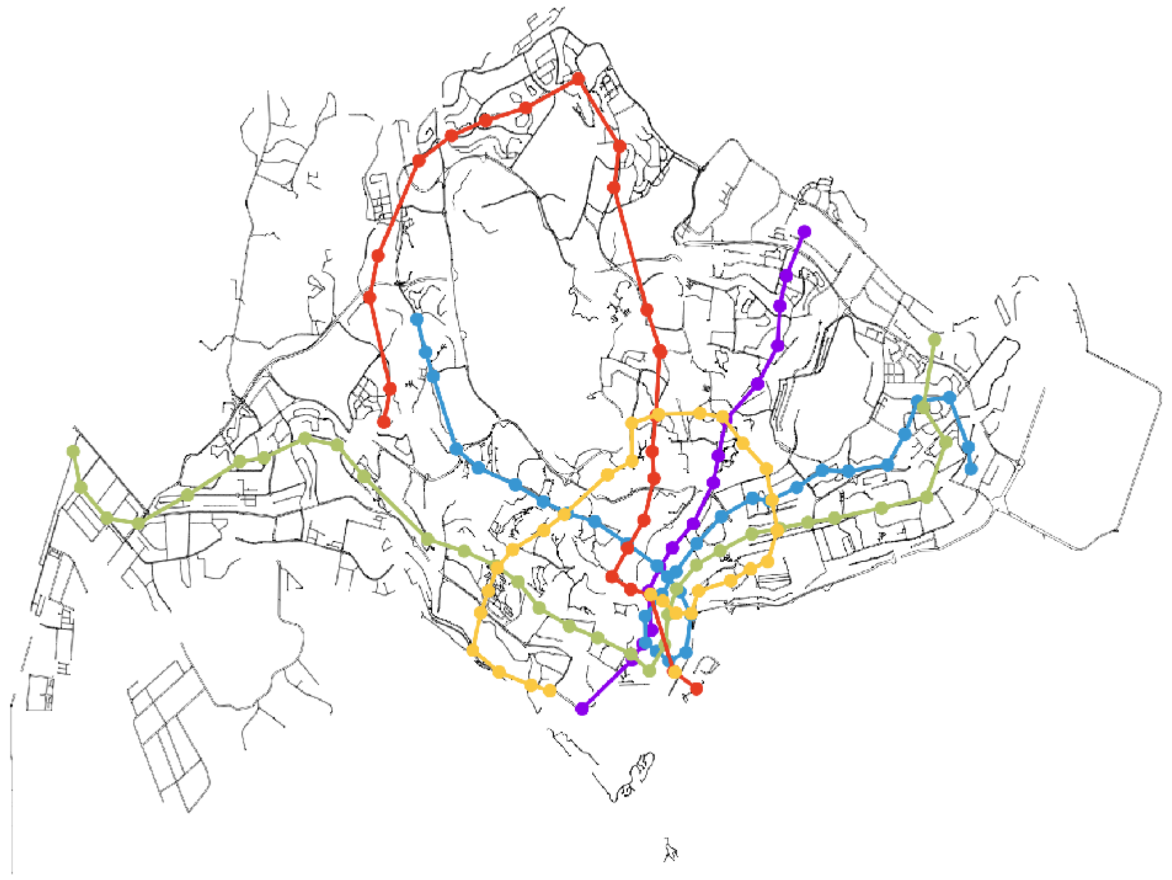
CSCI 357: Algorithmic Game Theory

Lecture 11: Matching Markets

Shikha Singh

Announcements and Logistics

- Student solutions to HW 1-3 on GLOW
  - Will upload solution to HW 4 as soon as late submissions come in
  - Planning to return feedback by tomorrow
  - HW 4 extra credit competition: will grade and announce on Monday
- Office hours after lecture today:
  - 4-5.30 pm
- Extra office hours tomorrow
  - 1 - 2.30 pm
- Will be in my office all day Saturday if you have questions as you study

Questions?

Midterm 1 Logistics

- Can pick up exam any time between 9 am and 7 pm from my office
- Fill out google form: https://tinyurl.com/357midterm
  - Please fill it out at least 2 hours in advance of your chosen start time
- Must return exam within 3.5 hours of pick up time:
  - For example, if you start at 9 am, must return the exam by 12.30 pm
- TCL 202 is booked for students who wish to use it
- I will be around if you have questions
- How to prepare: review all the problem sets, and then lectures

Questions?

Topics Summary

- **Game theory** (HW 1)
  - Dominated strategies, DSE, iterated elimination, pure Nash, Mixed Nash
- Single-parameter **auction design** (HW 2 and 3)
  - Myerson's lemma, critical bids, monotone allocations, surplus maximization
- **Sponsored ad auctions** (HW 4)
  - GSP auction, envy-free Nash, balanced bidding, GSP vs VCG
- **Bayes Nash and Revenue** (HW 5)
  - Analyzing auctions using Bayes Nash, applying revenue equivalence
- **VCG** mechanism (HW 5)

Questions?

Last Time

- Wrapped up general mechanism design
  - The VCG auction is DSIC and surplus maximizing
- Combinatorial auctions are challenging in practice
- Example: Spectrum auctions
- Takeaway: simultaneous ascending auctions are ideal for multiple items

Today

- Discuss decentralized matching markets
- Can be viewed as a simultaneous ascending auction, but the theory is more general

