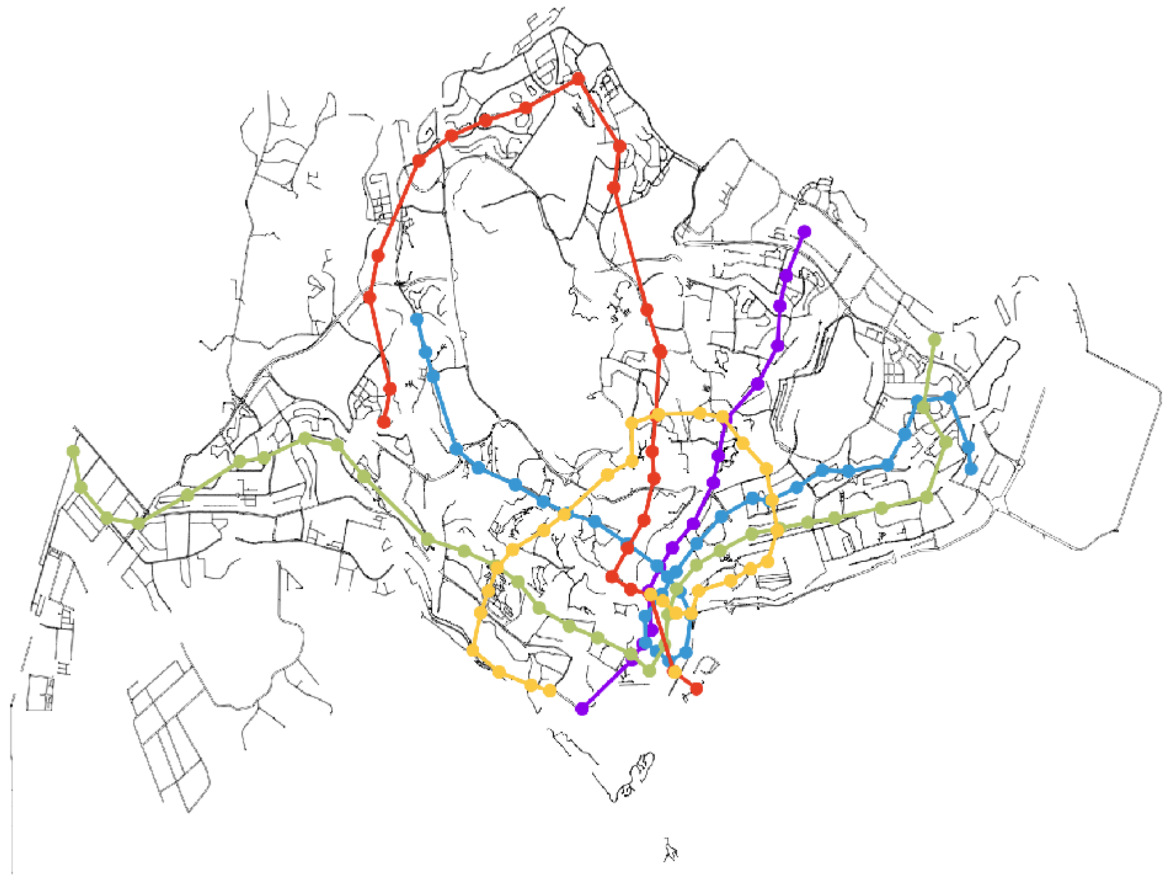
CSCI 357: Algorithmic Game Theory

Lecture 15: Voting & Social Choice 2

Shikha Singh

Announcements and Logistics

- Homework 6 is due this tonight at 11 pm
- Office hours in this room after lecture: 4-5.30 pm
- TA hours tonight 8-9.30 pm
- Homework 7 (Voting) will be released tomorrow
    - Partner assignment
    - Will send out a partner form
    - Let me know if you want me to help you find a partner

Questions?

Last Time

- A set A of alternatives and a $N = \{1,2,\ldots,n\}$ of voters
- Each agent $i \in N$ submits a list L_i (ranking over A)
- **Social-choice function** selects a single alternative for a given preferences profile, that is, $L_1, L_2, \ldots, L_n \mapsto a^*$ where $a^* \in A$
- **Majority rule:** elect the candidate with the majority of votes when there are only two alternates ($|A| = 2$)
- **Plurality rule**: elect the candidate with most 1st-place votes
- **Ranked-choice voting:** In each round eliminates the person with fewest first-place votes, and recurses
- Condorcet criterion and strategyproofness
    - Plurality and ranked-choice do not satisfy either

Borda Count

- Well known voting rule: often used in sports, also used in Eurovision song contest
- Voters submit their full ranked lists: an alternate gets $|A|$ for each first-choice vote, $|A|-1$ points for each second-choice vote, and so on and 1 point for each last-choice vote
- For our example:
  - a gets 15 points
  - b gets 12 points
  - c gets 10 points
  - d gets 13 points

	Voters #1,2	Voters #3,4	Voter #5
1st Choice	a	b	c
2nd choice	d	a	d
3rd choice	c	d	b
4th choice	b	c	a

- Borda count would elect a (in contrast to ranked-choice b)

Borda Count

- Is Borda count strategyproof?
  - **Idea**: incentive to rank closest competitor to preferred candidate last
- In example, what is the Borda score of a and b?
  - a's score: $2 \cdot 3 + 4 = 10$
  - b's score: $2 * 4 + 3 = 11$
- If voter 3 moves b to the last place
  - b's score: $8 + 1 = 9$
  - Thus, a will win now

Winner
b

1	2	3
b	b	a
a	a	b
c	c	c
d	d	d

→

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

Winner
a

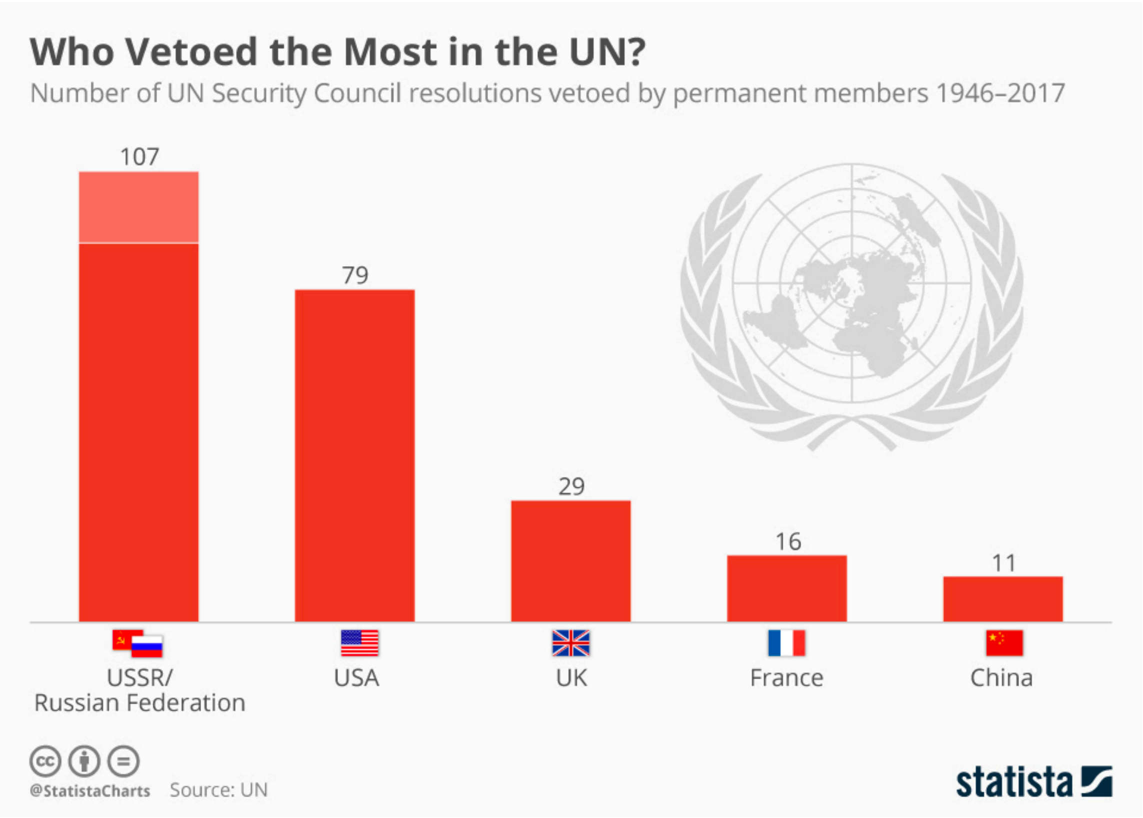
Positional Scoring Rules

- In general, you can have different ways to score each position
- For each vote, a **positional-scoring rule** on $m = |A|$ alternatives assigns a score of α_j to the alternative ranked in jth place. The alternative with maximum total score (across all votes) is selected.
    - Assume $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_m$ and $\alpha_1 > \alpha_m$
    - E.g., plurality gives 1 point for first-choice, zero for others
- Many positional scoring rules have been studied
    - Plurality can be thought of a positional voting rule, how?
    - Veto (HW 7) is also another example

