# OPTIMAL STOPPING IN A TWO-SIDED SECRETARY PROBLEM 

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#### Abstract

In the "secretary problem," well-known in the theory of optimal stopping, an employer is about to interview a maximum of $N$ secretaries about which she has no prior information. Chow et al. proved that with an optimal strategy the expected rank of the chosen secretary tends to $\prod_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{1 /(k+1)} \approx 3.87$.

We study a two-sided game-theoretic version of this optimal stopping problem, where men search for a woman to marry at the same time as women search for a man to marry. We find that in the unique subgame perfect equilibrium, the expected rank grows as $\sqrt{N}$ and that, surprisingly, the leading coefficient is exactly 1. We also discuss some possible variations.


## 1. Introduction

A mathematical result of such general appeal that it sometimes appears in ordinary newspapers (such as The Independent [4]) is the " $37 \%$ rule." This rule states that if you are walking down a one-way street of restaurants of which you have no knowledge beforehand and you have decided that, no matter what, you will pick the hundredth restaurant if you haven't picked one before that, then you should walk by 37 restaurants and then pick the first one that you like better than all the previous restaurants. This rule is optimal in the sense that you will maximize your chance of picking the best of the hundred restaurants.
1.1. The secretary problem. Although the restaurant setting may be the one most pertinent to every-day life, the problem has become known as the secretary problem in the mathematics literature. Instead of a hungry tourist judging restaurants, you are now an employer interviewing candidates for a position as your secretary. After each interview you must decide whether to hire this candidate or continue the search process, in which case you cannot return to this candidate. If

[^0]for some reason you will look at $N$ candidates at most, then you should never hire among the first $N / e$ candidates and then hire the first one that you like better than all the previous ones. Since $1 / e \approx 0.37$, this is the $37 \%$ rule. The history of the secretary problem has been nicely told by Ferguson [7].

The secretary problem is the prime example of a question of optimal stopping. The theory of optimal stopping was treated in a comprehensive way more than thirty years ago by Chow, Robbins and Siegmund [3], and more recently by Ferguson [6]. Our inspiration to the following piece of research came from a short paper by Steven Finch on mathematical constants from optimal stopping [8]. After mentioning the secretary problem and its constant $1 / e$, Finch continues with the following much less known variation, which is not treated in either of the above-mentioned books: What is the optimal stopping strategy if you want to minimize the expected rank $R_{N}$ of the chosen secretary? (The $N$ candidates are ranked so that the best one has rank 1, the second-best has rank 2, etc.) Lindley [10] found an optimal strategy, and Chow et al. [2] proved that using this strategy the expected rank $R_{N}$ tends to a constant as $N$ tends to infinity:

$$
\lim _{N \rightarrow \infty} R_{N}=\prod_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{1 /(k+1)} \approx 3.87
$$

We found this result doubly amazing - not only does it defy intuition that even with millions of candidates we can expect to end up with the fourth-best just by using a stopping rule, but it also seems unlikely that the constant could be expressed on such a closed form.
1.2. A two-sided secretary problem. The article in The Independent on the $37 \%$ rule was based on an interview with Peter Todd, a psychologist working on social heuristics, i.e. behavioral rules of thumb. By laboratory experiments as well as computer simulations, Todd and Miller [13, 5] have investigated simple heuristics for mate search where both sexes have the possibility of saying no to a partner. Mate search can therefore be described as a two-sided secretary problem. There are several natural variants of the exact specification. However, we have been unable to find any previous mathematical treatment of any variant of the two-sided secretary problem in the optimal stopping literature.

In this paper we are most interested in the following game theoretic setting of the problem. There is a large universe of $U$ men and $U$ women. Every woman has a personal total preference order on the men, and vice versa. In other words, for each woman $w$ there is a permutation of the set of men representing her personal rank order,
and similarly for each man. The preferences are unknown, only to be partially revealed through extensive dating. Each of these persons are willing to search for a mate for a maximum of $N$ rounds, where $N \ll U$ (think of $N$ in the tens or hundreds, and $U$ in the billions). In each round every woman will date a new man, and they can either agree on marrying (and leave the game) or at least one of them decides not to marry, in which case they proceed to the next round and will never date each other again. In the $N$ th round, the players have no other option than marrying. The condition that $N \ll U$ will guarantee that all players can count on new partners being available for all $N$ rounds, even though most will leave the game before the last round.

Now, where do the permutations come from? We make the usual assumption in the secretary problem that each permutation is drawn randomly among all possible permutations. We also assume that a random choice determines who will date who in each round (among all unseen partners still available). This means that from the viewpoint of any player entering round $r$, the rank of the next date relative the $r-1$ partners already observed is a random variable drawn from a uniform distribution on the set of ranks from 1 to $r$. Let us call this the next-player principle.