Week 7: Voting & Social Choice

Week 6: Matching Markets w/o Money

Week 5: Matching Markets w Money

Week 4: Bayesian Analysis & General Mechanism Design

Week 3: Application : Sponsored Ad Markets

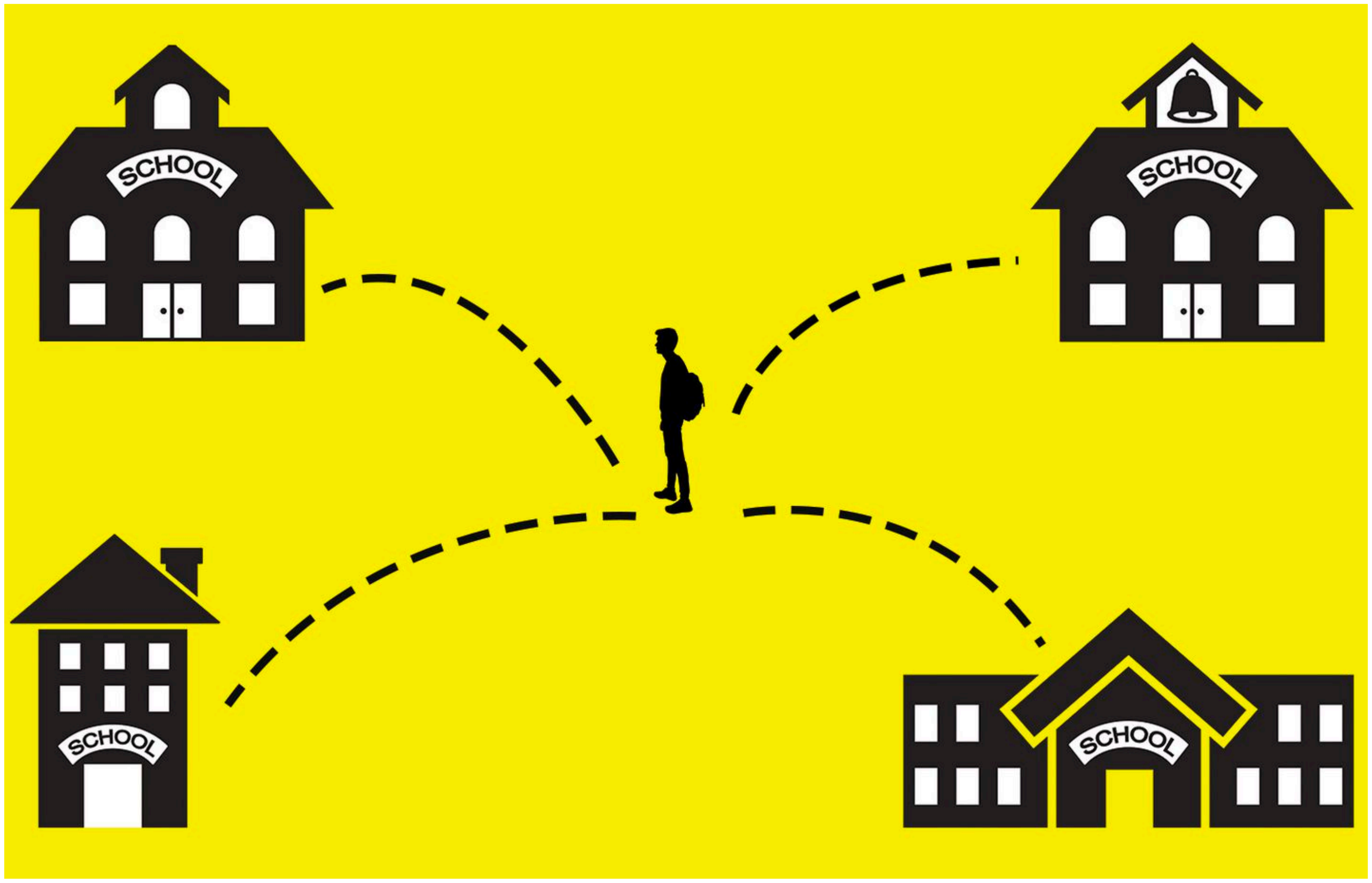
Week 2: DSIC Auctions

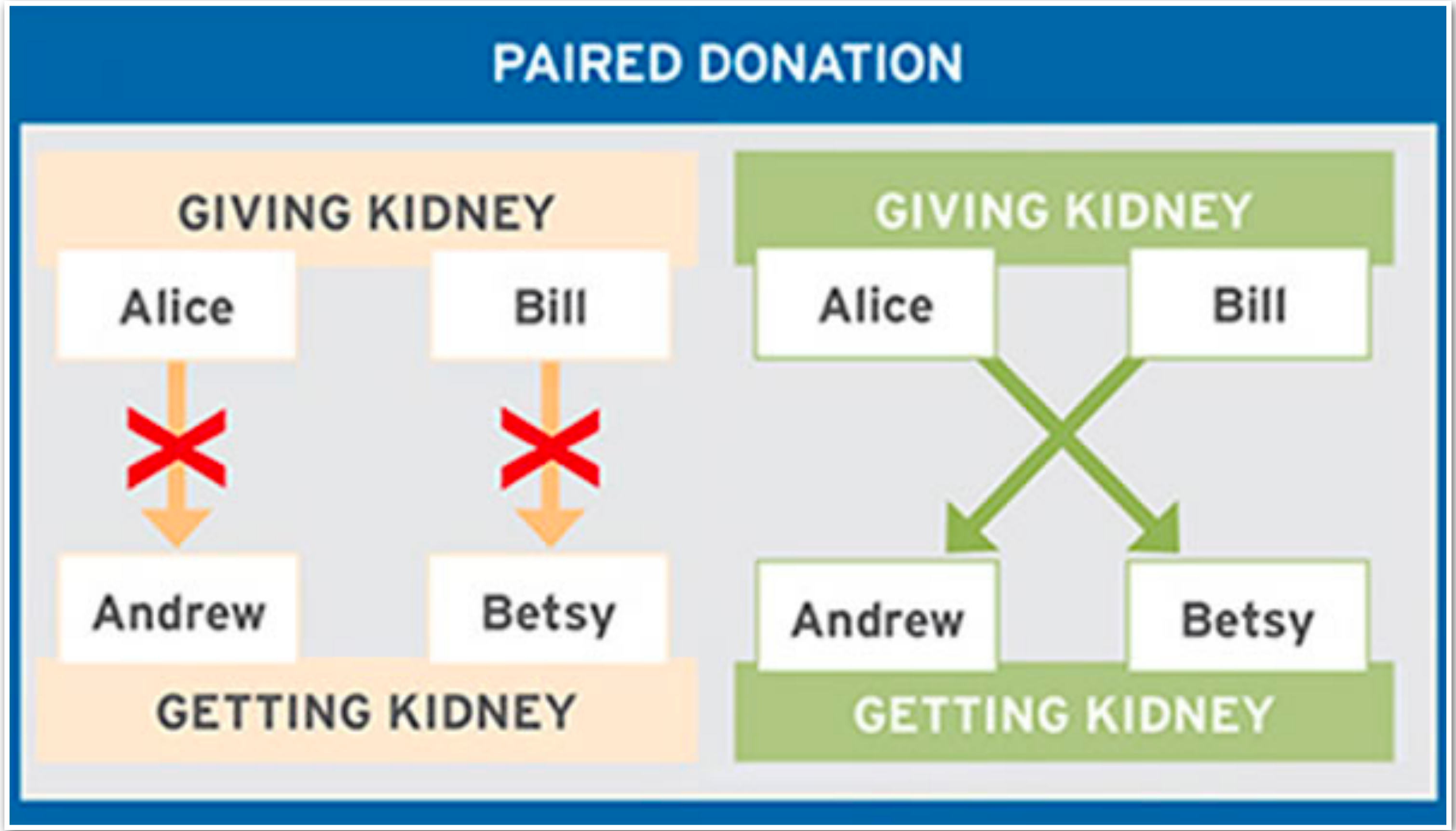
Week 1: Game Theory

Matching Markets (w Money)

Matching Markets with Money

Matching Markets Without Money





THE MATCH®
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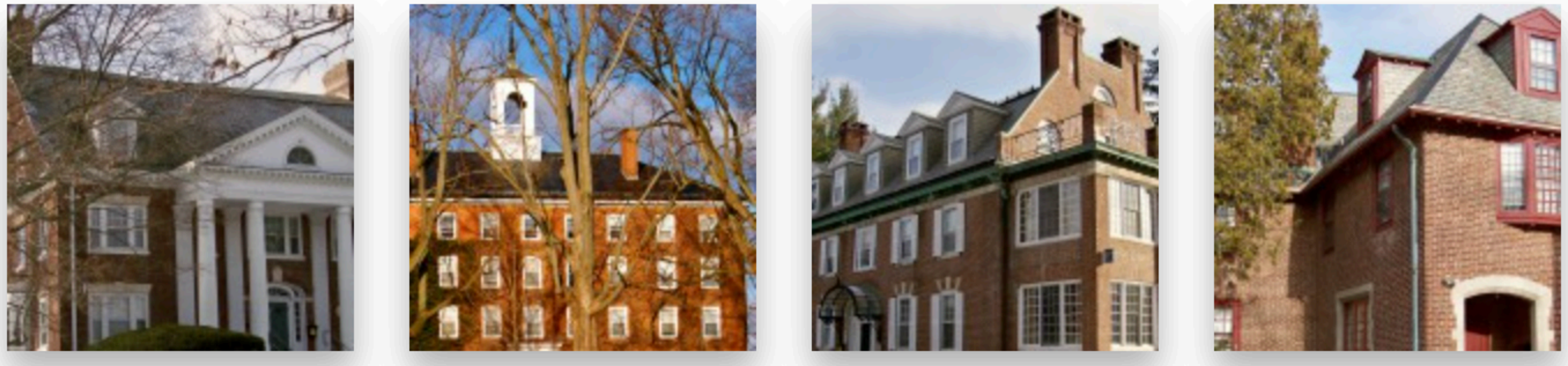
Housing & Residential Programs



Decentralized Market

- **Centralized** if transactions are decided by a central hub
    - For example, matching medical students to hospitals occurs through a central clearing house (NRMP)
    - College admissions in India are centralized
- A market is **decentralized** if participants are free to transact directly with each other, without any central coordination
    - College admissions in the US are decentralized
- What about digital markets?
    - Most are decentralized except Uber and Lyft
- Even decentralized markets may have some central coordination

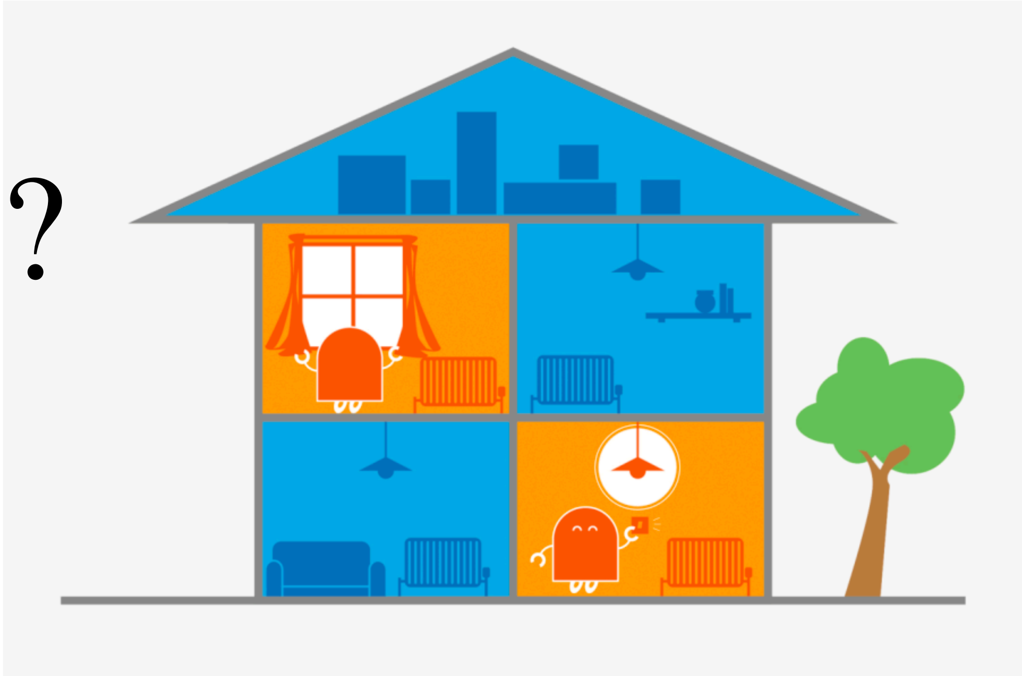
Multiple Item Matching Market

- We will discuss a **decentralized** asynchronous matching market where buyers are free to buy the item they wish
- Each buyer wants only **one** item & each item can be given to at most one buyer
- More formally, we have n potential buyers, m different items
- Assume $m \geq n$ (if this is not true, we can always create dummy items that everyone values at 0)
- Each buyer i has a private valuation $v_{ij} \geq 0$ for each item j
- **Examples**: matching houses to buyers, renters to Airbnb rooms, or any idiosyncratic items to buyers