Borda Count

- Does Borda satisfy the Condorcet criterion?
    - Question in Homework 7

Many Rules, Many Applications



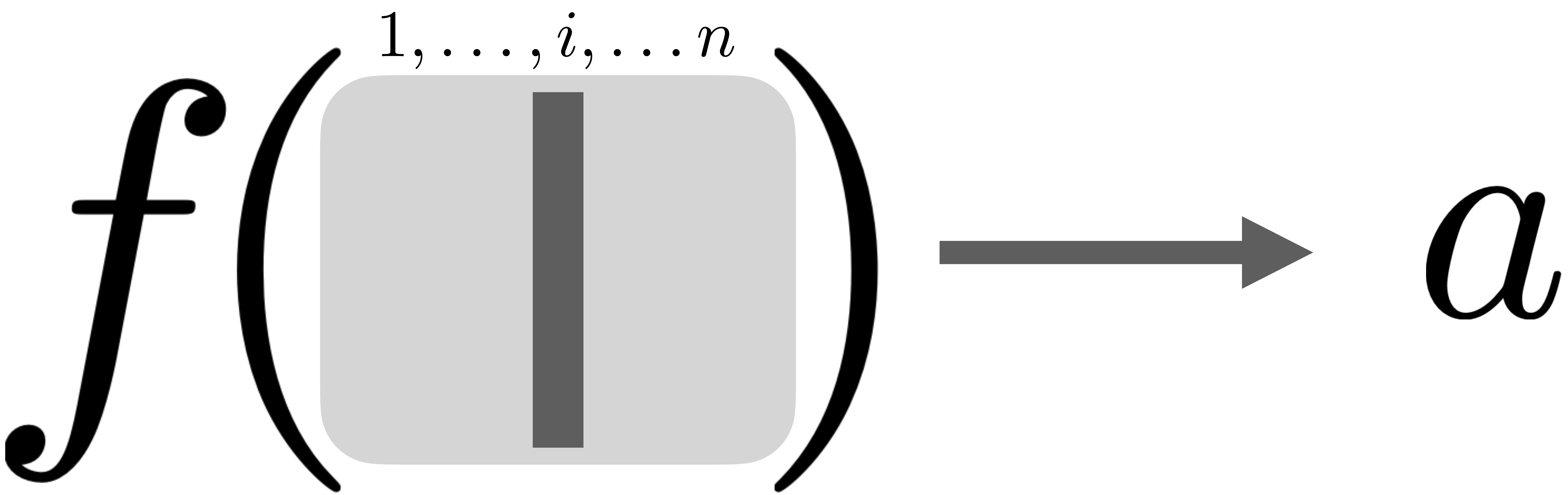
https://rohitvaish.in/Teaching/2022-Spring/Slides/Lec%202.pdf

One to Rule them All?

- For the same input profile, plurality, Borda and ranked-choice can all output a different winner!
    - Can you construct such an example?
- Changing the voting rule changes the outcome of the mechanism
- Leads to contention on which voting rule is the “best”
- Voting theorists have an "axiomatic" approach to study voting rules
- Identify "desirable" properties that one would like
- Compare rules based on that
- **Question**: Is there any voting rule that is strategyproof and reasonable?

Properties of Voting Rules

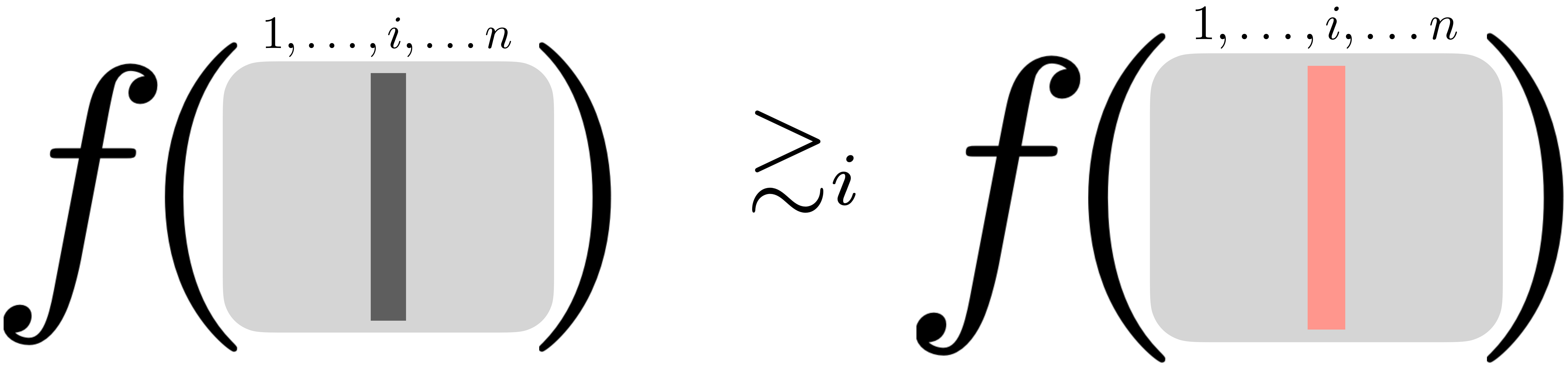
- **Onto**: For any candidate a, there exists an input profile where a wins



- Are Borda, plurality, ranked-choice etc onto?
    - Yes, can always construct a profile to make any candidate win

Properties of Voting Rules

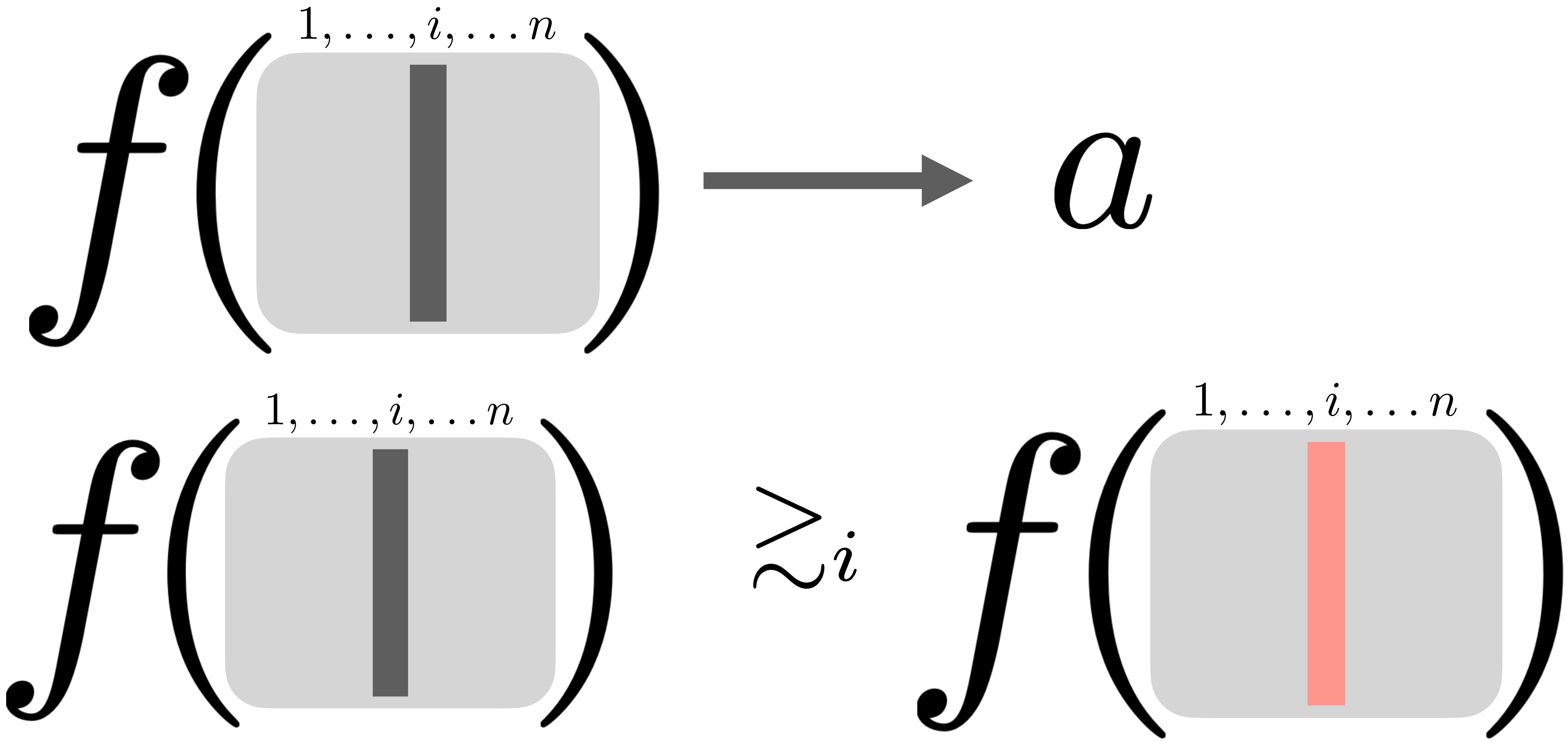
- **Strategyproof**: No voter can improve by misreporting preferences



- Are Borda, plurality, ranked-choice etc strategyproof?
  - No

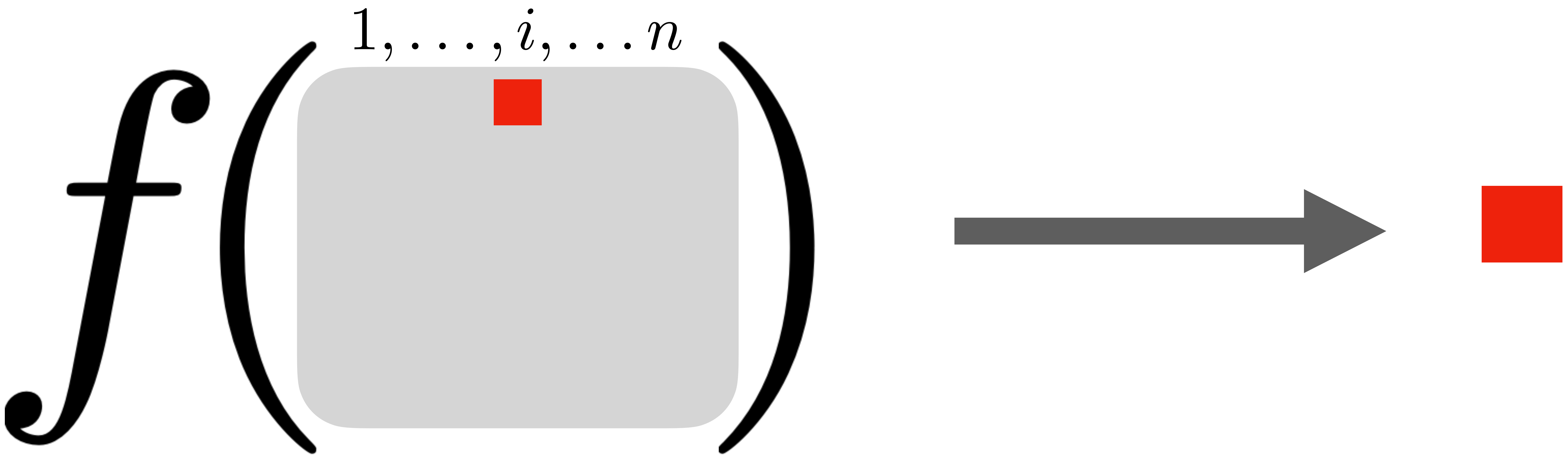
Onto and Strategyproof

- (3 or more alternatives) onto but not strategyproof? Borda, Plurality, Ranked-choice
- (3 or more alternatives) strategyproof but not onto? Constant or restricted majority



A Bad Voting Rule

- **Dictatorship** : A voting rule is **dictatorial** if there is a voter i such that the rule always elects i's first choice (regardless of others' preferences)



- Is a dictatorship straregyproof?
- Is a dictatorship onto?