Finally, what is the goal of the players? It is not very meaningful to maximize the probability of marrying the top-ranked mate, since the player will only ever see a very small portion of the universe, $N$ out of $U$. Instead, as in Lindley's variant of the secretary problem [10, 2] we assume that each player wants to minimize the expected rank of the mate among the $N$ partners the player would meet if she completed all $N$ rounds. Although the actual set of partners that a player would have met is not determined if she does not play the whole game, the expected rank can be computed by a simple formula that follows by induction from the next-player principle:

$$
R_{N}\left(r, R_{r}\right)=\frac{N+1}{r+1} R_{r}
$$

where $R_{N}$ is the expected rank in round $N$ of our current date in round $r$, ranked $R_{r}$ among the $r$ partners observed up to then. When several rank concepts are thrown around simultaneously, we will refer to this as the $N$-rank.

A strategy in this two-sided secretary game is a rule that says whether to propose marriage in round $r$ to a date of observed rank $R_{r}$. Remember that the player cannot be certain that the date will agree on marrying. (If players on the other side always agree, then we would be back in the one-sided secretary problem.)
1.3. A cooperative problem. To begin with, we can take the viewpoint of cooperative game theory where players can make a binding agreement to play a certain strategy.

Problem 1. If the players can make a binding agreement beforehand on a common strategy, what is the optimal choice? What is the expected $N$-rank under this strategy?

In this case the problem is a pure optimization problem of finding the strategy on which to agree. Given every player's strategy, the $N$-rank $R_{N}(r)$ a player can expect when she enters round $r$ can be computed by the following recurrence:
(1) $R_{N}(r)=P[$ marry $] \cdot \frac{N+1}{r+1} \cdot E\left[R_{r} \mid\right.$ marry $]+(1-P[$ marry $]) \cdot R_{N}(r+1)$.

The boundary condition is $R_{N}(N)=(N+1) / 2$ since at the last round each player can expect to obtain an average partner. As before, $R_{r}$ denotes the rank of your current date among the $r$ partners you have seen. We desire to compute $R_{N}(1)$, the expected $N$-rank when we enter the game.

We have not solved Problem 1 completely, but we will motivate the following conjecture which defines a new optimal stopping constant:

Conjecture 2. In the cooperative two-sided secretary game, the asymptotics of the expected $N$-rank when entering the game is given by

$$
\lim _{N \rightarrow \infty} \frac{R_{N}(1)}{\sqrt{N}}=\sqrt{27 / 32} \approx 0.92
$$

1.4. A noncooperative problem. Now switch to the noncooperative version of the game, where each player tries to optimize her own outcome.

Problem 3. What is the optimal strategy in the noncooperative twosided secretary game, and what is the expected $N$-rank under this strategy?

In noncooperative game theory, a strategy can only be optimal given the strategies of the other players. If each player plays a strategy that is optimal given the strategies of everybody else, then we have a Nash equilibrium. A common problem in game theory is that there exist very many Nash equilibria (such as "always marry in the first round"). The solution proposed by Selten [12] is to assume that perfectly rational players will coordinate on a subgame perfect equilibrium, that is, an equilibrium where all players' strategies are optimal in every possible subgame. In the two-sided secretary game, finding the unique subgame
perfect equilibrium is a recursive optimization problem: In the last round, every player will accept marriage since it is always preferable to not marrying at all; in the next to last round, a player will want to marry if her current date is better than what she would expect to obtain in the last round, etc. We refer to this as the optimal strategy, which will be the same for all players.

In this case we have been able to solve the problem completely. The answer is surprisingly simple.

Theorem 4. In the noncooperative two-sided secretary game, the asymptotics of the expected $N$-rank when entering the game is given by

$$
\lim _{N \rightarrow \infty} R_{N}(1) / \sqrt{N}=1
$$

This is our main result.
1.5. The two-sided secretary problem with universal rank symmetry. Due to the risk of rejections from good partners, each player in the two-sided game is at a clear disadvantage compared to an employer in a one-sided secretary problem. This phenomenon is inherent to two-sidedness, but it is reasonable to ask how small the disadvantage can be made if preferences are dependent on each other so that the probability of you liking me increases if I like you.

An extreme case in this direction is universal rank symmetry. By this we mean that each woman $w$ and man $m$ have the same universal rankings of each other, so that if $m$ is ranked by $w$ as the $R$ th best man in the universe, then $w$ is ranked by $m$ as the $R$ th best woman in the universe. (In combinatorial terms, the women's permutations of the men constitute a $U \times U$ Latin square since no two women can give the same rank to the same man. The women's Latin square then uniquely determines the men's Latin square.)

