w'_e prefer this matching to M while a_i, b_i, c_e , and one of t_e, w_e, w'_e prefer M and all the other vertices are indifferent. Since $k \in \{0, \dots, 99\}$, there are at least 100 matchings that are tied with M . \square

Proof of Lemma 15

Lemma 15. *Let M^* be any matching in G that satisfies the following three conditions:*

1. *Every vertex gadget is either in **red** or **blue** state.*
2. *For every edge gadget, at least one of its adjacent vertex gadgets is in **blue** state.*
3. *For each edge $e = (i, j)$ where $i < j$, if the vertex gadget Z_i is in **blue** state, then $M^* \cap Y_e = F_e$ otherwise $M^* \cap Y_e = L_e$.*

Then (i) M^ is popular in G and (ii) any matching that contains an inter-gadget edge loses to M^* .*

Before we prove this lemma, we describe the LP framework for popular matchings in roommates instances given in [25]. It will be convenient to consider the graph \tilde{G} which is G augmented with self-loops, i.e., every vertex considers itself its *bottom-most* acceptable choice, i.e., it finds being matched to itself more preferable than being unmatched. We can henceforth restrict our attention to perfect matchings in \tilde{G} .

For any matching N in G , define the matching \tilde{N} in \tilde{G} as $\tilde{N} := N \cup \{(u, u) : u \text{ is left unmatched in } N\}$. For any vertex u and neighbors x, y of u in G , the function $\text{vote}_u(x, y)$ will be useful to us. This is defined as follows:

$$\text{vote}_u(x, y) = \begin{cases} 1 & \text{if } u \text{ prefers } x \text{ to } y; \\ -1 & \text{if } u \text{ prefers } y \text{ to } x; \\ 0 & \text{if } u \text{ is indifferent between } x \text{ and } y. \end{cases}$$

To show property (i) on the popularity of M^* , we will use the following weight function on the edge set of \tilde{G} . For any edge $(u, v) \in E$, define $\text{wt}(u, v) = \text{vote}_u(v, \tilde{M}^*(u)) + \text{vote}_v(u, \tilde{M}^*(v))$.

So the weight of edge (u, v) is the sum of the votes of u and v for each other over their respective partners in \tilde{M}^* . For any self-loop (u, u) , let $\text{wt}(u, u) = \text{vote}_u(u, \tilde{M}^*(u))$. So $\text{wt}(u, u)$ is 0 if $\tilde{M}^*(u) = u$, else it is -1 . The definition of wt implies that for any perfect matching \tilde{N} in \tilde{G} , we have $\text{wt}(\tilde{N}) := \sum_{(u, v) \in \tilde{N}} \text{wt}(u, v) = \Delta(N, M^*)$. Thus, in order to show that M^* is popular in G , it suffices to show that for every perfect matching \tilde{N} in \tilde{G} , $\text{wt}(\tilde{N}) \leq 0$. Below we will establish this inequality for a maximum weight perfect matching.

Consider the following LP that computes a max-weight perfect matching in $\tilde{G} = (V, \tilde{E})$ under the edge weight function wt . We will be using **LP3** for analysis only – we do not have to solve it.

$$\text{maximize } \sum_{e \in \tilde{E}} \text{wt}(e) \cdot x_e \tag{LP3}$$

subject to

$$\begin{aligned} \sum_{e \in \tilde{\delta}(u)} x_e &= 1 \quad \forall u \in V \\ \sum_{e \in E[S]} x_e &\leq \lfloor |S|/2 \rfloor \quad \forall S \in \Omega \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}. \end{aligned}$$

Here $\tilde{E} = E \cup \{(u, u) : u \in V\}$ and $\tilde{\delta}(u) = \delta(u) \cup \{(u, u)\}$, where $\delta(u) \subseteq E$ is the set of edges incident to vertex u . Also Ω is the collection of all odd-sized sets $S \subseteq V$. Note that $E[S]$ is the set of edges in E with both endpoints in S .