[Gibbard '73, Satterthwaite '75]

When there are 3 or more alternatives, a voting rule is strategyproof and onto if and only if it is dictatorial.

Impossibility Result

- **Gibbard-Satterthwaite theorem**.
  When there are 3 or more alternatives, a voting rule is strategyproof and onto if and only if it is dictatorial
- We only a dictatorial mechanism is strategyproof and onto
- Need to show, SP + Onto $\implies$ dictatorship
- We will only prove it for $n = 2$ voters. Break into several steps
  - SP $\implies$ Monotone
  - SP + Onto $\implies$ Pareto optimality
  - **GS Proof:** Monotone + Pareto optimal $\implies$ dictatorship

Monotonicity

- **Definition**. Suppose a is the current winner (on profile L). For all input profiles L', in which for all voters, any candidate who was ranked below a in L is still ranked below a in L', then a should continue to win in L'.
    - Support of a either increases or stays the same: a's outcome cannot get worse
- **Theorem**. Strategyproof $\Longleftrightarrow$ monotone

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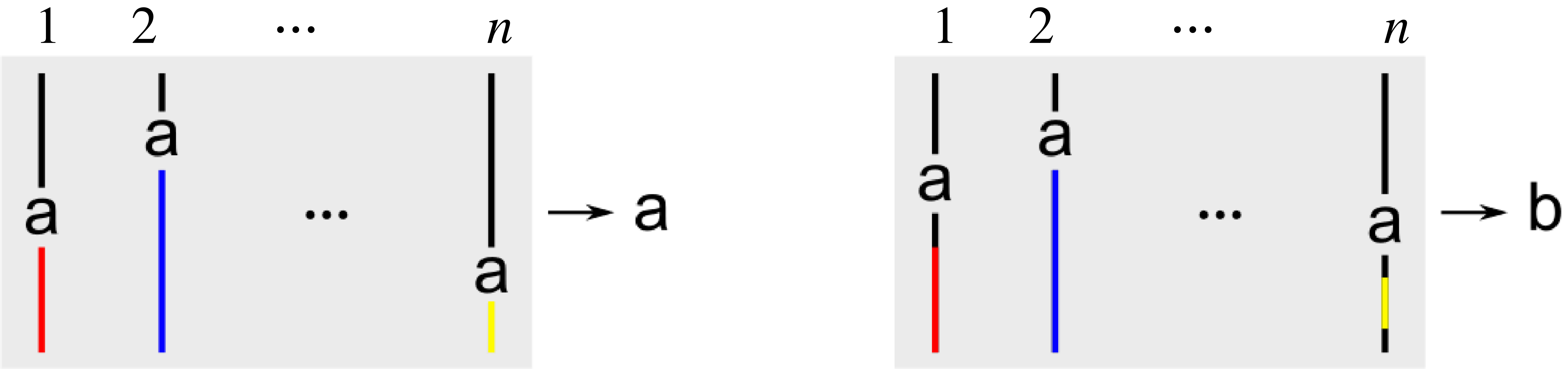
Strategyproof $\implies$ Monotone

- Suppose a rule is strategyproof but not monotone



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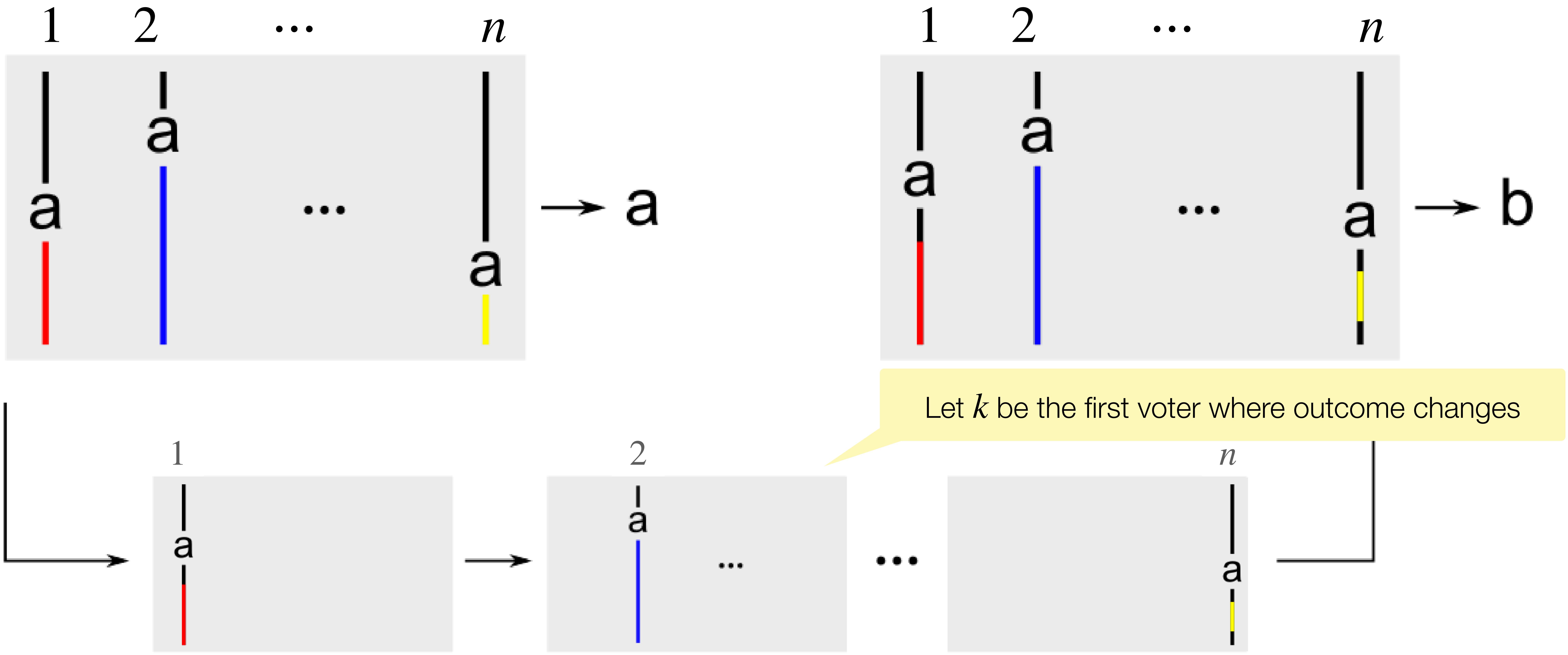
Strategyproof $\Longrightarrow$ Monotone

- Suppose a rule is strategyproof but not monotone



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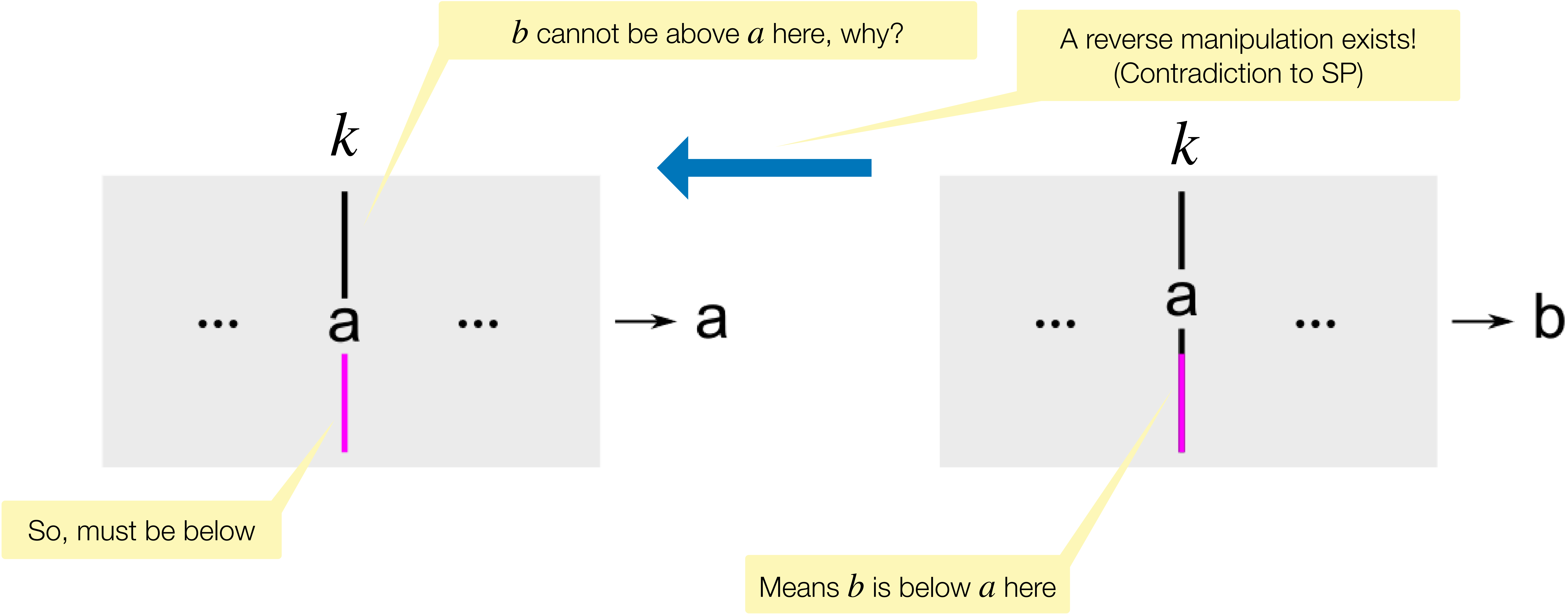
Strategyproof $\Longrightarrow$ Monotone