Problem 5. What is the optimal strategy in the two-sided secretary game with universal rank symmetry, and what is the expected $N$-rank under this strategy?

We conjecture that universal rank symmetry radically changes the outcome, so that the expected $N$-rank tends to a small constant:

Conjecture 6. In the two-sided secretary game with universal rank symmetry, the asymptotics of the expected $N$-rank when entering the game is given by

$$
\lim _{N \rightarrow \infty} R_{N}(1)=\text { constant }<5
$$

1.6. Outline of paper. We will work through the problems in the above order, starting with our partial solution to the cooperative version. We will then treat the noncooperative problem in the same way, but here we have been able to compute the asymptotics using a method inspired by Chow et al. [2]. After partially treating the case of universal rank symmetry, we conclude with some discussion and open problems.

## 2. Problem 1: The optimal strategy under binding AGREEMENT

Let us assume that the players agree on a strategy, i.e. a set of thresholds $s_{1}, s_{2}, \ldots, s_{N-1}$ together with the rule that you must propose marriage to your date in round $r$ if and only if his observed rank satisfies $R_{r} \leq s_{r}$. Without loss of generality we can always choose the threshold $s_{r}$ to be an integer between 0 and $r$. Then, according to the next-player principle, the probability that you will propose in round $r$ is $s_{r} / r$. Hence, the probability of an agreement to marry is

$$
P[\text { marry }]=\left(\frac{s_{r}}{r}\right)^{2},
$$

and the expected observed rank of your partner, given that you propose, is

$$
E\left[R_{r} \mid \text { marry }\right]=\frac{s_{r}+1}{2} .
$$

We can plug these expressions into the fundamental recurrence (1). Setting $n=N-r$ (the number of rounds remaining) and setting $\rho_{n}=$ $R_{N}(N-n) \cdot 2 /(N+1)$, the recurrence takes the shape

$$
\begin{equation*}
\rho_{n}=\left(\frac{s_{N-n}}{N-n}\right)^{2} \cdot \frac{s_{N-n}+1}{N-n+1}+\left(1-\left(\frac{s_{N-n}}{N-n}\right)^{2}\right) \cdot \rho_{n-1} \tag{2}
\end{equation*}
$$

with boundary condition $\rho_{0}=1$. We want to minimize $\rho_{N-1}$. All factors and terms are nonnegative, so for each $n$ from 1 to $N-1$ we simply want to agree on the threshold $s_{N-n}$ that will minimize $\rho_{n}$. To find this threshold, differentiate the expression and find the zero to be $s_{N-n}=\frac{2}{3}\left(-1+(N-n+1) \cdot \rho_{n-1}\right)$. Then adjust the threshold to the closest integer between 0 and $r$. For example, setting $n=1$ we see that in the next-to-last round we shall propose if our current date is among the best two thirds of all partners we have seen.

For $n \ll N$ we obtain the approximative recurrence

$$
\rho_{n} \approx \rho_{n-1}-\frac{4}{27} \rho_{n-1}^{3}, \quad \rho_{0}=1
$$

Making the ansatz $\rho_{n}=A \cdot(n+B)^{C}$ and approximating the difference $\rho_{n}-\rho_{n-1}$ by the derivative, we obtain the solution $\rho_{n}=\sqrt{27 / 8} \cdot(n+$ 4) ${ }^{-1 / 2}$.

In the unlikely event that this crude approximation works all the way for $n=0,1, \ldots, N-1$, then a player's expected $N$-rank when entering the game would be $R_{N}(1)=\frac{N+1}{2} \rho_{N-1} \approx \sqrt{27 N / 32}$. Amazingly, computer calculations seem to vindicate that $\lim _{N \rightarrow \infty} R_{N}(1) / \sqrt{N} \approx$ $0.92 \approx \sqrt{27 / 32}$, see the graph in Figure 1


Figure 1. The graph of $R_{N}(1) / \sqrt{N}$ for the two-sided secretary game under optimal binding agreement.

To summarize this section: We have found a recurrence giving the optimal strategy under cooperation, and we conjecture that the asymptotic behavior is given by $\lim _{N \rightarrow \infty} R_{N}(1) / \sqrt{N}=\sqrt{27 / 32}$. It is possible that this problem might yield to the same method that we use in the next section.

## 3. Problem 2: The optimal noncooperative strategy

The strategy on which the players agreed in the previous section is not a Nash equilibrium. This is obvious from the observation that in the next-to-last round all players agreed to propose to any partner among the best two thirds, while a rational player who is not under any binding agreement would not accept a partner worse than average, since she would expect an average partner in the last round.