Example Market

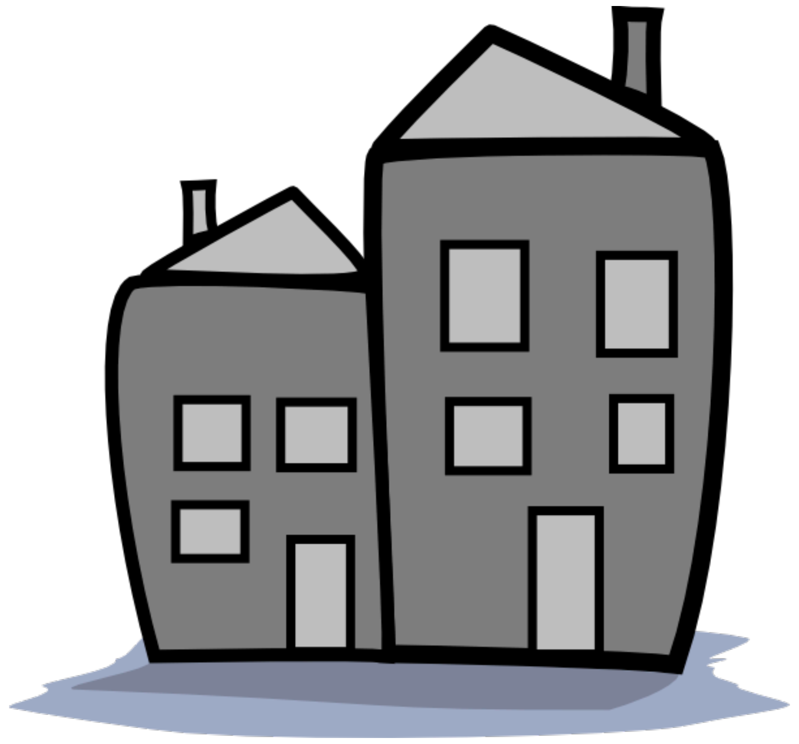
Prices | Zoe | Valuations

p_1?



12, 2, 4

p_2?



Valuations listed are in **order** of houses **(top down)**

Ch…s

8, 7, 6

p_3?

Jing

7, 5, 2

Prices and Utility

- If the price of item j is p_j, the utility that person i receives from getting item j is

$$u_{ij} = v_{ij} - p_j$$

- **Goal of buyers:** choose items to maximize their utility
- **Questions.**
  - What prices do we expect to see in a market where each buyer selfishly chooses items to maximize their utility?
  - Does there exists **prices** and **a way to match buyers to items** (find a matching) such that each buyer gets an item that maximizes their utility
  - Do these prices **"clear the market"**: sell all items that have any demand
- Imagine an ascending price clock and bidders dropping out of contention

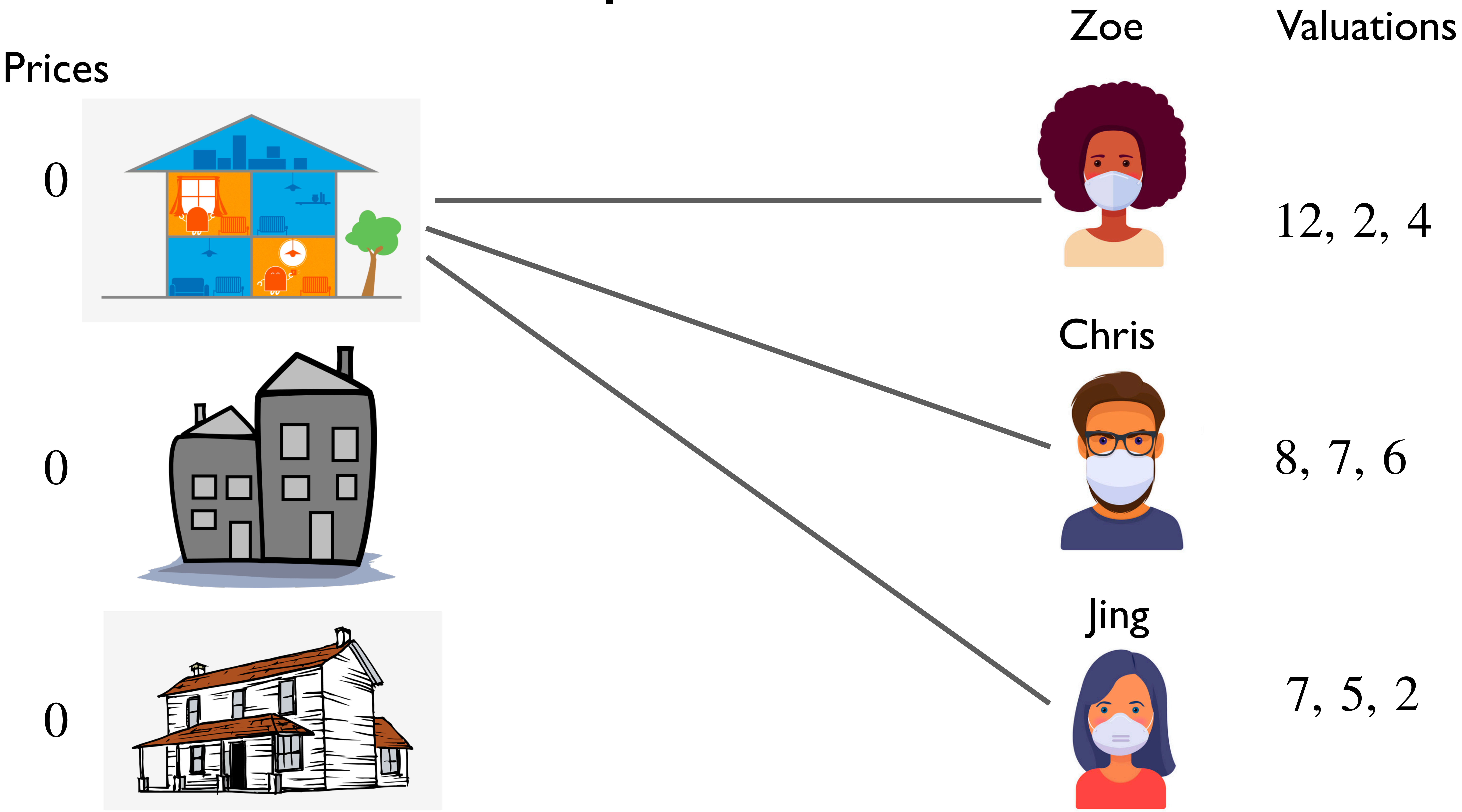
Social Welfare

- **(Vocabulary)**. **Welfare** and **surplus** is used interchangeably)
- Let $M(i)$ denote the item matched with buyer i or $\varnothing$ if none
- Our goal has been to design mechanisms that maximize social welfare, that is, find a matching M of buyers to items that maximizes $\sum_{i=1}^{n} v_{iM(i)}$
- Goal: an outcome that achieves good guarantees but we have no control over it
- **Question.** If we let the market run its course (with buyers demanding items they prefer and sellers responding to demand) what prices and allocation do we expect to see?
    - How good is the social welfare of such an outcome?

Preferred Items & Graph

- Given prices $\mathbf{p} = (p_1, \ldots, p_m)$ for the items, the preferred items for buyer i are all the items that maximize its utility
- Let $u_i^* = \max_{\text{all items } j} (v_{ij} - p_j)$ be the maximum utility i can obtain given $\mathbf{p}$
- Let the set of preferred items P_j of buyer i given the prices $\mathbf{p}$ be all items that maximize its utility, that is,
  - $P_j = \{j \mid v_{ij} - p_j = u_i^*\}$ assuming $u_i^* \geq 0$
  - If $u_i^* < 0$ then $P_j = \emptyset$
- Create a **preferred-item graph** (given prices $\mathbf{p}$) where nodes are items and buyers and there is an edge between buyers and their preferred items