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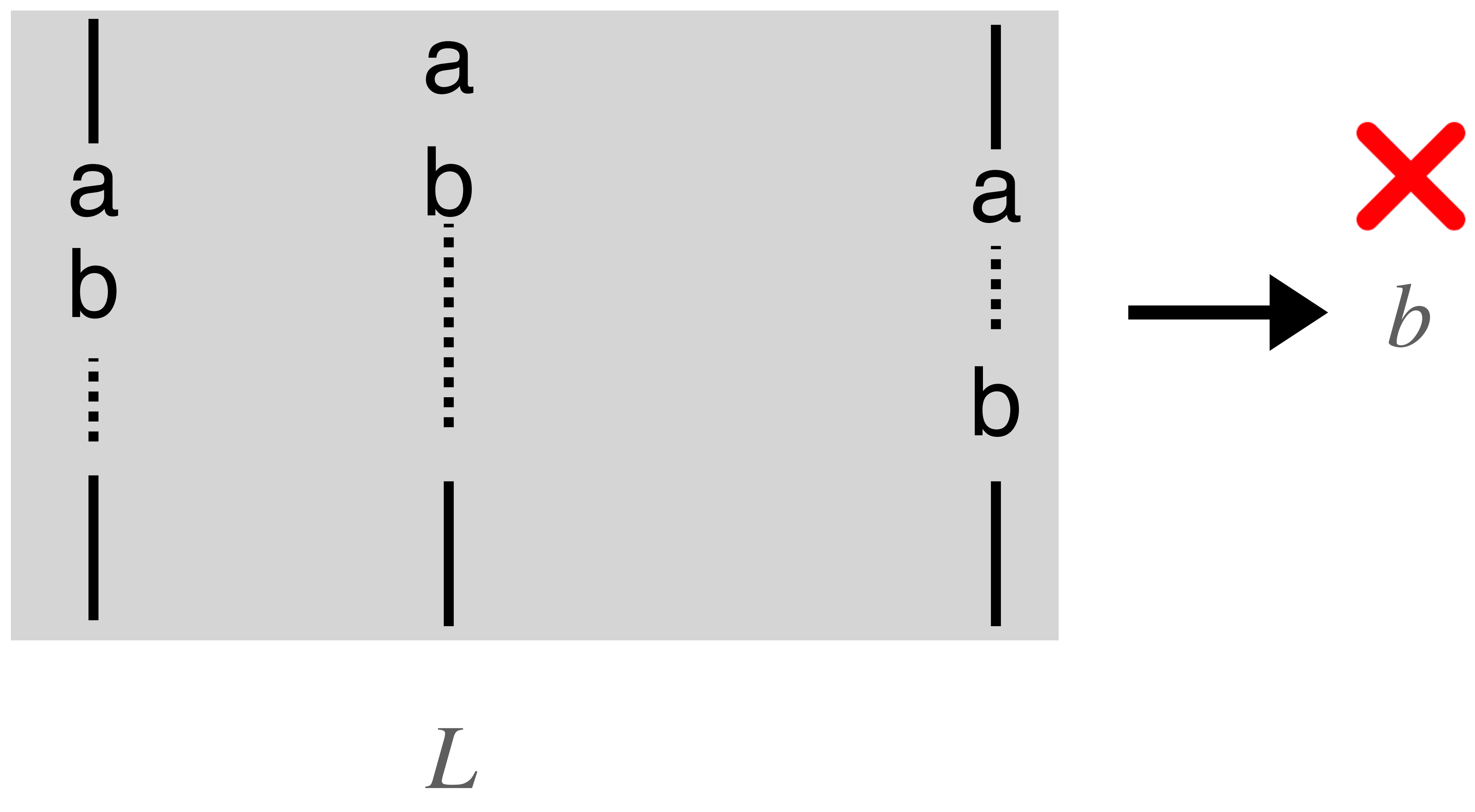
Pareto Optimality

- **Definition**. Given preference profile L, if there is an alternative a that every voter prefers to b, then $f(L) \neq b$.
- **Lemma**. SP + Onto $\Longrightarrow$ Pareto optimality



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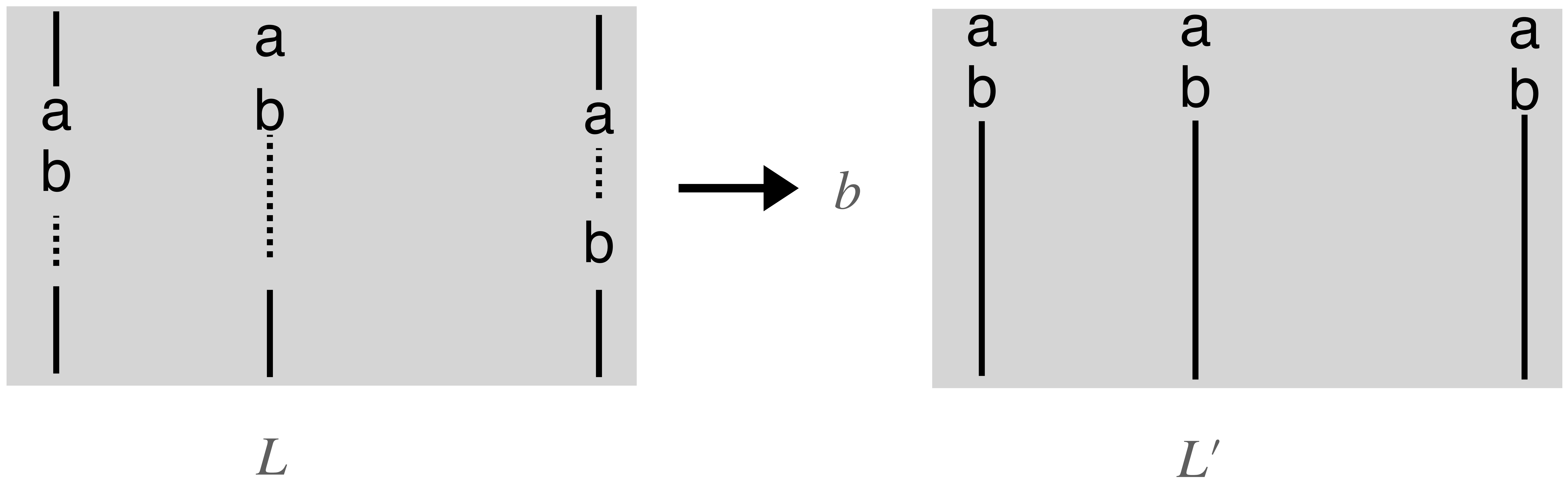
- **Lemma**. SP + Onto $\Longrightarrow$ Pareto optimality
- **Proof**. Suppose $f(L) = b$. Consider L' below. $f(L') = ?$



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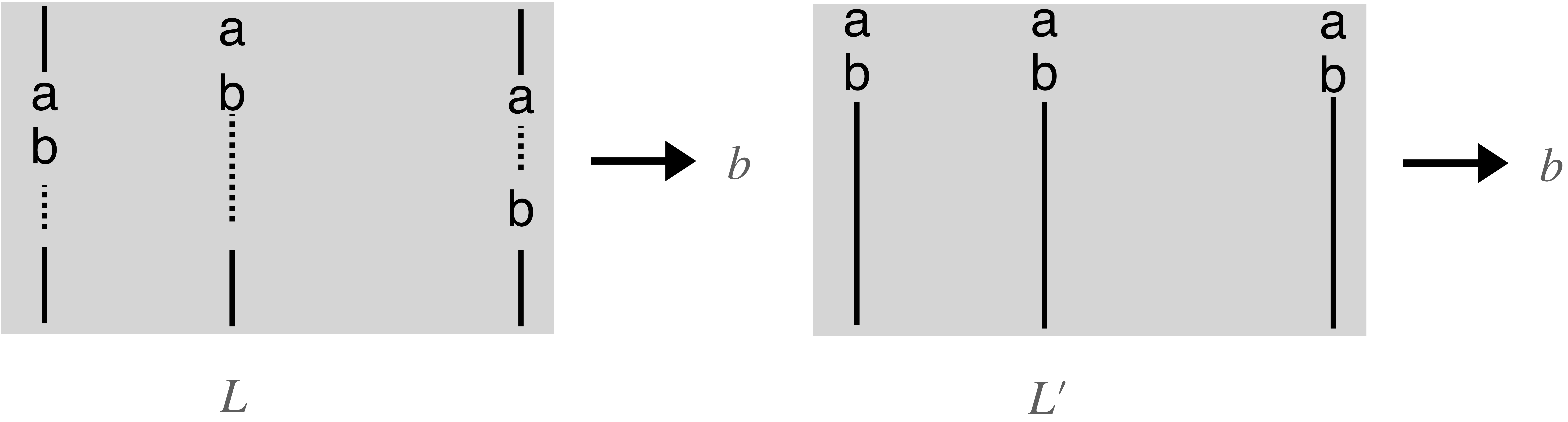

Pareto Optimality

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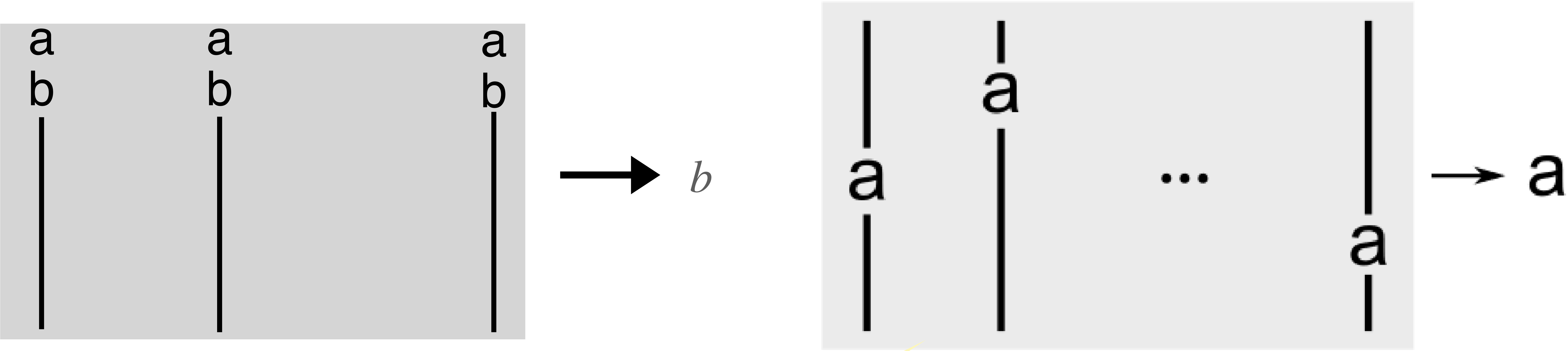
- **Lemma**. SP + Onto $\Longrightarrow$ Pareto optimality
- **Proof**. Suppose $f(L) = b$. By onto, there exists a profile L'' where a wins.