LP3 computes $\max_{\tilde{N}} \text{wt}(\tilde{N}) = \max_N \Delta(N, M^*)$ where \tilde{N} is any perfect matching in \tilde{G} . Thus M^* is popular if and only if the optimal value of **LP3** is 0. The dual of **LP3** is given by **LP4** below.

$$\text{minimize } \sum_{u \in V} y_u + \sum_{S \in \Omega} \lfloor |S|/2 \rfloor \cdot z_S \tag{LP4}$$

subject to

$$\begin{aligned} y_u + y_v + \sum_{S \in \Omega : u, v \in S} z_S &\geq \text{wt}(u, v) \quad \forall (u, v) \in E \\ y_u &\geq \text{wt}(u, u) \quad \forall u \in V \quad \text{and} \quad z_S \geq 0 \quad \forall S \in \Omega. \end{aligned}$$

By LP-duality, M^* is popular if and only there exists a feasible solution (\vec{y}, \vec{z}) to **LP4** such that $\sum_{u \in V} y_u + \sum_{S \in \Omega} \lfloor |S|/2 \rfloor \cdot z_S = 0$. We will prove M^* 's popularity in G by an appropriate assignment of values to (\vec{y}, \vec{z}) .

Regarding property (ii), our dual solution will be an optimal solution to **LP4**. Moreover, every inter-gadget edge will be *slack* for this dual optimal solution. Hence complementary slackness conditions will imply that $\text{wt}(\tilde{N}) < 0$ for any matching N in G that contains an inter-gadget edge; in other words, $\Delta(N, M^*) < 0$ for such a matching N , i.e., N loses to M^* .

Proof. (of Lemma 15) Let $z_S = 0$ for all $S \in \Omega$. We will assign y -values as described below.

Consider any edge $e = (i, j)$. Suppose the vertex gadget Z_i is in **blue** state where $i = \min\{i, j\}$. So $M^* \cap Y_e = F_e$.

$$- \text{ let } y_{s_e} = y_{t_e} = y_{s'_e} = -1;$$

- let $y_{s'_e} = y_{t''_e} = y_{t'_e} = 1$;
- let $y_{v_e} = y_{w_e} = 1$ and $y_{v'_e} = y_{w'_e} = -1$;
- let $y_{c_e} = 1$ and $y_{d_e} = -1$ while $y_{c'_e} = -1$ and $y_{d'_e} = 1$.

Suppose the vertex gadget Z_i is in **red** state where $i = \min\{i, j\}$. So $M^* \cap Y_e = L_e$.

- let $y_{s_e} = y_{t_e} = y_{t''_e} = -1$;
- let $y_{s'_e} = y_{s'_e} = y_{t'_e} = 1$;
- let $y_{v_e} = y_{w_e} = 1$ and $y_{v'_e} = y_{w'_e} = -1$;
- let $y_{c_e} = -1$ and $y_{d_e} = 1$ while $y_{c'_e} = 1$ and $y_{d'_e} = -1$.

For a vertex gadget Z_i in **blue** state, we will set y -values as given below.

- let $y_{a_i} = y_{b_i} = 1$; let $y_{a'_i} = y_{b'_i} = -1$;
- let $y_{u_i^k} = 0$ for all $k \in \{0, \dots, 99\}$.

For any vertex gadget Z_i in **red** state, we will set y -values as follows.

- let $y_{a_i} = y_{a'_i} = 1$; $y_{b_i} = y_{b'_i} = -1$;
- let $y_{u_i^k} = 0$ for all $k \in \{0, \dots, 99\}$.

It is easy to check that the above assignment of y -values along with $\vec{z} = \vec{0}$ is a feasible solution to **LP4**. The self-loop constraints are easily seen to hold since we have $y_v \geq -1 = \text{wt}(v, v)$ for all vertices v other than the u_i^k vertices. Also, $y_{u_i^k} = 0 = \text{wt}(u_i^k, u_i^k)$ for all i and k .

We will check feasibility for the case when Z_i is in **blue** state and Z_j is in **red** state. The other cases are totally analogous. For every edge (u, v) , we need to show that $y_u + y_v \geq \text{wt}(u, v)$.