The thresholds that define the subgame perfect equilibrium strategy in this noncooperative case are determined by what you can expect to get if you decline to marry. You should accept marriage in round $r$ if and only if the expected $N$-rank if you marry now is less than or equal to the expected $N$-rank if you do not marry:

$$
\frac{N+1}{r+1} \cdot R_{r} \leq R_{N}(r+1)
$$

and the threshold $s_{r}$ should be the largest integer value of $R_{r}$ satisfying the above inequality, so that

$$
s_{r}=\left\lfloor\frac{r+1}{N+1} \cdot R_{N}(r+1)\right\rfloor .
$$

We can assume all other players to reason in the same way, so that we have $P$ [marry] $=\left(s_{r} / r\right)^{2}$ as in the previous section. Consequently, we can just plug this new value of $s_{r}$ into the recurrence (21). The same approximations now yield

$$
\rho_{n} \approx \rho_{n-1}-\frac{1}{8} \rho_{n-1}^{3}, \quad \rho_{0}=1,
$$

with an approximate solution of $\rho_{n}=2(n+4)^{-1 / 2}$. As in the cooperative case, the validity of this approximation is supported by computer calculations indicating that

$$
\lim _{N \rightarrow \infty} R_{N}(1) / \sqrt{N}=1
$$

This is our Theorem 4. We will prove this result using the same approach as Chow et al. [2], although we will encounter different technical difficulties than they did.
3.1. Proof of Theorem 4. For convenience, introduce $i=r-1$ (so that the range is $i=0,1, \ldots, N-1)$ and $c_{i}=R_{N}(i+1)$. We wish to prove that $c_{0} / \sqrt{N} \rightarrow 1$. Define the unrounded threshold

$$
t_{i}=\frac{i+1}{N+1} \cdot c_{i}
$$

so that $s_{i}=\left\lfloor t_{i}\right\rfloor$. Then recurrence (11) can be rewritten as

$$
\begin{equation*}
t_{i-1}=\frac{s_{i}^{2}\left(s_{i}+1\right)+2\left(i^{2}-s_{i}^{2}\right) t_{i}}{2 i(i+1)} ; \quad t_{N-1}=\frac{N}{2} \tag{3}
\end{equation*}
$$

For any $i$, define $\alpha_{i}=t_{i}-s_{i}$. Then $0 \leq \alpha_{i}<1$, and (3) becomes

$$
\begin{equation*}
t_{i-1}=\frac{\left(t_{i}-\alpha_{i}\right)^{2}\left(t_{i}-\alpha_{i}+1\right)+2\left(i^{2}-\left(t_{i}-\alpha_{i}\right)^{2}\right) t_{i}}{2 i(i+1)} \tag{4}
\end{equation*}
$$

It is trivial to see that $a_{i} t_{i}+\left(1-\alpha_{i}\right)\left[t_{i}^{2}+\alpha_{i}\left(t_{i}-\alpha_{i}\right)\right] \geq 0$. If we add this number to the nominator of (4) we obtain the upper bound

$$
\begin{equation*}
t_{i-1} \leq T_{i}\left(t_{i}\right) \stackrel{\text { def }}{=} \frac{-t_{i}^{3}+2 t_{i}^{2}+2 i^{2} t_{i}}{2 i(i+1)} \tag{5}
\end{equation*}
$$

Similarly, if we subtract $\alpha_{i}\left[\left(t_{i}-1\right)^{2}+\alpha_{i}\left(t_{i}-\alpha_{i}\right)\right]+\left(t_{i}-\alpha_{i}\right)+\alpha_{i}^{2} \geq 0$ from the nominator of (4) we obtain the lower bound

$$
\begin{equation*}
t_{i-1} \geq \tau_{i}\left(t_{i}\right) \stackrel{\text { def }}{=} \frac{-t_{i}^{3}+t_{i}^{2}+\left(2 i^{2}-1\right) t_{i}}{2 i(i+1)} \tag{6}
\end{equation*}
$$

It is easy to show that, if $i \geq 2$, the functions $T_{i}(t)$ and $\tau_{i}(t)$ are both increasing for all $t$ in the range $[0, \sqrt{2 / 3} i]$, and that this interval includes all values $t$ can possibly attain, so we can use these recursive inequalities in order to prove explicit lower and upper bounds.