Preferred-Item Graph



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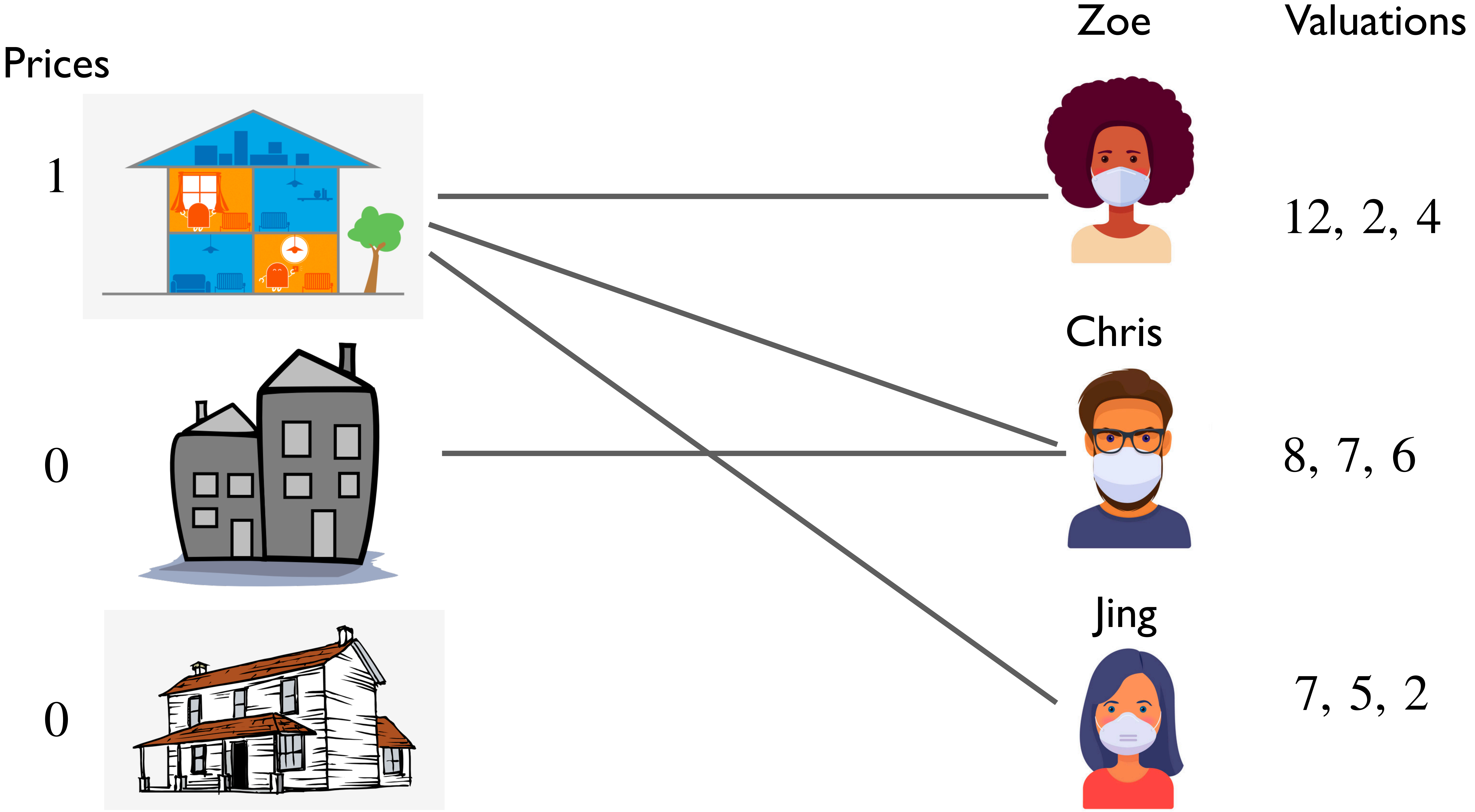

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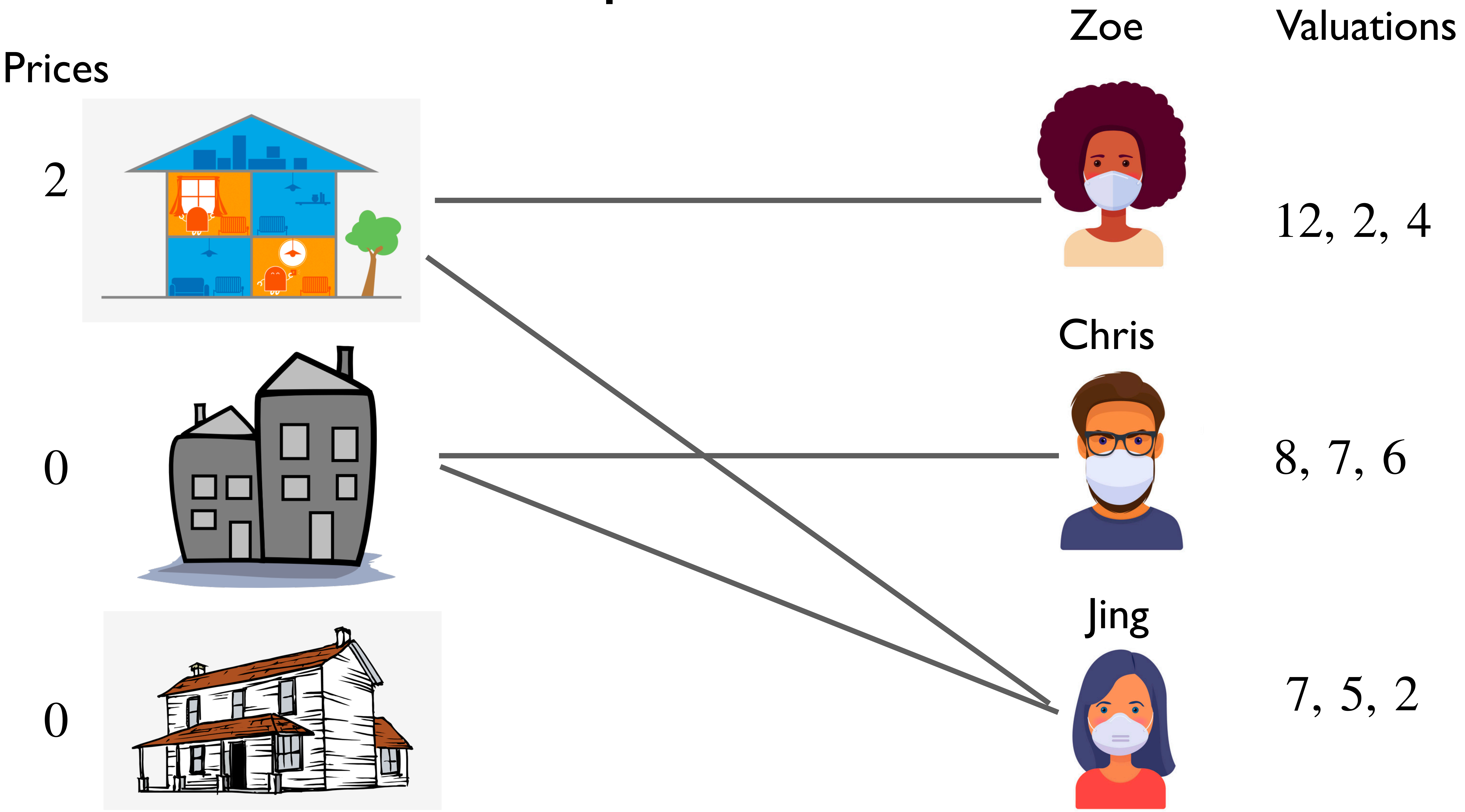

Market-Clearing Prices

- A selection of prices $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ is **market-clearing** if:
  - **Condition 1.** There is a **matching** in the preferred-item graph such that **all buyers are matched** to an item
  - **Condition 2.** If an item j is not matched to any buyer, then its price $p_j = 0$, in other words, every item with non-zero price $p_j > 0$ must get sold
- This means that at market-clearing prices, each buyer can come by and pick some item that maximizes its utility
  - Assume tie-breaks can occur in a coordinated way
- Furthermore, at these **prices the market will "clear"**
  - Only items left behind are those that are not desirable (price 0)

Matching. A subset of edges M form a matching if no two edges in M is incident on the same node

Market-Clearing Prices

What does this condition remind you off?

- **Condition 1** says that, given the prices, all buyers:
  - Get a utility maximizing item

$$\underbrace{v_{ij} - p_j}_{\text{Utility from receiving } j \text{ at price } p_j} \geq \underbrace{v_{ij'} - p_{j'}}_{\text{Utility from receiving } j' \text{ at price } p_{j'}}$$

- Why do we need **Condition 2?**
- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
  - Only good that is allowed to be not sold are those with $p_j = 0$

Market-Clearing Prices

Outcome must be **envy free**!

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  - Get a utility maximizing item

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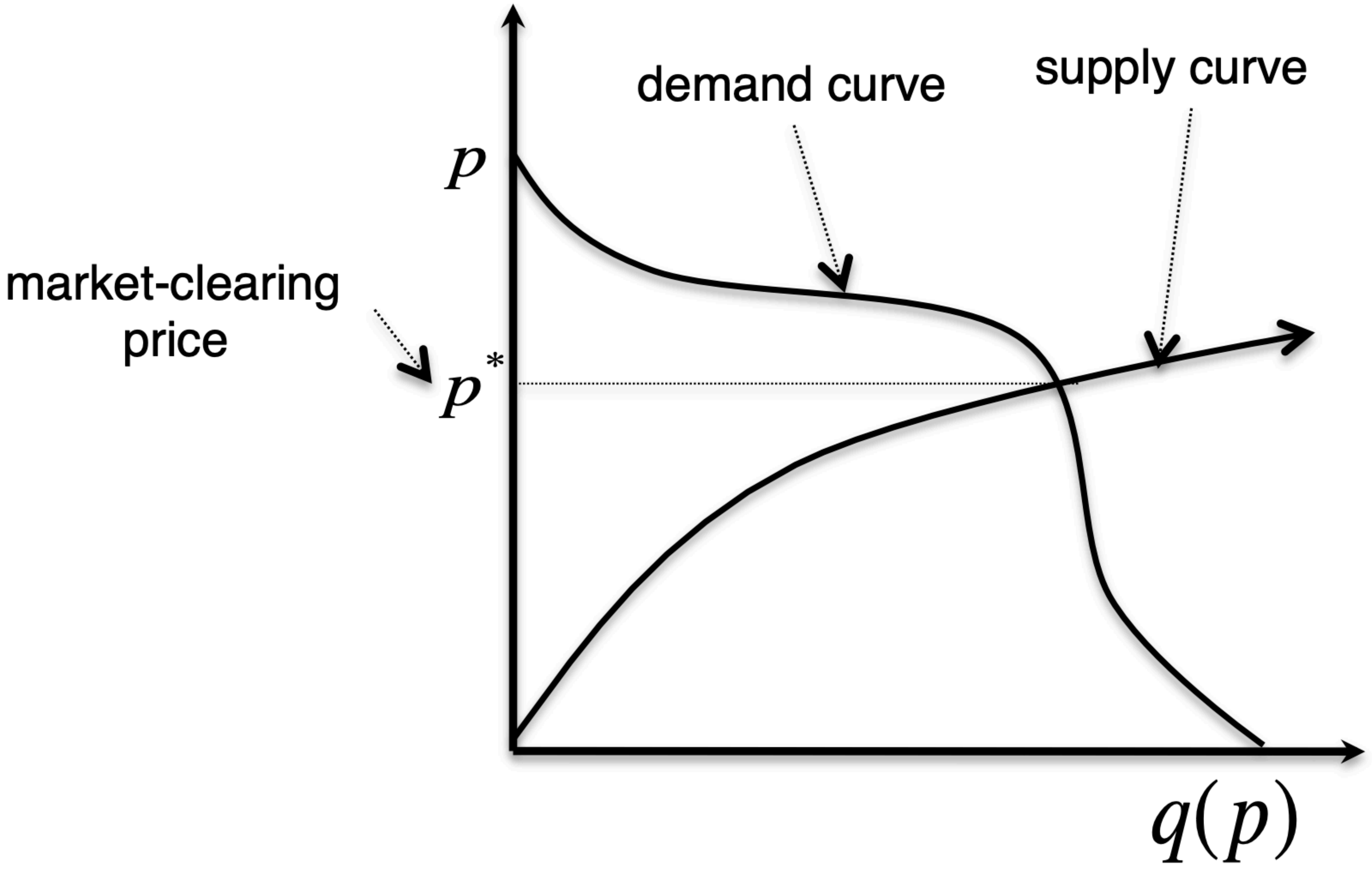
Market-Clearing Prices