L' L''

L' to L, a's support only goes up, by monotonicity b cannot win.

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Impossibility Result

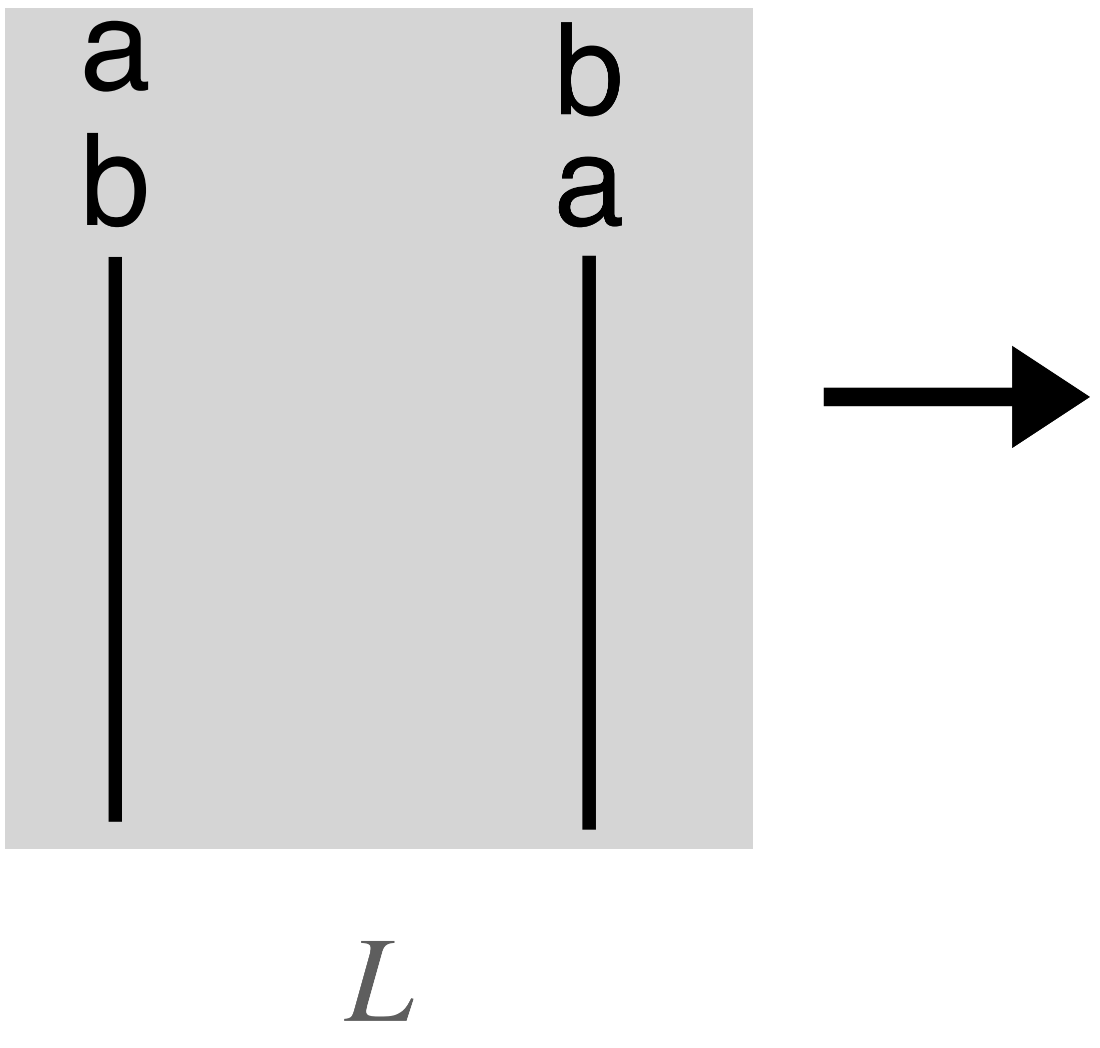
- **Gibbard-Satterthwaite theorem**.
  When there are 3 or more alternatives, a voting rule is strategyproof and onto if and only if it is dictatorial
- We only a dictatorial mechanism is strategyproof and onto
- Need to show, SP + Onto $\Longrightarrow$ dictatorship
- We will only prove it for $n = 2$ voters. Break into several steps
  - SP $\Longrightarrow$ Monotone
  - SP + Onto $\Longrightarrow$ Pareto optimality
  - **GS Proof:** Monotone + Pareto optimal $\Longrightarrow$ dictatorship ← We're here

GS Proof for $n = 2$

- Need to show: when we have 2 voters, and any number of alternatives, then monotone + Pareto optimality implies that one of the voters is a dictator (for each alternative)
- Break into two parts:
- **Claim 1**. Consider a monotone and Pareto-optimal rule f with two voters and alternatives $a, b \in A$. Then either voter 1 is a dictator for a or voter 2 is a dictator for b (wrt f).
- **Claim 2**. Consider a monotone and Pareto-optimal rule f with two voters and alternatives $a, b \in A$. Then either voter 1 is a dictator for a or voter 2 is a dictator for b (wrt f).

GS Proof for $n = 2$

- **Claim 1**. Consider a monotone and Pareto-optimal rule f with two voters and alternatives $a, b \in A$. Then either voter 1 is a dictator for a or voter 2 is a dictator for b (wrt f).
- Proof. Consider an input profile L.
- What can we say about $f(L)$?



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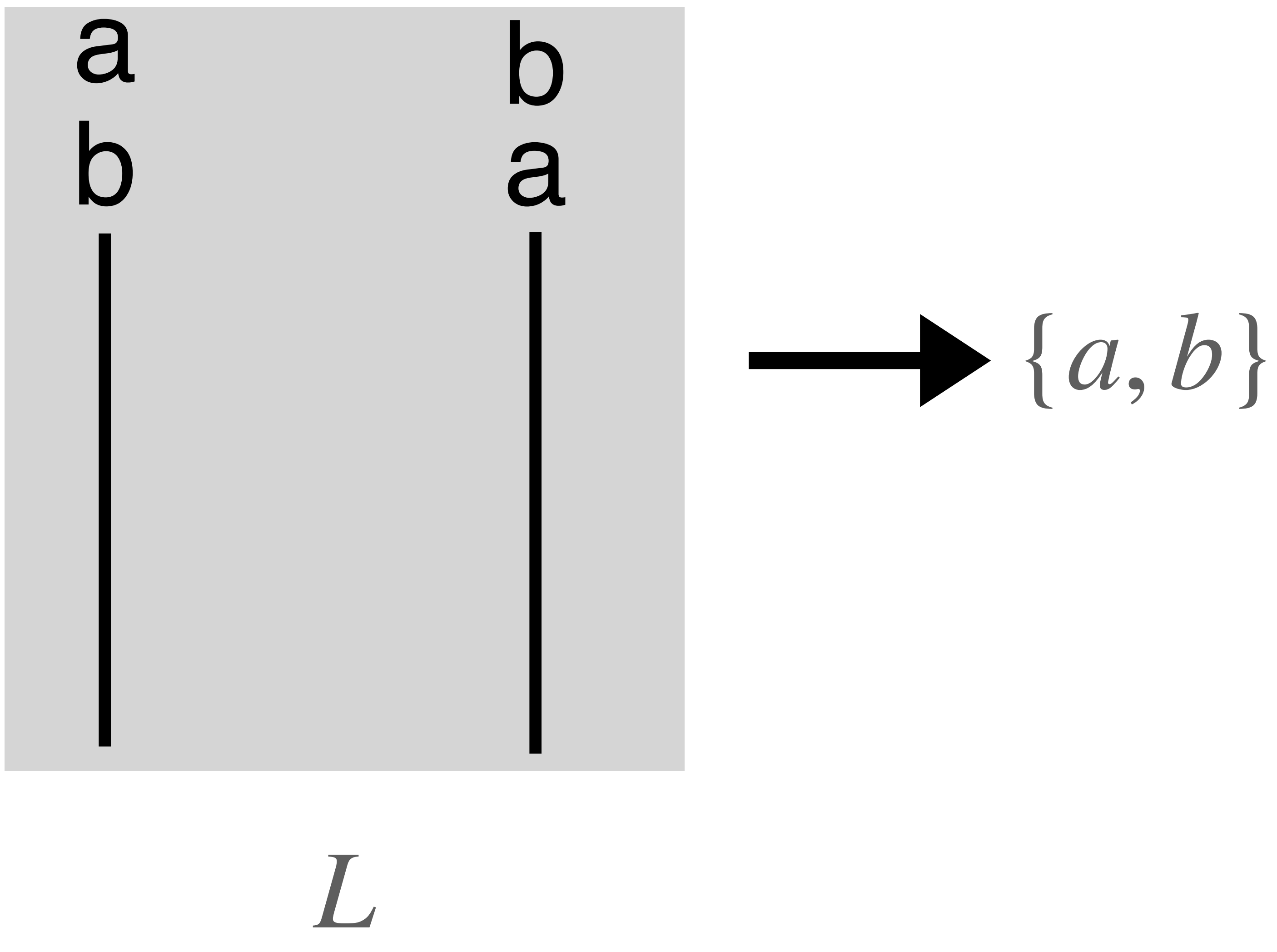
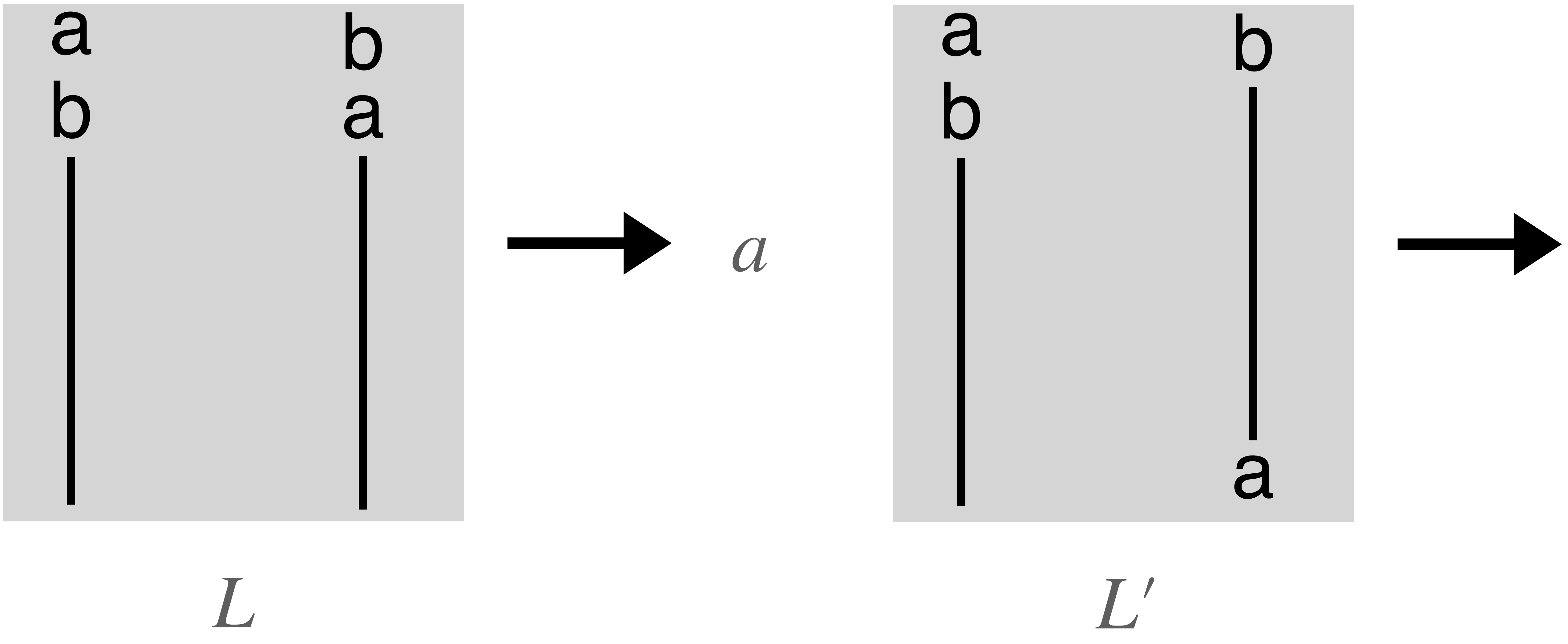