Lemma 7. Let $f(N)$ be any real function with $\lim _{N \rightarrow \infty} f(N)=\infty$. For sufficiently large $N$ the upper bound

$$
t_{i} \leq \frac{i+\sqrt{i}}{\sqrt{N-i+3}}
$$

holds for all $i$ in the interval $f(N) \leq i \leq N-1$.
Proof. Proof by backwards induction. The lemma holds for $i=N-1$, since

$$
t_{N-1}=N / 2 \leq \frac{N-1+\sqrt{N-1}}{\sqrt{4}}
$$

for all $N \geq 2$. Assuming the lemma holds for a given $i$ we have

$$
t_{i-1} \leq T_{i}\left(t_{i}\right) \leq T_{i}\left(\frac{i+\sqrt{i}}{\sqrt{N-i+3}}\right)
$$

thanks to (5) and the fact that $T_{i}$ is an increasing function. Hence, to conclude the induction step we only need to prove that

$$
T_{i}\left(\frac{i+\sqrt{i}}{\sqrt{N-i+3}}\right) \leq \frac{i-1+\sqrt{i-1}}{\sqrt{N-i+4}}
$$

This can be verified through several steps of computations in Maple (see Appendix).

The lower bound is trickier. We have not been able to find a lower bound for which the induction step works, but by adding a constant term of 0.148 to the denominator we obtain a lower bound that works from $N-22$ and downwards.

Lemma 8. For sufficiently large $N$, the lower bound

$$
t_{i} \geq \frac{i+1}{\sqrt{N-i+3}+0.148}
$$

holds for all $i$ in the interval $\sqrt{N}+1 \leq i \leq N-22$.
Proof. For large $N$ we can compute an approximation of $t_{N-22}$ by using the recurrence 21 steps, keeping only the most significant term in each step: If $t_{N-k}=a_{k} N+o(N)$ we have $s_{N-k}=a_{k} N+o(N)$ too, so our recurrence (3) gives $t_{N-k-1}=\frac{-a_{k}^{3}+2 a_{k}}{2} N+o(N)$. Maple gives

$$
t_{N-22} \approx 0.19427 N \geq 0.19425 N-4.07925=\frac{(N-22)+1}{\sqrt{22+3}+0.148}
$$

Again we proceed by backwards induction. Assuming the lemma holds for a given $i$ we have

$$
t_{i-1} \geq \tau_{i}\left(t_{i}\right) \geq \tau_{i}\left(\frac{i+1}{\sqrt{N-i+3}+0.148}\right)
$$

thanks to (6) and the fact that $\tau_{i}$ is an increasing function. It remains for us to prove that

$$
\tau_{i}\left(\frac{i+1}{\sqrt{N-i+3}+0.148}\right) \geq \frac{i}{\sqrt{N-i+4}+0.148}
$$

for large $N$. Again, several steps in Maple verifies this inequality, see Appendix.

We are now in a position to finish the proof of Theorem 4. In the case $t_{i}<1$ we always have $s_{i}=0$ so no partners are ever accepted and the expected $N$-rank $c_{i}$ remains constant down to $c_{0}$. Let $i_{\text {crit }}$ be the greatest $i$ with $t_{i}<1$, so that $c_{0}=c_{1}=\cdots=c_{i_{\text {crit }}}$.

Lemma 8 gives that, for large $N$,

$$
t_{\lceil\sqrt{N}+1\rceil} \geq \frac{\lceil\sqrt{N}+1\rceil+1}{\sqrt{N-\lceil\sqrt{N}+1\rceil+3}+0.148}>1
$$

so $i_{\text {crit }}<\sqrt{N}+1$. Lemma 7 with $f(N)=N^{1 / 2}-N^{1 / 3}$ gives, for large $N$,

$$
\begin{gathered}
t_{\left\lceil N^{1 / 2}-N^{1 / 3}\right\rceil} \leq \frac{\left\lceil N^{1 / 2}-N^{1 / 3}\right\rceil+\sqrt{\left\lceil N^{1 / 2}-N^{1 / 3}\right\rceil}}{\sqrt{N-\left\lceil N^{1 / 2}-N^{1 / 3}\right\rceil+3}} \\
=\sqrt{\frac{N-2 N^{5 / 6}+o\left(N^{5 / 6}\right)}{N-N^{1 / 2}+o\left(N^{1 / 2}\right)}}<1
\end{gathered}
$$

so $i_{\text {crit }}+1 \geq N^{1 / 2}-N^{1 / 3}$.
Since $N^{1 / 2}-N^{1 / 3} \leq i_{\text {crit }}+1<\sqrt{N}+2$, Lemma 7 with $f(N)=$ $N^{1 / 2}-N^{1 / 3}$ gives that

$$
1 \leq t_{i_{\text {crit }}+1} \leq \frac{i_{\text {crit }}+1+\sqrt{i_{\text {crit }}+1}}{\sqrt{N-i_{\text {crit }}+2}} \leq \frac{\sqrt{N}+2+\sqrt{\sqrt{N}+2}}{\sqrt{N-\sqrt{N}+1}} \rightarrow 1
$$