- $y_{a_i} + y_{b_i} = 2 = \text{wt}(a_i, b_i)$ and $y_{a'_i} + y_{b'_i} = -2 = \text{wt}(a'_i, b'_i)$.
- $y_{a_i} + y_{b'_i} = 0 = \text{wt}(a_i, b'_i)$ and $y_{a'_i} + y_{b_i} = 0 = \text{wt}(a'_i, b_i)$.
- $y_{a_i} + y_{u_i^k} = 1 > 0 = \text{wt}(a_i, u_i^k)$ for $k \in \{0, \dots, 99\}$.
- $y_{a_j} + y_{b_j} = 0 = \text{wt}(a_j, b_j)$ and $y_{a'_j} + y_{b'_j} = 0 = \text{wt}(a'_j, b'_j)$.
- $y_{a_j} + y_{b'_j} = 0 = \text{wt}(a_j, b'_j)$ and $y_{a'_j} + y_{b_j} = 0 = \text{wt}(a'_j, b_j)$.
- $y_{a_j} + y_{u_j^k} = 1 > 0 = \text{wt}(a_j, u_j^k)$ for $k \in \{0, \dots, 99\}$.

When Z_i is in **blue** state, recall that $M^* \cap Y_e = F_e$. So we have:

- $y_{s_e} + y_{t''_e} = 0 = \text{wt}(s_e, t''_e)$ and $y_{s'_e} + y_{t_e} = 0 = \text{wt}(s'_e, t_e)$.
- $y_{s_e} + y_{t'_e} = 0 = \text{wt}(s_e, t'_e)$ and $y_{s'_e} + y_{t_e} = -2 = \text{wt}(s'_e, t_e)$.
- $y_{s'_e} + y_{t'_e} = 0 = \text{wt}(s'_e, t'_e)$ and $y_{s''_e} + y_{t''_e} = 2 = \text{wt}(s''_e, t''_e)$.
- $y_{s_e} + y_{v_e} = 0 = \text{wt}(s_e, v_e)$ and $y_{s_e} + y_{v'_e} = -2 = \text{wt}(s_e, v'_e)$.
- $y_{t_e} + y_{w_e} = 0 = \text{wt}(t_e, w_e)$ and $y_{t_e} + y_{w'_e} = -2 = \text{wt}(t_e, w'_e)$.
- $y_{v_e} + y_{v'_e} = 0 = \text{wt}(v_e, v'_e)$ and $y_{w_e} + y_{w'_e} = 0 = \text{wt}(w_e, w'_e)$.
- $y_{s_e} + y_{c_e} = 0 = \text{wt}(s_e, c_e)$ and $y_{t_e} + y_{c'_e} = -2 = \text{wt}(t_e, c'_e)$.
- $y_{c_e} + y_{d_e} = 0 = \text{wt}(c_e, d_e)$ and $y_{c'_e} + y_{d'_e} = 0 = \text{wt}(c'_e, d'_e)$.

Finally we will check that the inter-gadget edges are covered.

- $y_{b_i} + y_{d_e} = 0 > -1 = \text{wt}(b_i, d_e)$ and $y_{b_j} + y_{d'_e} = 0 > -1 = \text{wt}(b_j, d'_e)$.

Observe that we have $y_u + y_v = 0$ for every edge $(u, v) \in M^*$ and $y_{u_i^k} = 0$ for every vertex u_i^k left unmatched in M^* . Thus the objective function of **LP4** with the above setting of (\vec{y}, \vec{z}) evaluates to 0. Hence M^* is popular in G . This finishes the proof of property (i) of Lemma 15.

We will now show property (ii). Note that condition 2 in the statement of Lemma 15 is necessary to show property (ii): Indeed, we know from the proof of Lemma 14 that without this condition, i.e., if both Z_i and

Z_j are in **red** state for some edge (i, j) , there are matchings containing inter-gadget edges that do *not* lose to our matching.

We described above a vector \vec{y} such that $(\vec{y}, \vec{0})$ is a solution to LP4 and we will now use complementary slackness to show property (ii). Observe that $(\vec{y}, \vec{0})$ is an optimal solution to LP4 since its objective function value is 0 which is the same as $\text{wt}(M^*)$ since $\Delta(M^*, M^*) = 0$.