GS Proof for $n = 2$

- **Claim 1**. Consider a monotone and Pareto-optimal rule f with two voters and alternatives $a, b \in A$. Then either voter 1 is a dictator for a or voter 2 is a dictator for b (wrt f).
- Proof. Without loss of generality, suppose $f(L) = a$
- Consider L' which is same as L except 2 moves a to last place
- By monotonicity over other candidates, $f(L')$ cannot be anything other than a



GS Proof for $n = 2$

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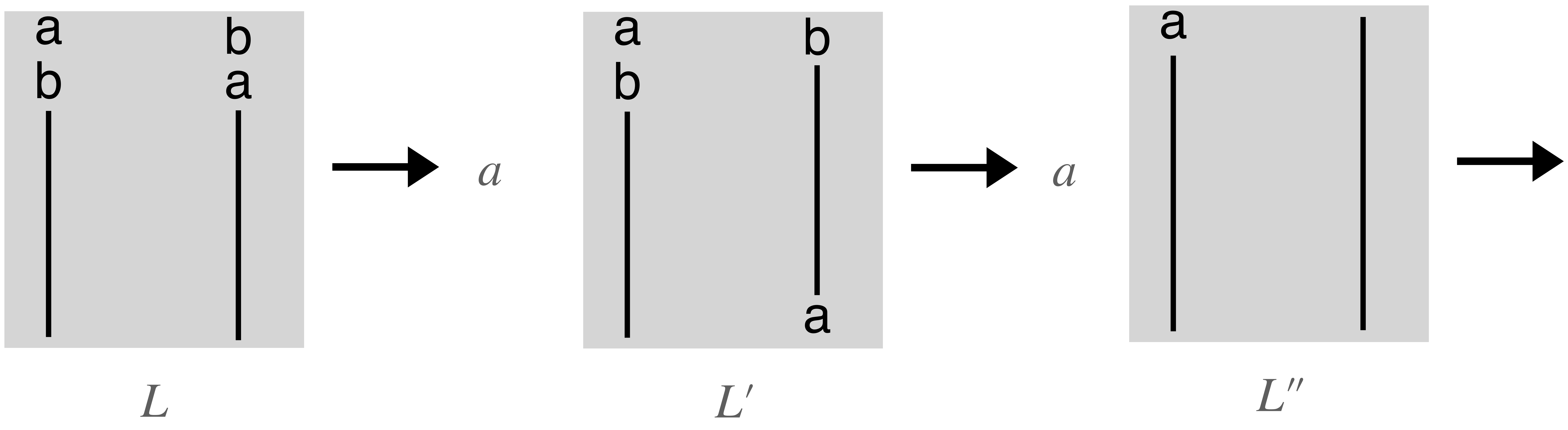
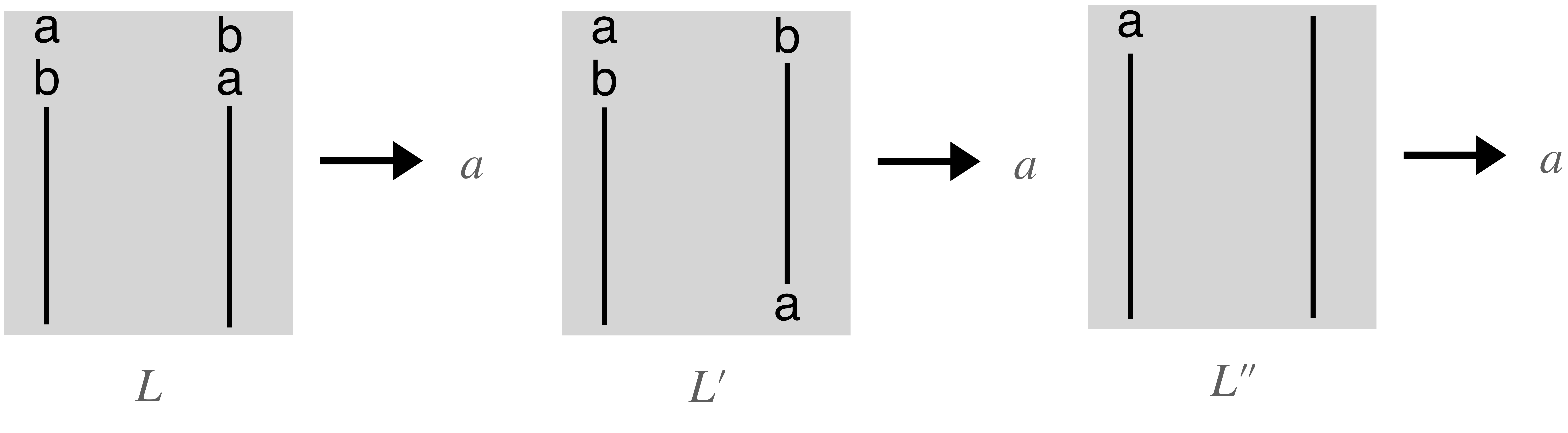
- Proof. Without loss of generality, suppose $f(L) = a$
- Now consider L'' where 1 ranks a at the top, all other rankings are arbitrary
- Then, $f(L'') = a$ by monotonicity between L' and L'' wrt a



GS Proof for $n = 2$

- **Claim 1**. Consider a monotone and Pareto-optimal rule f with two voters and alternatives $a, b \in A$. Then either voter 1 is a dictator for a or voter 2 is a dictator for b (wrt f).
- Proof. Without loss of generality, suppose $f(L) = a$
- Thus, 1 is a dictator for a.
- Analogously, we can assume $f(L) = b$ and show 2 is a dictator for b ∎



GS Proof for $n = 2$

- **Claim 2**. Consider a strategyproof and onto rule f with two voters, then one of them must be a dictator for each alternative $a \in A$.
- **Proof**. Consider a triple (a, b, x), where $a, b \in A$ and $x \in A \backslash \{a, b\}$
- Applying our earlier claim to (a, b):
  - Either 1 must be a dictator for a or 2 must be a dictator for b
  - Wlog assume 1 must be a dictator for a
- Applying our earlier claim to (b, x)
  - Either 1 must be a dictator for b or 2 must be a dictator for x
- Since 1 is already a dictator for a, 2 cannot be a dictator for x, why?
  - Thus 1 must be a dictator for both a, b
- Similarly, considering (x, a): 1 must be a dictator for x as well
- Applying this to all triples, concludes the proof ∎

Arrow's Impossibility Theorem

- The GS theorem is closely related to and can be derived from an even more famous impossibility result: Arrow's theorem
- **Arrow's impossibility theorem**. With three or more alternatives, no social-rank function satisfies the following three properties:
  - Non-dictatorship
  - Unanimity
  - Independence of irrelevant alternatives (IIA)
- Unanimity means if every voter ranks a over b, then the social-rank function should rank a over b
- IIA means that, for every pair a, b of alternatives, the relative order of a over b in the output ranking should be a function of only the relative order of a, b in each voter's list and not depend on the position of any "irrelevant" alternative c in anyone's preferences

Plurality does not satisfy IIA (e.g., Bush vs Gore outcome was affected by Nader)

Arrow's and GS

- One can also derive the Gibbard-Satterthwaite theorem from Arrow's theorem, using a reduction argument
- Suppose we have a non-trivial and strategyproof voting rule
  - Use it to construct a a voting rule that satisfies the three conditions in Arrow's theorem
- Intuitively, not satisfying IIA can lead to opportunities for strategic manipulation
- You also need to ensure technicalities like Arrow's theorem is a result about social-ranking functions (voting rules that produce a full ranked list) while the GS theorem holds even for social choice functions (voting rules that elect a winner)