as $N \rightarrow \infty$. Then $s_{i_{\text {crit }}+1}=1$, and by (3) we get $t_{i_{\text {crit }}} \rightarrow 1$.
Recall that, by definition, $t_{i}=c_{i}(i+1) /(N+1)$ so that $c_{i_{\text {crit }}}=$ $t_{i_{\text {crit }}}(N+1) /\left(i_{\text {crit }}+1\right)$. Thus, we have

$$
\frac{c_{0}}{\sqrt{N}}=\frac{c_{i_{\text {crit }}}}{\sqrt{N}}=\frac{N+1}{i_{\text {crit }}+1} \cdot \frac{t_{i_{\text {crit }}}}{\sqrt{N}} \rightarrow 1
$$

since $t_{i_{\text {crit }}} \rightarrow 1$ and $N^{1 / 2}-N^{1 / 3}-1 \leq i_{\text {crit }}<\sqrt{N}+1$.
3.2. A social dilemma. This game illustrates the game-theoretic concept of a social dilemma. All players would like to agree on the cooperative strategy, where you accept to marry quite a lot of partners for the good of the group. However, when an individual player finds herself in a position where the strategy calls on her to marry a date that she finds below her expectations, she is tempted to reject this partner and optimize her own good instead. But if all players do that, then the expected outcome is worse for all of them; for large $N$ the expected change for the worse is about 8 percent.

## 4. Problem 3: The optimal strategy under universal rank SYMMETRY

If the preferences of all players satisfy universal rank symmetry, then it should be easier for the players to find mutually acceptable agreements.

As in the previous problem, a rational player will accept a partner in round $r$ if the observed rank does not exceed the threshold $s_{r}=$ $\left\lfloor\frac{r+1}{N+1} \cdot R_{N}(r+1)\right\rfloor$. The new circumstance in the current setting is that the events "your observed rank of me is at most $s_{r}$ " and "my observed rank of you is at most $s_{r}$ " are no longer independent. Hence the probability $P$ [marry $]=P\left[\right.$ both ranks $\left.\leq s_{r}\right]$ must be found by
other means. Integration over the unknown global rank, which is the same for both players, yields

$$
\begin{equation*}
P[\text { marry }]=\sum_{k=0}^{s_{r}-1} \sum_{\ell=0}^{s_{r}-1} \frac{\binom{r-1}{k}\binom{r-1}{\ell}}{\binom{2 r-1)}{k+\ell}(2 r-1)} \tag{7}
\end{equation*}
$$

Similarly, the expected rank of a partner given the fact that you both agree to marry is no longer given by the arithmetic mean in the rank interval $\left[1, s_{r}\right]$ but is more favorable:

$$
\begin{equation*}
E\left[R_{r} \mid \text { marry }\right]=\frac{r}{s_{r}} \sum_{k=0}^{s_{r}-1}(k+1)\binom{r-1}{k} \sum_{\ell=0}^{s_{r}-1} \frac{\binom{r-1}{\ell}}{\binom{2 r-1)}{k+\ell}(2 r-1)} \tag{8}
\end{equation*}
$$

The recurrence resulting from plugging these expressions into (1) determines the optimal strategy under universal rank symmetry.

We have not yet been able to find the limit of the expected $N$-rank in this case, but computer calculations indicate that it approaches a small constant (less than 5). Thus, making the preferences dependent of each other in this way seems to have changed the behavior of the expected rank so that it resembles the behavior in the one-sided problem.

## 5. Discussion and open problems

For now, we have to leave our conjectures of the asymptotic behavior of the expected $N$-rank in Problem 1 and 3 as open problems. Another open problem is to find the optimal strategy when the universe is small, $U=N$, so that you do not know during a date whether there will be any more dates or if all possible partners will already be married if you reject the one you are currently entertaining. In this case, another reasonable version is obtained if we lift the restriction on dating the same person twice so that in each round one would simply select a random date among all remaining unmarried players of the opposite sex.

In the context of the two-sided secretary problem, it is reasonable to briefly discuss the theory of stable matching, pioneered by Gale and Shapley [9] and later comprehensively treated by Roth and Sotomayor [11. Also this theory studies men and women who have preferences on each other. The difference is that the stable matching theory assumes that each player knows all her preferences in advance, while in the secretary problem the preferences are only revealed slowly as the player meets new potential partners. On the other hand, in the secretary problem we have no guarantees that the marriages will be stable when new partners are met later in life. It would be interesting to study some measure of how stable these marriages will be under various conditions.