We will show that every inter-gadget edge is *slack* with respect to $(\vec{y}, \vec{0})$. Notice that slackness of every inter-gadget edge with respect to $(\vec{y}, \vec{0})$ implies, by complementary slackness, that any matching N that includes an inter-gadget edge cannot be an optimal solution to LP3. That is, $\text{wt}(\tilde{N}) = \Delta(N, M^*) < 0$, or, in other words, N loses to M^* . So for each edge $e = (i, j)$, where $i < j$, it suffices to check that the inter-gadget edges (d_e, b_i) and (d'_e, b_j) are slack.

Whether Z_i is in **blue** state or in **red** state, we have $\text{wt}(d_e, b_i) = -1$ since b_i prefers $M^*(b_i) \in \{a_i, a'_i\}$ to d_e while d_e is indifferent between $M^*(d_e) = c_e$ and b_i . Similarly, whether Z_j is in **blue** state or in **red** state, we have $\text{wt}(d'_e, b_j) = -1$ since b_j prefers $M^*(b_j) \in \{a_j, a'_j\}$ to d'_e while d'_e is indifferent between $M^*(d'_e) = c'_e$ and b_j .

Recall that $y_{b_i}, y_{b_j}, y_{d_e}, y_{d'_e} \in \{\pm 1\}$. So in the constraint for any inter-gadget edge in LP4, the left side is an even number while the right side is -1 . Thus every constraint in LP4 for an inter-gadget edge is slack. Hence any matching that contains an inter-gadget edges loses to M^* . \square

Proof of Lemma 18

Lemma 18. *If M is a Copeland winner in G , then M does not use any inter-gadget edge.*

Proof. The proof is similar to that of Lemma 16. Below, we will outline the main steps involved in the proof.

Suppose, for contradiction, that M uses an inter-gadget edge, say (b_i, d_e) (see Fig. 3). Let \mathcal{X} denote the set of all vertex and edge gadgets in the matching instance. Consider a partitioning of \mathcal{X} into *single*, *double*, *triple*, and *auxiliary* gadgets as follows:

- Each *double* (resp., *triple*) gadget is a pair of an edge gadget and an adjacent vertex gadget (Y_e, Z_i) (resp., a triple of an edge and two adjacent vertex gadgets (Y_e, Z_i, Z_j)) where $e = (i, j)$ is an edge such that M contains an inter-gadget edge between Y_e and Z_i (resp., two inter-gadget edges between Y_e and each of Z_i and Z_j). While there is an edge gadget that shares an inter-gadget edge with one of (resp., both) its adjacent vertex gadgets, we make a double (resp., triple) gadget out of these two (resp., three) gadgets. Note that a vertex gadget can be included in at most one double/triple gadget in this manner.
- Next, if there is an edge gadget that is adjacent to two vertex gadgets both of which are in **red** state and are still unclaimed by any edge gadget, then this edge gadget claims both these vertex gadgets; such a triple of edge gadget and its adjacent vertex gadgets is classified as an *auxiliary* gadget. Note that such an edge gadget does not share an inter-gadget edge with either of the vertex gadgets.
- All remaining vertex and edge gadgets are classified as *single* gadgets.

Observe that

$$\text{loss}(M) + \text{ties}(M) \geq \prod_{X \in \mathcal{X}} (\text{loss}(M \cap X) + \text{ties}(M \cap X)),$$

where X denotes any single, double, triple or auxiliary gadget.