- **Condition 2** says that the outcome is "market clearing" in the sense that every good that is desired is sold
  - Only good that is allowed to be not sold are those with $p_j = 0$
- Why is this condition important?
  - Notice that we can trivially satisfy Condition 1 by setting all prices to be ∞
  - At that price, no buyer wants any item
- But is this a good outcome?
  - No one gets anything: no welfare/surplus generated!
- Need prices to **clear market** and to **optimize surplus** generated

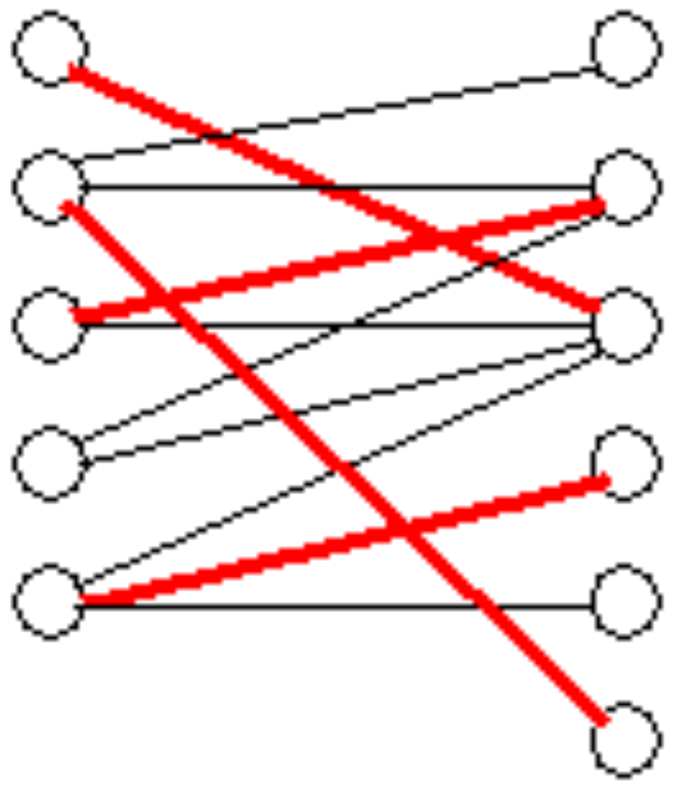
Economics Point of View

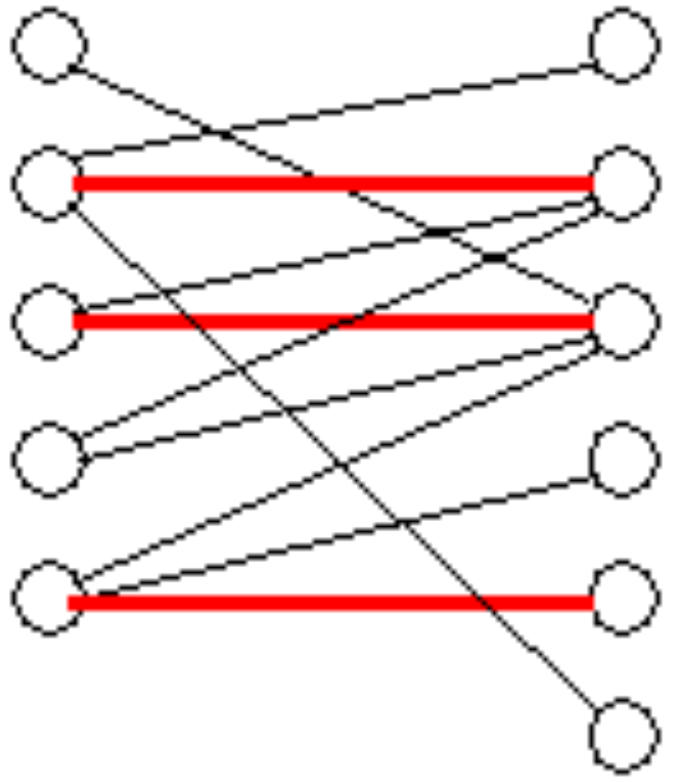
- Market clearing prices in economics are prices at which supply is equal to demand
- **Demand curve:** as price increases, typically demand goes down
- **Supply curve:** As price increases, typically supply $s(p)$ increases
- Price where they meet: market clearing



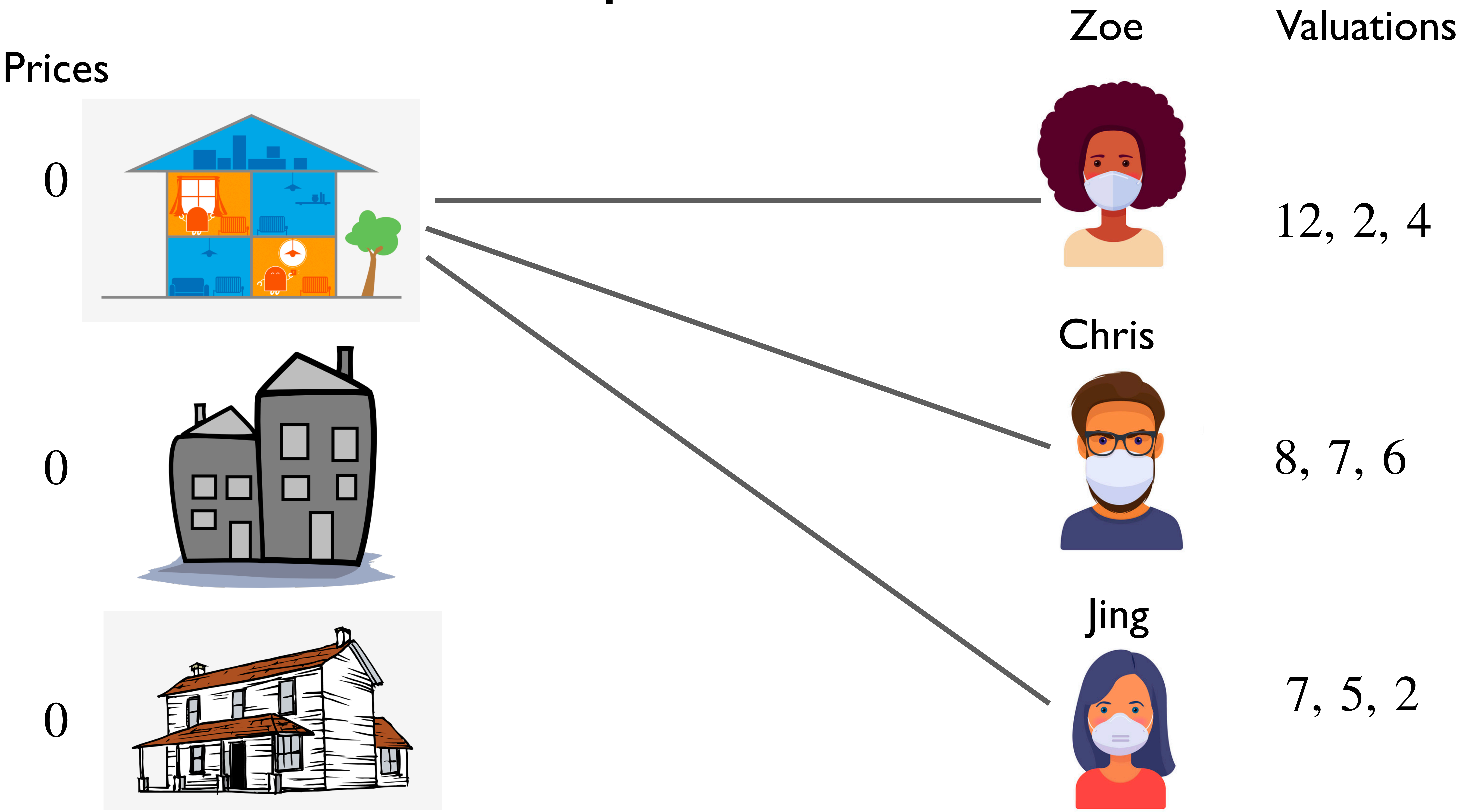
Market-Clearing Prices

- **Reminder (matching definition).** A subset of edges M form a matching if no two edges in M is incident on the same node
  - An independent set of edges
- Looking for a **buyer-perfect matching** (a matching that "covers" all buyer nodes)
- Since the edges in the preferred-item graph depend on prices of items, the question is,
  - What prices cause a buyer-perfect matching to exist?
- **Question.** How do we know when a buyer-perfect matching is not possible?