In a recent study, Caldarelli and Capocci [1] have studied stable marriages under the assumption that people's preferences are influenced by a commonly appreciated trait such as beauty. They find that this condition favors the very beautiful players while all others are worse off than in a world where preferences are random. Universal rank symmetry is an assumption in the opposite direction. True preferences are likely to reflect a mixture of randomness, a common sense of beauty, and an "I-like-you-if-you-like-me" component like our universal rank symmetry. Modelling such sets of permutations is a challenge to combinatorialists.

## Appendix

Here we prove two lemmas that are needed in the proof of Theorem 4.
Lemma 9. Let $f(N)$ be any real function with $\lim _{N \rightarrow \infty} f(N)=\infty$. Then, for sufficiently large $N$, the inequality

$$
T_{i}\left(\frac{i+\sqrt{i}}{\sqrt{N-i+3}}\right) \leq \frac{i-1+\sqrt{i-1}}{\sqrt{N-i+4}}
$$

holds for all $f(N) \leq i \leq N-1$.
Proof. After the substitution $z=\sqrt{N-i+3}$ our inequality transforms to

$$
\begin{equation*}
T_{i}\left(\frac{i+\sqrt{i}}{z}\right) \leq \frac{i-1+\sqrt{i-1}}{\sqrt{z^{2}+1}} \tag{9}
\end{equation*}
$$

and the condition $i \leq N-1$ transforms to $z \geq 2$. We will prove that the inequality (9) holds for $z \geq 1$ and sufficiently large $i$. A large $N$ implies a large $i$ since $f(N) \leq i$, so the lemma will follow.

Evaluating $T_{i}$ in (19) yields

$$
g(i, z) \stackrel{\text { def }}{=} \frac{i-1+\sqrt{i-1}}{\sqrt{z^{2}+1}}-\frac{-\frac{(i+\sqrt{i})^{3}}{z^{3}}+\frac{2(i+\sqrt{i})^{2}}{z^{2}}+\frac{2 i^{2}(i+\sqrt{i})}{z}}{2 i(i+1)}>0 .
$$

We see that
$g(i, 1)=(\sqrt{2}-1) i^{3}+i^{5 / 2}+(\sqrt{2(i-1)}+1)\left(i^{2}+i\right)-3 i^{3 / 2}-(2+\sqrt{2}) i$
which is positive for large $i$. Since $g(i, z)$ is continuous it suffices to show that $g(i, z) \neq 0$ for $z \geq 1$ and $i$ large. The zeros of $g(i, z)$ are the same as the zeros of

$$
\left(\frac{i-1+\sqrt{i-1}}{\sqrt{z^{2}+1}}\right)^{2}-\left(\frac{-\frac{(i+\sqrt{i})^{3}}{z^{3}}+\frac{2(i+\sqrt{i})^{2}}{z^{2}}+\frac{2 i^{2}(i+\sqrt{i})}{z}}{2 i(i+1)}\right)^{2} .
$$

We multiply this by $(2 i(i+1))^{2}\left(z^{2}+1\right) z^{6}$ and obtain a polynomial of degree 6 in $z$ :

$$
\begin{aligned}
q(z) & \stackrel{\text { def }}{=} 4\left(2\left(i^{5}+i^{4}-i^{3}-i^{2}\right) \sqrt{i-1}-2 i^{11 / 2}-i^{4}-i^{3}\right) z^{6} \\
& -8\left(i^{5}+3 i^{9 / 2}+3 i^{4}+i^{7 / 2}\right) z^{5} \\
& +4\left(2 i^{11 / 2}+5 i^{5}+4 i^{9 / 2}-4 i^{7 / 2}-6 i^{3}-4 i^{5 / 2}-i^{2}\right) z^{4} \\
& +4\left(-i^{5}-i^{9 / 2}+4 i^{4}+8 i^{7 / 2}+5 i^{3}+i^{5 / 2}\right) z^{3} \\
& +\left(3 i^{6}+10 i^{11 / 2}+9 i^{5}-4 i^{9 / 2}-15 i^{4}-22 i^{7 / 2}-25 i^{3}-16 i^{5 / 2}-4 i^{2}\right) z^{2} \\
& +4\left(i^{5}+5 i^{9 / 2}+10 i^{4}+10 i^{7 / 2}+5 i^{3}+i^{5 / 2}\right) z \\
& -\left(i^{6}+6 i^{11 / 2}+15 i^{5}+20 i^{9 / 2}+15 i^{4}+6 i^{7 / 2}+i^{3}\right)
\end{aligned}
$$

We must show that $q(z) \neq 0$ for $z \geq 1$ and $i$ large. Using a computer program like Maple this is just a matter of verification:

- Compute the fourth derivative $q^{(4)}(z)=a_{2}(i) z^{2}+a_{1}(i) z+a_{0}(i)$.
- Verify that $\left(\frac{a_{1}(i)}{2 a_{2}(i)}\right)^{2}-\frac{a_{0}(i)}{a_{2}(i)}<0$ for large $i$. This implies that $q^{(4)}(z)$ has no real roots.
- Check that $q^{(4)}(0)>0$ for large $i$ so that $q^{(4)}(z)>0$ everywhere.