Consider any double or triple gadget X . Since the matching M uses at least one inter-gadget edge (b_i, d_e) in X , the vertex a_i must be matched with either the vertex b'_i or one of the vertices in u_i^0, \dots, u_i^{99} (otherwise, if a_i is unmatched, then M will not be Pareto optimal (see footnote 6) and it is easy to see that every Copeland winner has to be Pareto optimal). In both cases, the matching $\{(a_i, u_i^{k'}), (a'_i, b'_i)\}$ is tied with M where $k' \in \{0, \dots, 99\}$. Thus, there are at least 100 matchings that defeat or tie with $M \cap (Y_e \cup Z_i)$. In other words, for every double or triple gadget X , the value of $\text{loss}(M \cap X) + \text{ties}(M \cap X)$ is at least 100. We therefore have

$$\text{loss}(M) + \text{ties}(M) \geq 2^{n'} \cdot 3^{n''} \cdot 10^{m'} \cdot 100^{t_2} \cdot 100^{t_3} \cdot 100^a, \quad (7)$$

where n' (resp., n'') is the number of vertices present as single gadgets in **red** (resp., **blue**) state, m' is the number of edges that are present as single gadgets, t_2 is the number of double gadgets, t_3 is the number

of triple gadgets, and a is the number of auxiliary gadgets in the aforementioned partition. As done in Lemma 16, we once again used Lemma 12 in bounding $\text{loss}(M \cap Y_e) + \text{ties}(M \cap Y_e)$ by 10 for every edge gadget Y_e that is present as a single gadget, and used Observation 1 in bounding $\text{loss}(M \cap Z_i) + \text{ties}(M \cap Z_i)$ by 2 (resp., 3) for every vertex gadget Z_i (acting as single gadget) that is in **red** (resp., **blue**) state. Additionally, we used Lemma 14 to obtain the corresponding bound for an auxiliary gadget.

We will now construct an alternative matching M^* that has a higher Copeland score than M to derive the desired contradiction. Starting with M , let us remove any inter-gadget edges from each double/triple gadget and convert both $M \cap Z_i$ and $M \cap Z_j$ in the triple gadget (or just $M \cap Z_i$ in case of a double gadget) to **blue** state and replace $M \cap Y_e$ with F_e . Additionally, for each auxiliary gadget (Y_e, Z_i, Z_j) , we convert both $M \cap Z_i$ and $M \cap Z_j$ to **blue** state and replace $M \cap Y_e$ with F_e .

Note that for any edge $e = (i, j)$ in a single gadget, either Z_i or Z_j is now in **blue** state. If it is $\min\{i, j\}$ that is in **blue** state, then let $M^* \cap Y_e = F_e$, else let $M^* \cap Y_e = L_e$. The rest of the gadgets are in the same state as under M .

Notice that M^* satisfies the conditions in Lemma 15. Thus, $\text{loss}(M^*) = 0$ and any matching that ties with M^* must not use any inter-gadget edge. Therefore, the number of matchings that tie with M^* across the entire graph G is simply the product of matchings that tie with it on individual single, double, triple, and auxiliary gadgets. Hence,

$$\text{loss}(M^*) + \text{ties}(M^*) = \text{ties}(M^*) = 2^{n'} \cdot 3^{n''} \cdot 10^{m'} \cdot 30^{t_2} \cdot 90^{t_3} \cdot 90^a, \quad (8)$$

where n', n'', m', t_2, t_3 and a are defined as in (7). For any double or triple gadget, by Observation 1 and Lemma 11, the entire contribution to $\text{ties}(M^*)$ due to these two gadgets is 30 (or 90 in case of three gadgets) and to $\text{loss}(M^*)$ is 0. Additionally, for an auxiliary gadget, the value of $\text{ties}(M^*)$ is equal to $3 \times 3 \times 10 = 90$.

Comparing (7) and (8) along with the fact that $\text{loss}(M) \geq 0 = \text{loss}(M^*)$, we get that $\text{score}(M^*) > \text{score}(M)$. Indeed,

$$\text{loss}(M^*) + \text{ties}(M^*)/2 = 2^{n'-1} \cdot 3^{n''} \cdot 10^{m'} \cdot 30^{t_2} \cdot 90^{t_3} \cdot 90^a, \text{ and}$$

$$\text{loss}(M) + \text{ties}(M)/2 \geq \frac{1}{2} (\text{loss}(M) + \text{ties}(M)) \geq 2^{n'-1} \cdot 3^{n''} \cdot 10^{m'} \cdot 100^{t_2} \cdot 100^{t_3} \cdot 100^a.$$

This contradicts the fact that M is a Copeland winner. Thus, a Copeland winner must not use any inter-gadget edge. \square