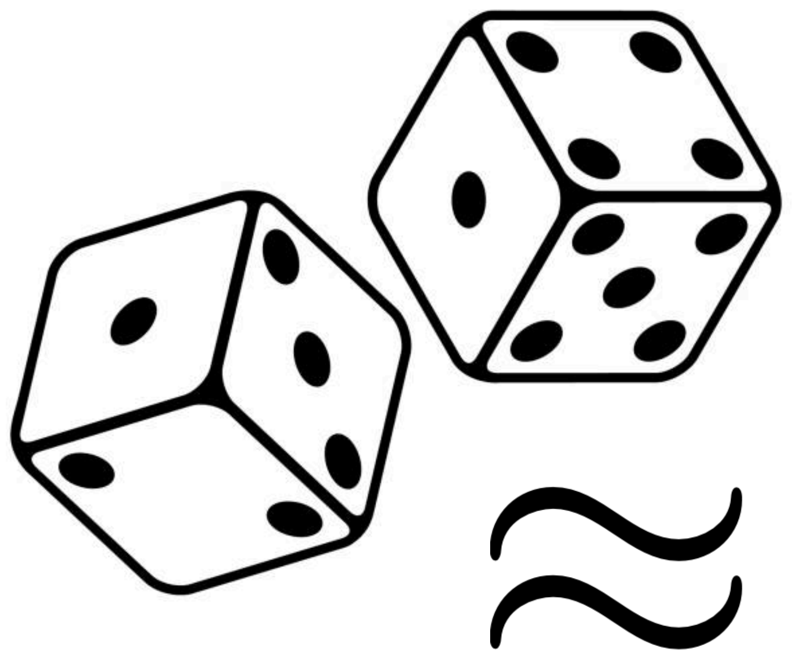
Takeaways

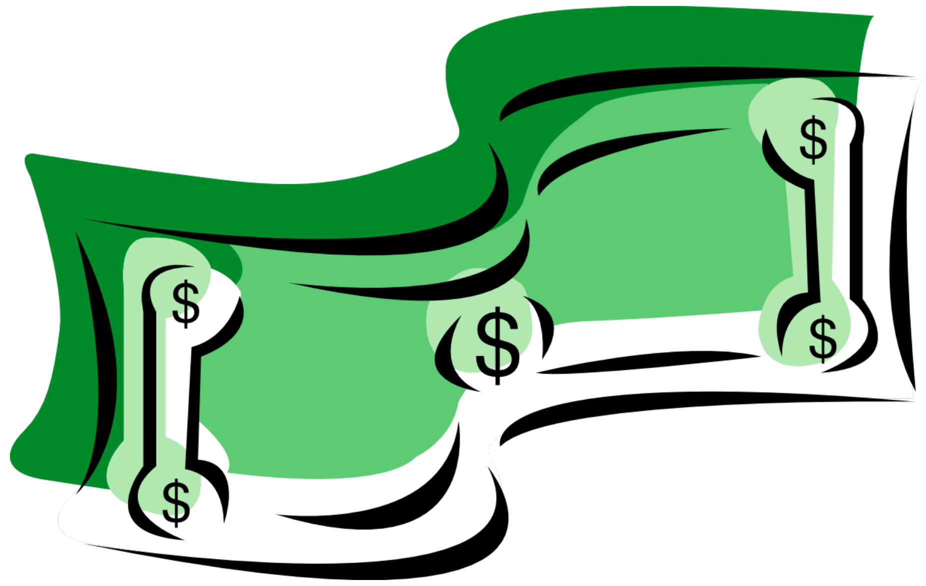
- When when we have two voters, and more than two alternatives, any voting rule that is reasonable (onto and non-dictatorial) is manipulable!
- Does this mean we should give up on strategyproofness entirely?
- How have we been managing to design strategyproof mechanism rules so far?

> *"The GS theorem seems to quash any hope of designing incentive-compatible social-choice functions.The whole field of Mechanism Design attempts escaping from this impossibility result using various modifications.": Nisan*