This is immediately evident from the expression for $q(z)$ above.

- Check that $q^{\prime \prime \prime}(1), q^{\prime \prime}(1), q^{\prime}(1)$, and $q(1)$ are positive for large $i$.

This shows that $q(z)>0$ for all $z \geq 1$ for sufficiently large $i$.
Lemma 10. For sufficiently large $N$, the inequality

$$
\tau_{i}\left(\frac{i+1}{\sqrt{N-i+3}+0.148}\right) \geq \frac{i}{\sqrt{N-i+4}+0.148}
$$

holds for all $i$ in the interval $\sqrt{N}+1 \leq i \leq N-22$.
Proof. Put $\varepsilon=0.148$. After the substitution $z=\sqrt{N-i+3}+\varepsilon$ our inequality transforms to

$$
\begin{equation*}
\tau_{i}\left(\frac{i+1}{z}\right) \geq \frac{i}{\sqrt{(z-\varepsilon)^{2}+1}+\varepsilon} \tag{10}
\end{equation*}
$$

The interval $\sqrt{N}+1 \leq i \leq N-22$ transforms to

$$
\begin{equation*}
5 \leq z-\varepsilon \leq \sqrt{(i-1)^{2}-i+3} \tag{11}
\end{equation*}
$$

Evaluating $\tau_{i}$ in (10) and multiplying by $2 i z^{3}\left(\sqrt{(z-\varepsilon)^{2}+1}+\varepsilon\right)>0$ yields

$$
\begin{aligned}
p(i) & \stackrel{\text { def }}{=}\left(\left(2 z^{2}-1\right) \sqrt{(z-\varepsilon)^{2}+1}-\left(2 z^{3}+\left(1-2 z^{2}\right) \varepsilon\right)\right) i^{2} \\
& +\left((z-2)\left(\sqrt{(z-\varepsilon)^{2}+1}+\varepsilon\right) i\right. \\
& -\left(z^{2}-z+1\right)\left(\sqrt{(z-\varepsilon)^{2}+1}+\varepsilon\right) \\
& \geq 0,
\end{aligned}
$$

a quadratic polynomial in $i$.
Let us first show that the leading coefficient of $p(i)$ is positive, i. e.

$$
\left(2 z^{2}-1\right) \sqrt{(z-\varepsilon)^{2}+1}>2 z^{3}+\left(1-2 z^{2}\right) \varepsilon
$$

After squaring, expanding, and collecting we get $4 \varepsilon z^{3}-3 z^{2}-2 \varepsilon z+1>$ 0 . The zeros of this third-degree polynomial in $z$ are $z_{1} \approx-0.592$, $z_{2} \approx 0.559$, and $z_{3} \approx 5.100$, all of which are less than $5+\varepsilon$.

There are always real roots to $p(i)$ since its constant term is negative for $z \geq 5+\varepsilon$. For each $z$, let $i(z)$ be the greater of the two roots of $p(i)$. We can, of course, write down an explicit formula for $i(z)$ and with a computer program like Maple it is easy to check that $\lim _{z \rightarrow \infty} i(z)-z=$ $1-\varepsilon$. Thus, for any $\delta>0$ there is a $Z_{\delta}$ such that $p(i)>0$ for $i>z+1-\varepsilon+\delta$ for all $z>Z_{\delta}$. We choose $\delta=1 / 4$. In the interesting interval (11) we have $z+1-\varepsilon+1 / 4 \leq \sqrt{(i-1)^{2}-i+3}+3 / 4<i$, where the last inequality follows from $(i-1)^{2}-i+3<(i-3 / 4)^{2}$ which is true if $i \geq 3$.

In the interval $Z_{1 / 4}<z \leq \varepsilon+\sqrt{(i-1)^{2}-i+3}$ we know that $p(i)>$ 0 so we only have to worry about the interval $5+\varepsilon \leq z \leq Z_{1 / 4}$. Since $i(z)$ is a continuous function we can define $I=\sup \{i(z): 5+\varepsilon \leq$ $\left.z \leq Z_{1 / 4}\right\}$. Then $p(i)>0$ in the interval (11) for all $i>I$. A large $N$ implies a large $i$ since $\sqrt{N}+1 \leq i$, so the lemma follows.

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