Preferred-Item Graph



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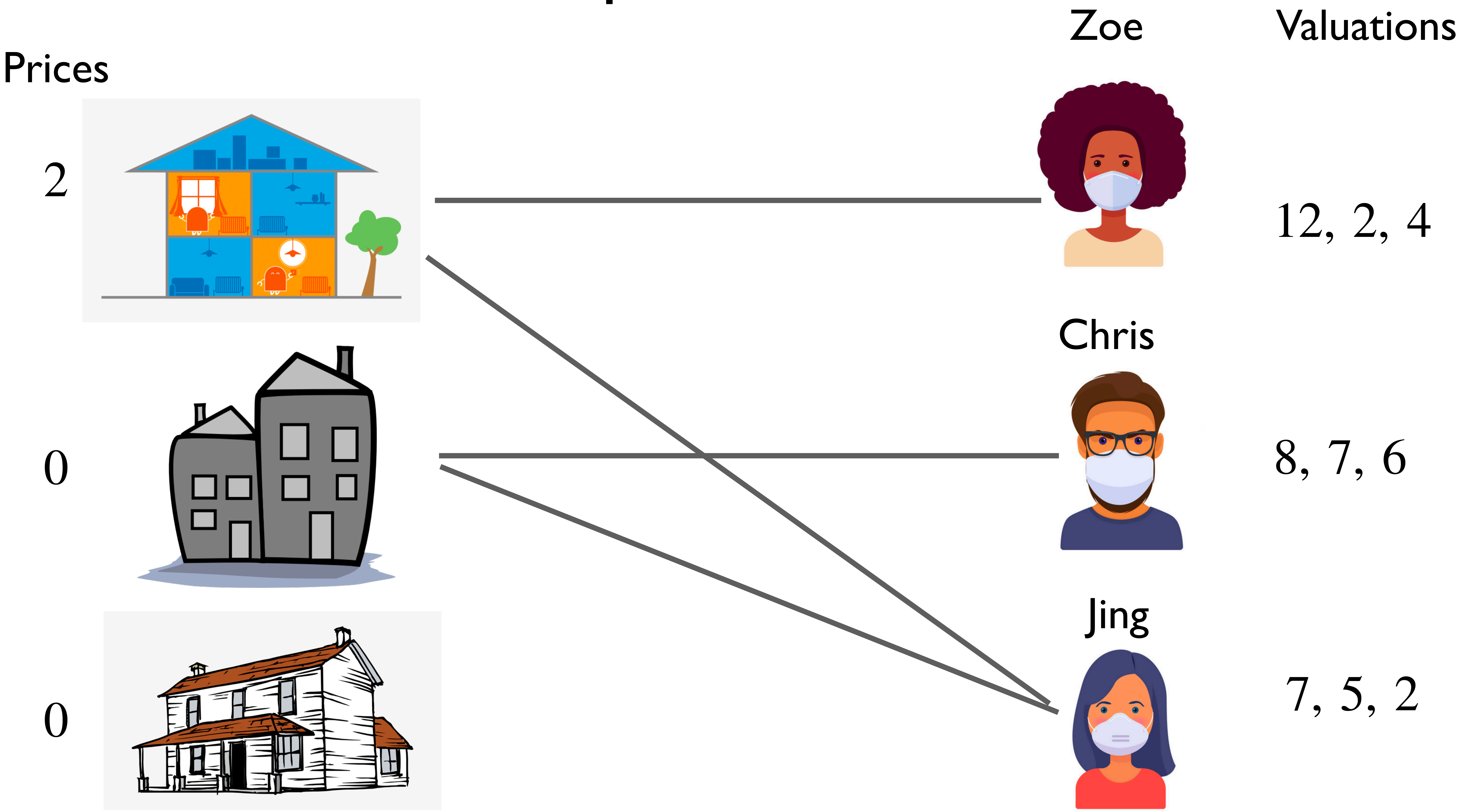
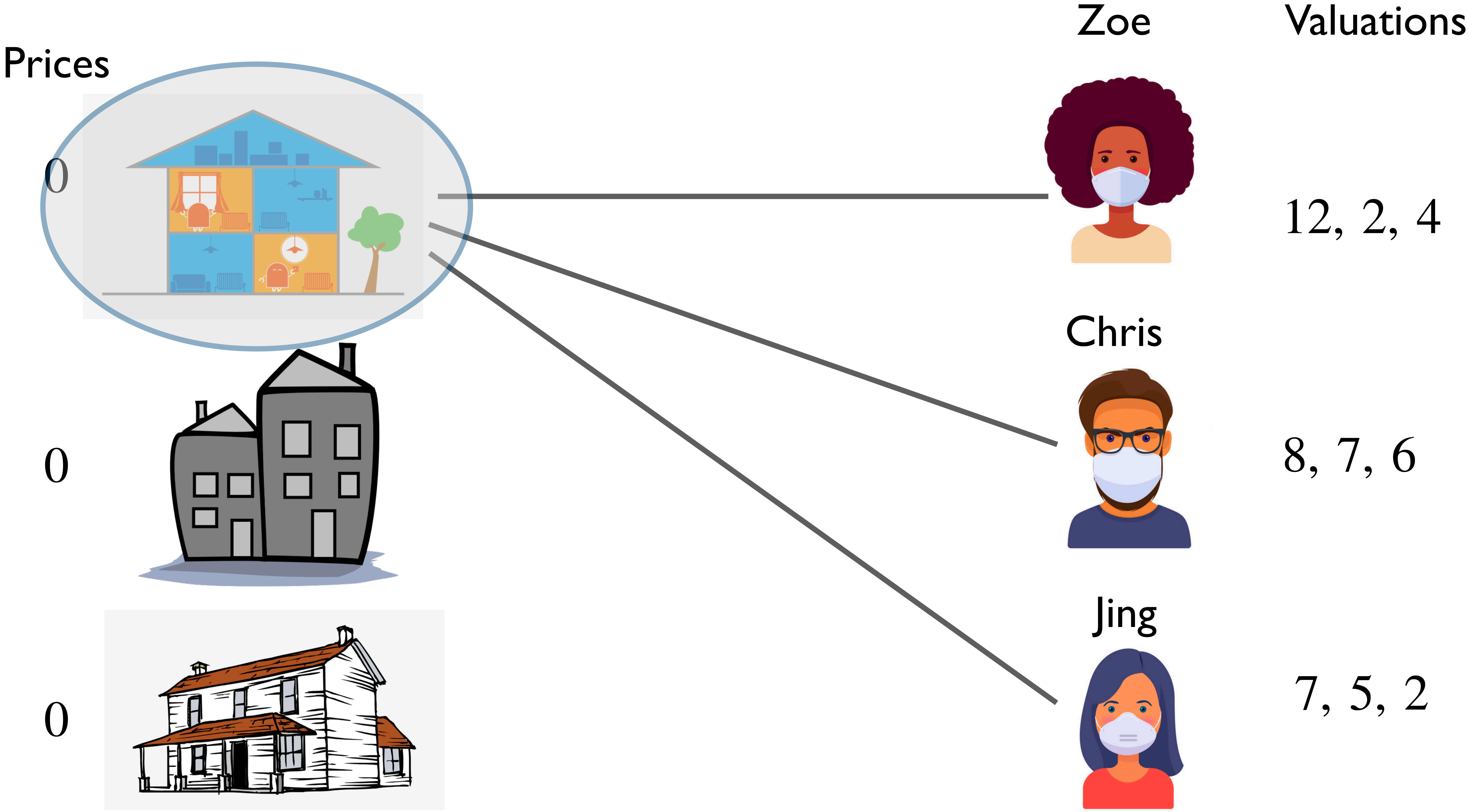


Hall's Theorem

- Let S be a subset of nodes, then the **neighborhood** $N(S)$ is the set of all nodes that adjacent to nodes in S
- In a bipartite graph (X, Y) has a Y perfect matching **if and only if** for every $S \subseteq Y$ with neighborhood $N(S)$ the following holds:

  $|N(S)| \geq |S|$ (neighborhood is at least as large)
- Thus, when a Y-perfect matching is not possible, there exists a subset $T \subseteq Y$ that violates the above condition, that is,
  - Such a set $N(T)$ is called a **constricted set** set
- If there is no buyer-perfect matching: can always find a constricted set
  - "Over-demanded" items at current price