Circumventing GS

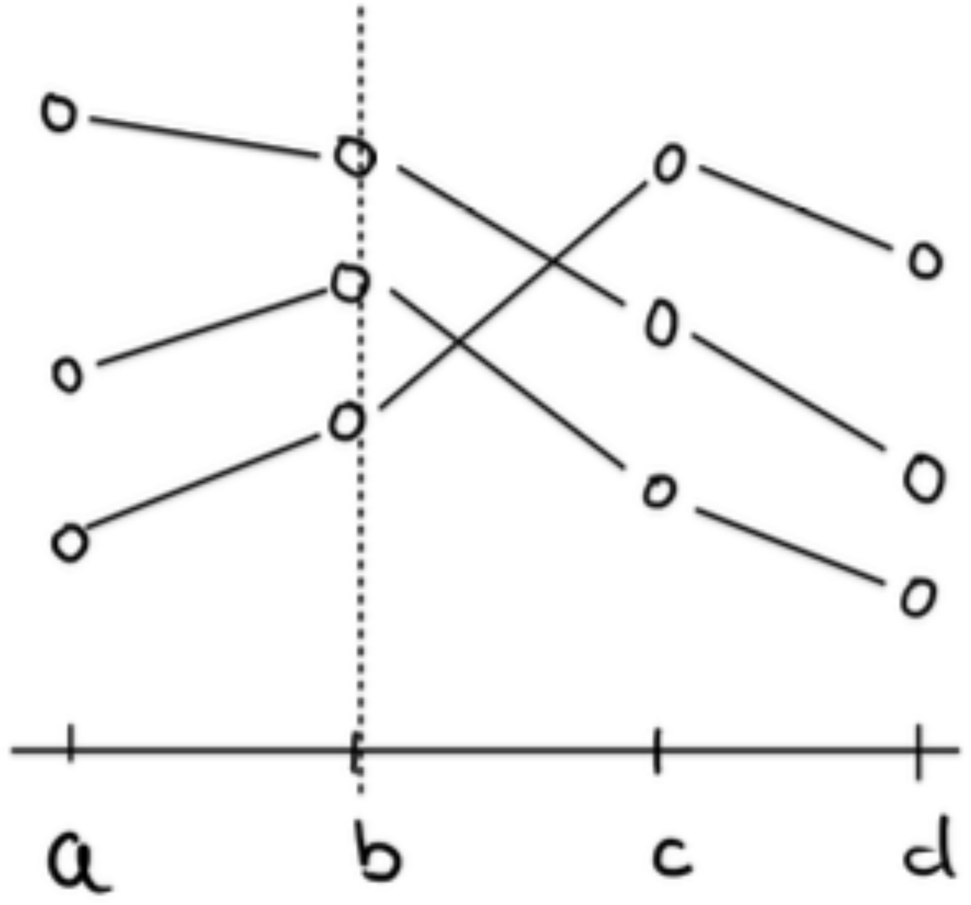
Randomness and approximation



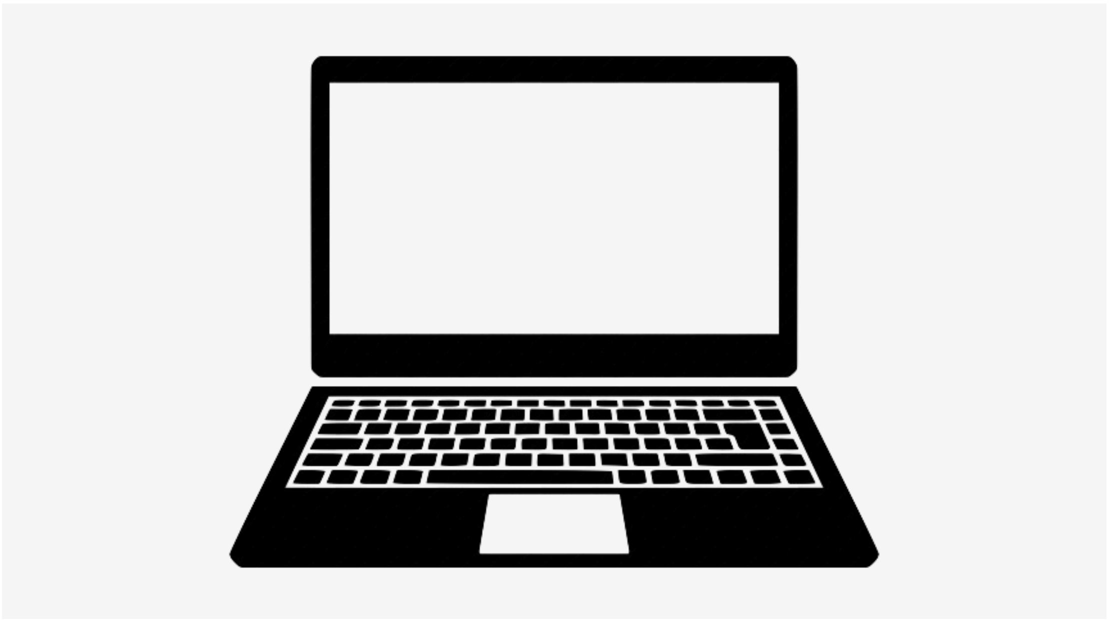


Money

Incomplete information

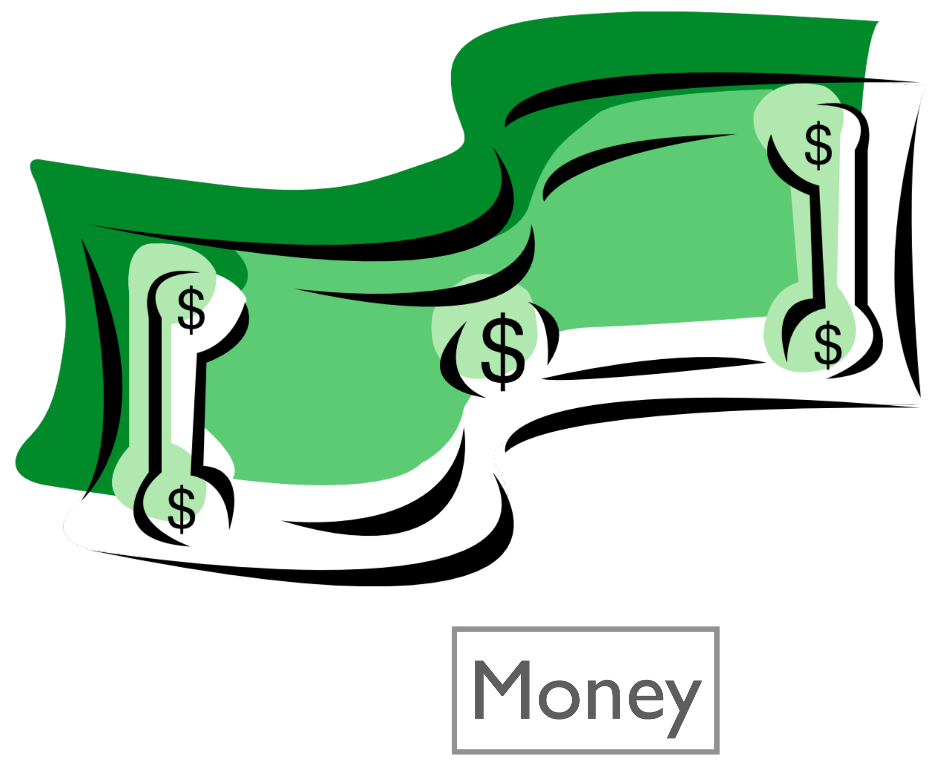


Restricted preference profiles



Computational complexity

Circumvent GS: Money



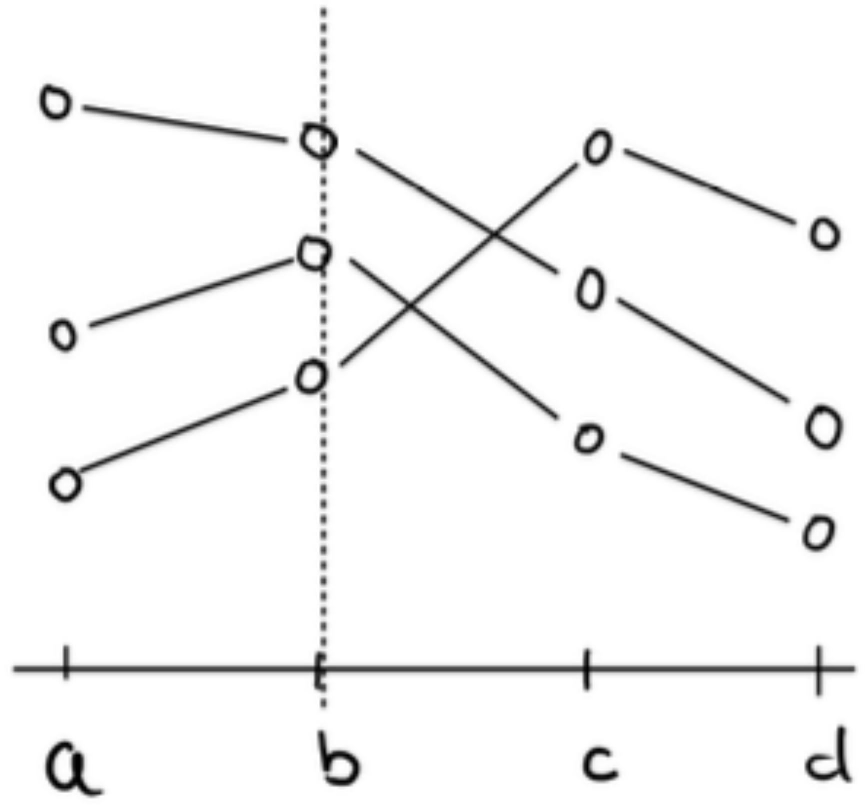
- Mechanism's goal was to output an **allocation** (mapping of items to agents)
- Set of alternatives $A = \{\text{all possible allocations}\}$
- Agents have preferences over allocation (their own, or in general over all)
- Agents "vote" (express their preferences) by bidding on allocations
- Similarities: Myerson proved strategyproof iff monotone allocation wrt bids
- Design strategyproof mechanisms by charging appropriate payments
- Similarly, if money or transfer is possible in some voting applications, can circumvent GS using mechanisms similar to VCG

Circumvent GS: Restricted Preferences

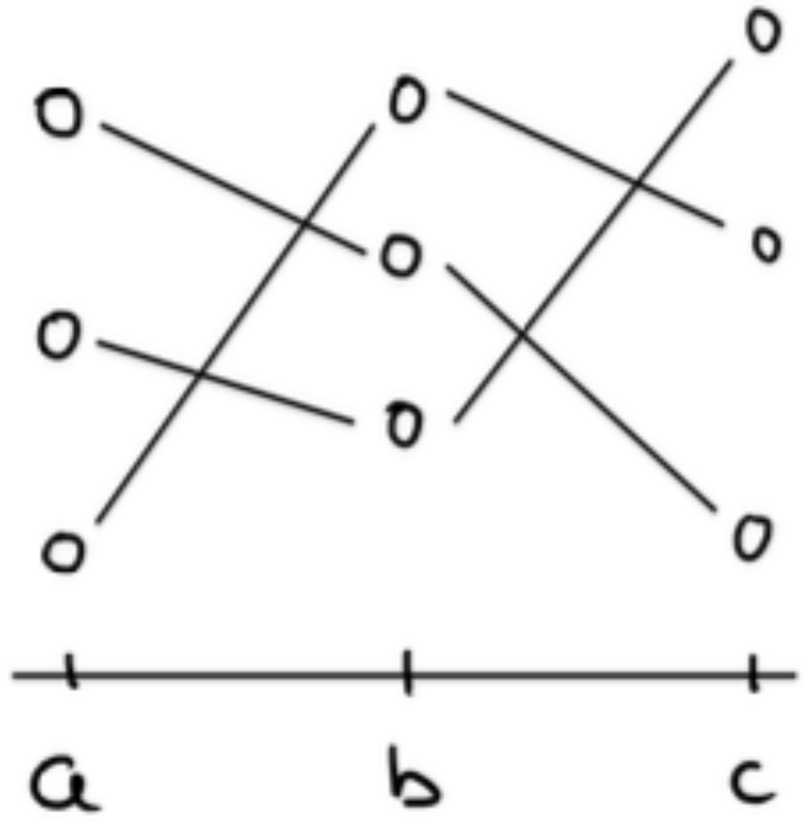
- In matching mechanisms from last week, we did not have money
- We were able to design strategyproof mechanisms for one-sided matching
- Preferences of agents were restricted
    - Did not have preferences over all possible matchings
    - Just care about their own match
- There are other ways we can restrict preferences
- Most common restriction on preferences considered in the voting landscape:
    - Single-peaked preferences

Single-Peaked Preferences

- Imagine that the candidates are points on a real line
  - Line could represent the political spectrum
- A voter i has single-peaked preferences if there is a "peak" $p_i \in \mathbb{R}$ such that the voters prefers candidates closer to her peak
- Idea is that single-peaked preferences are a reasonable approximation of voter's preferences



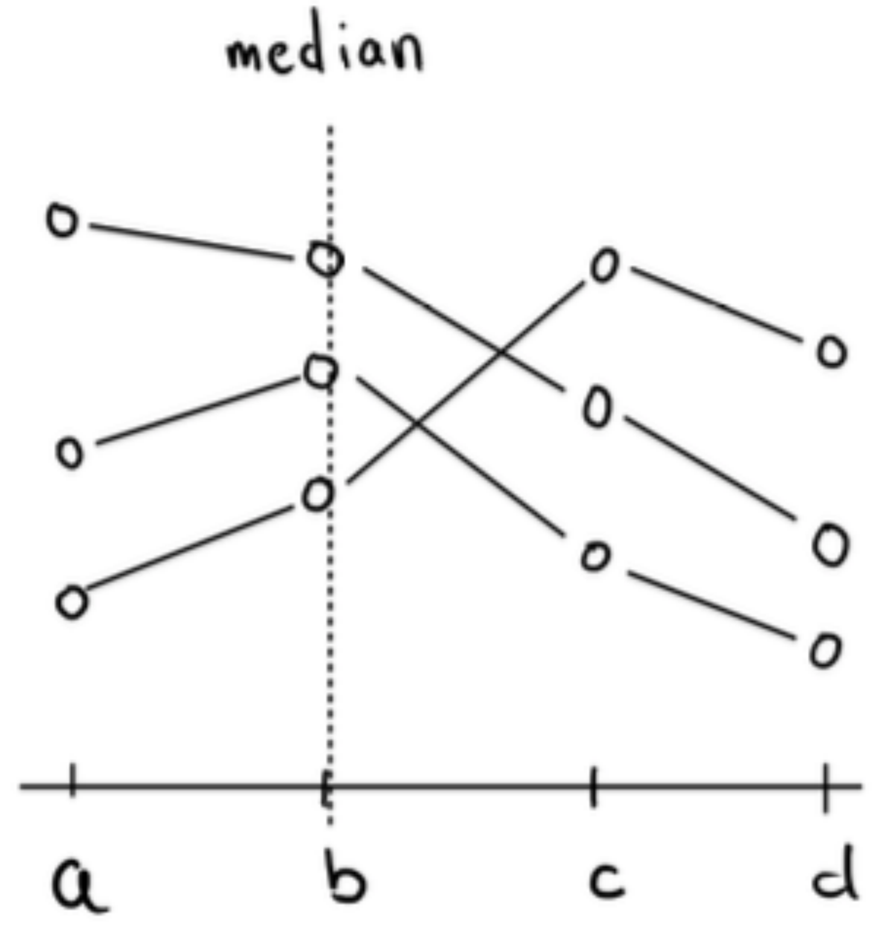
Single-peaked



Not single-peaked

Single-Peaked Preferences

- Imagine that the candidates are points on a real line
  - Line could represent the political spectrum
- A voter i has single-peaked preferences if there is a "peak" $p_i \in \mathbb{R}$ such that the voters prefers candidates closer to her peak
- Idea is that single-peaked preferences are a reasonable approximation of voter's preferences
- Given single-peaked preferences, how do we select a candidate?
  - Average rule?
  - Median rule?
- Turns out, **median voter rule** is individual and group strategyproof and satisfies the Condorcet criterion



Single-peaked