Preferred-Item Graph



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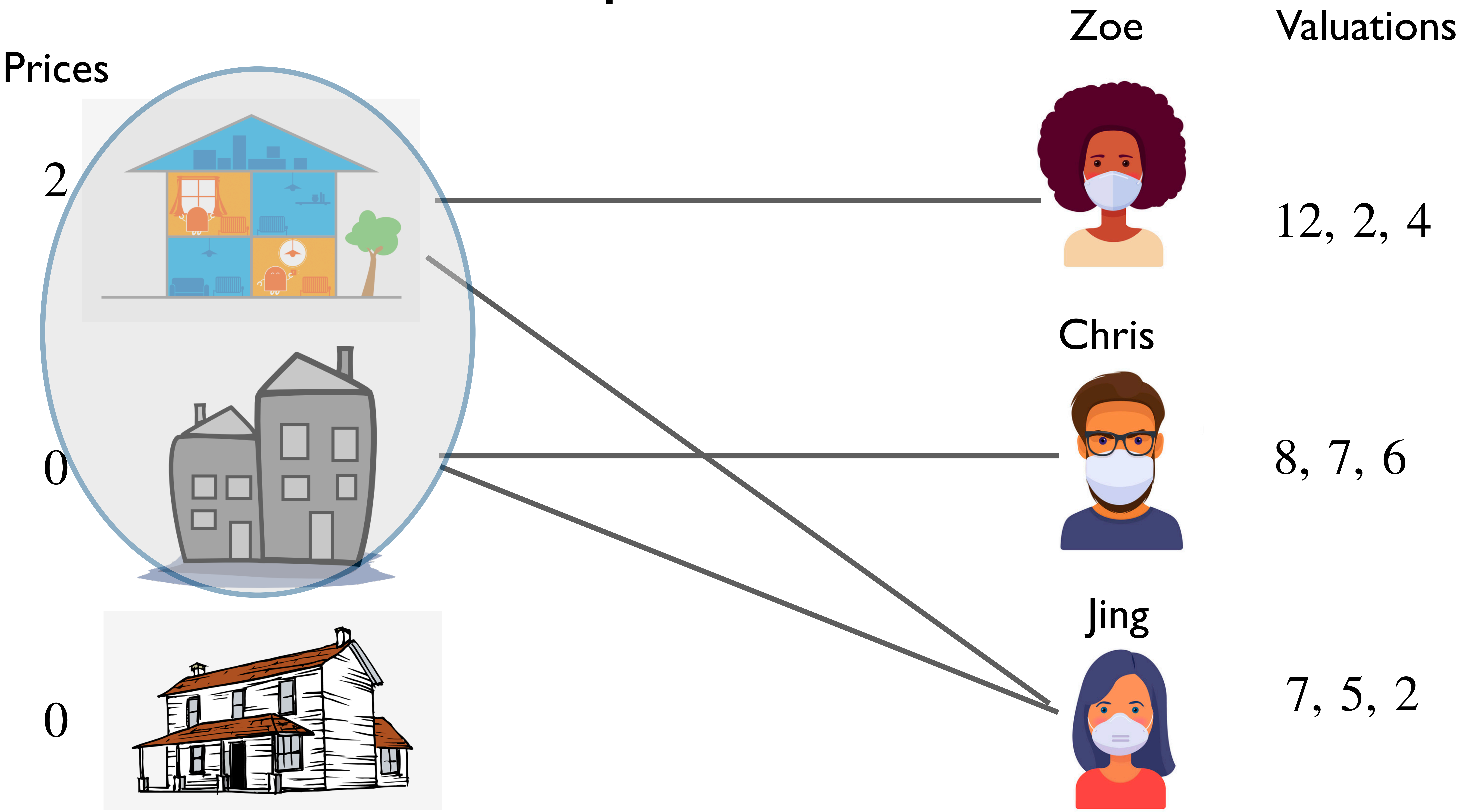

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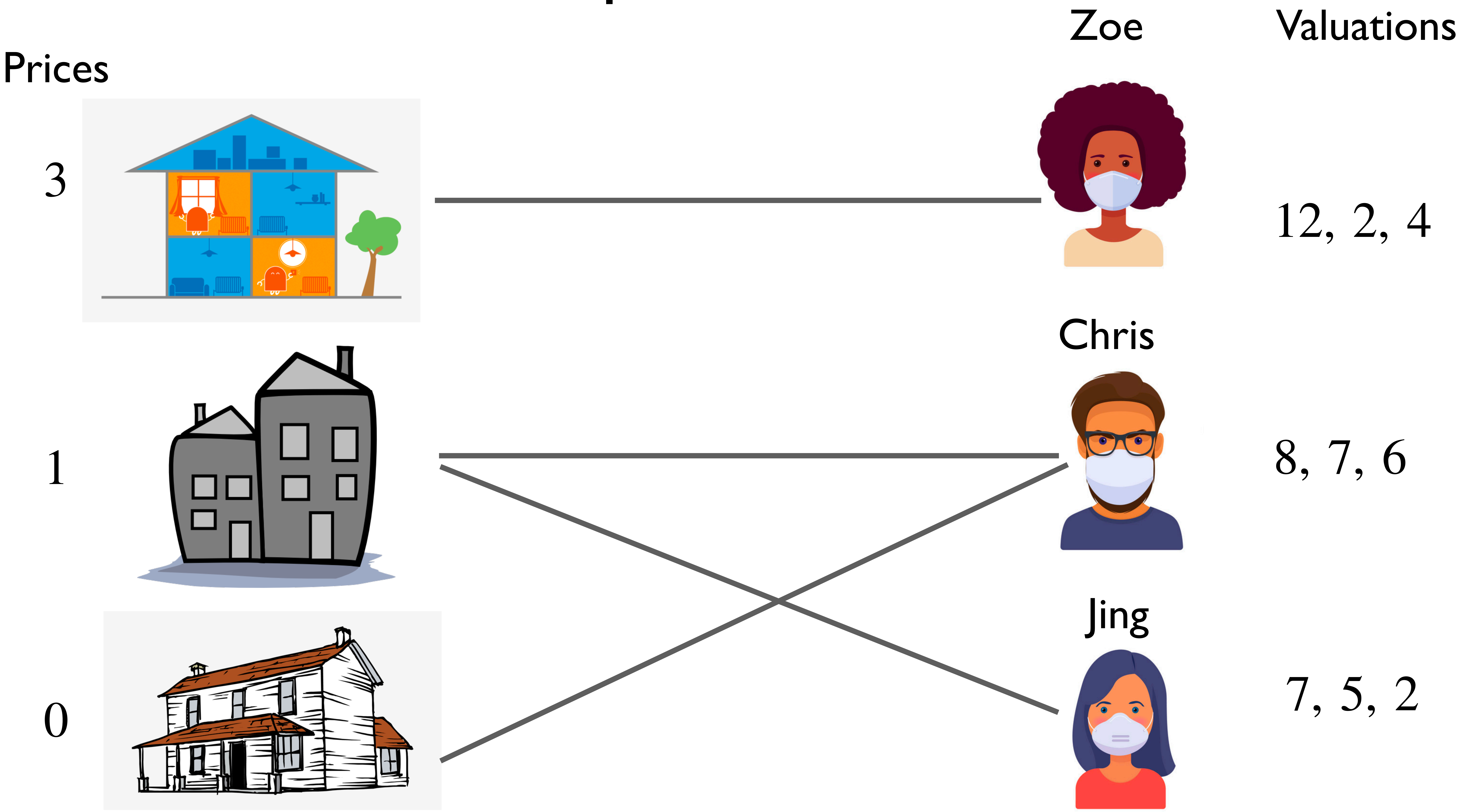

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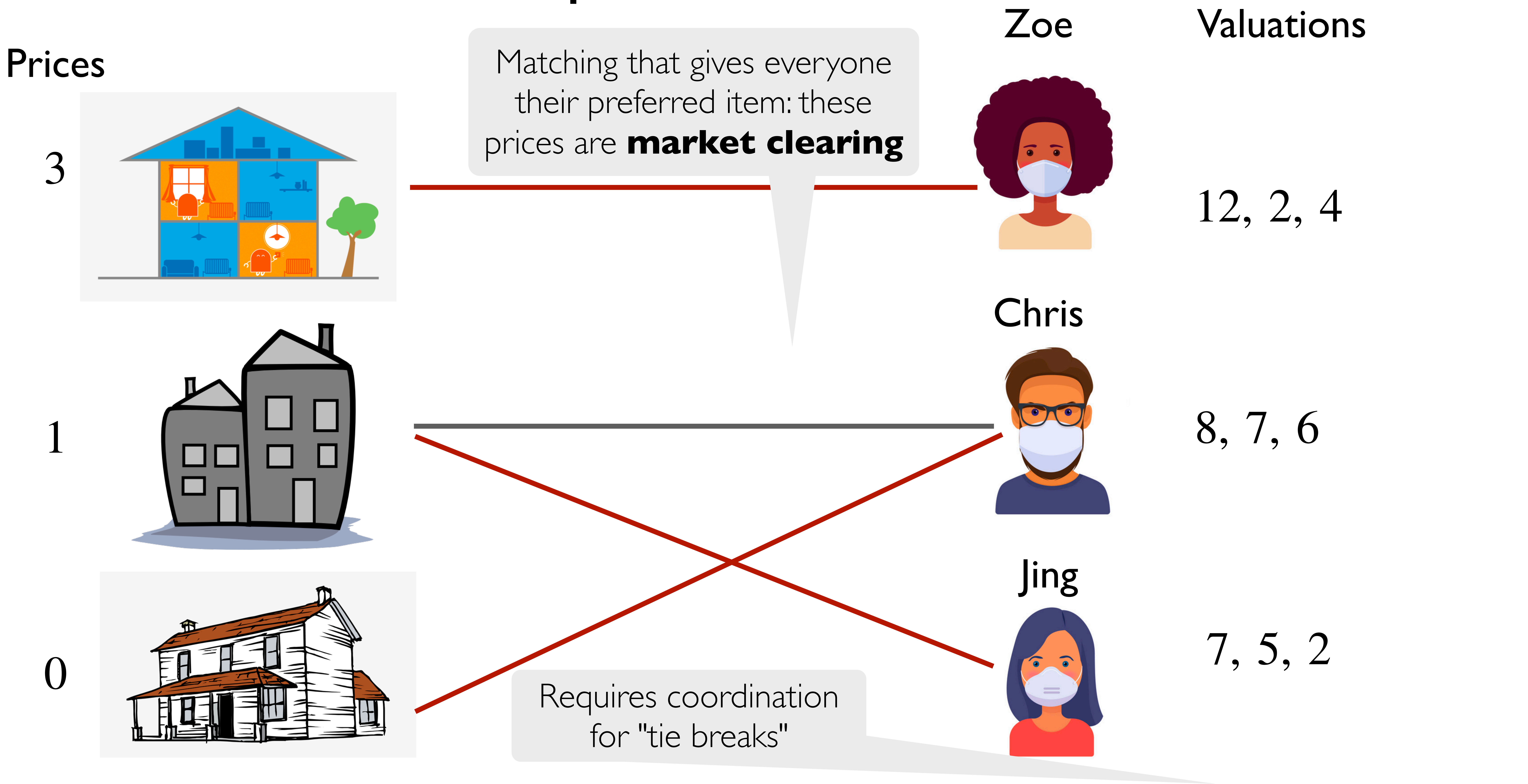

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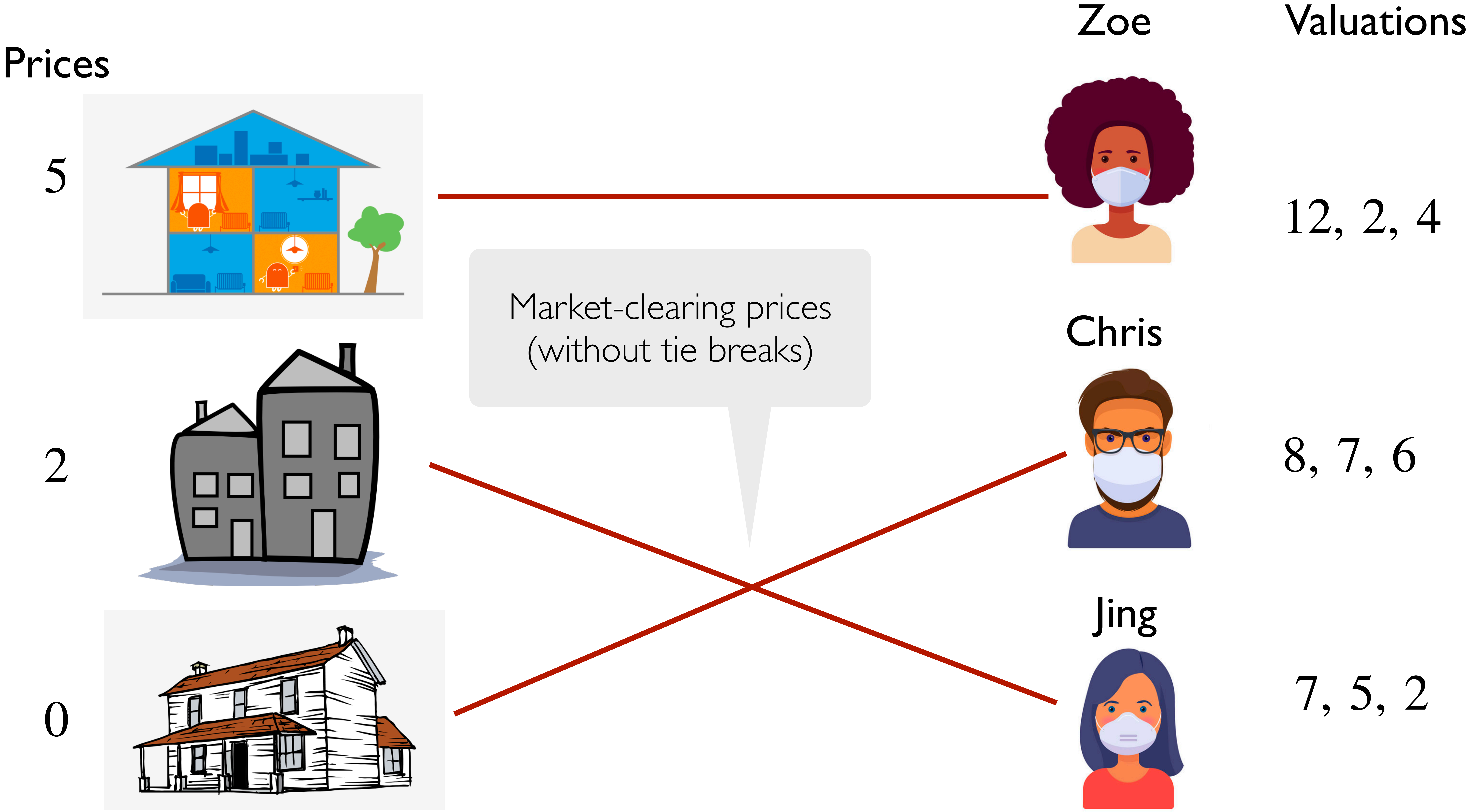
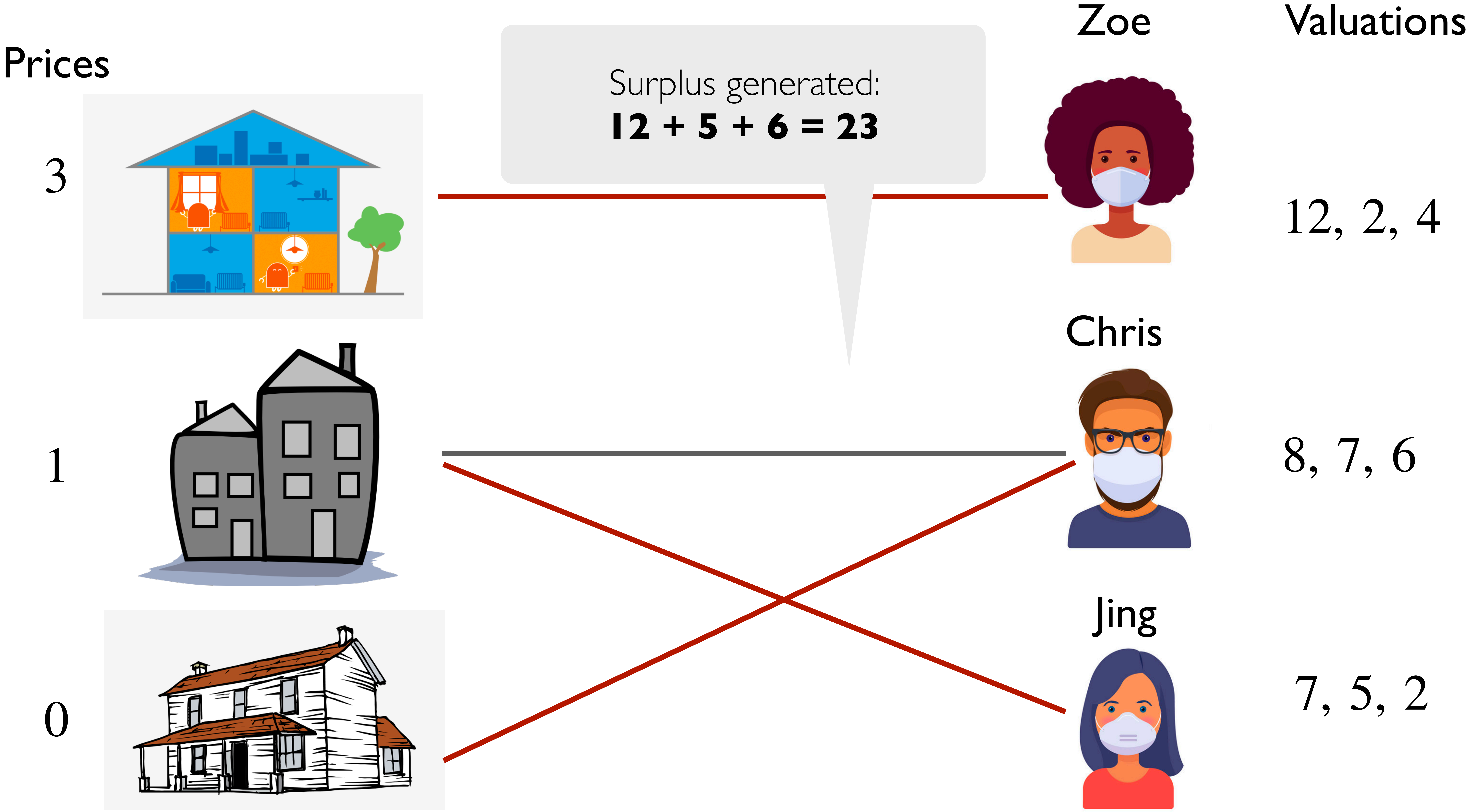


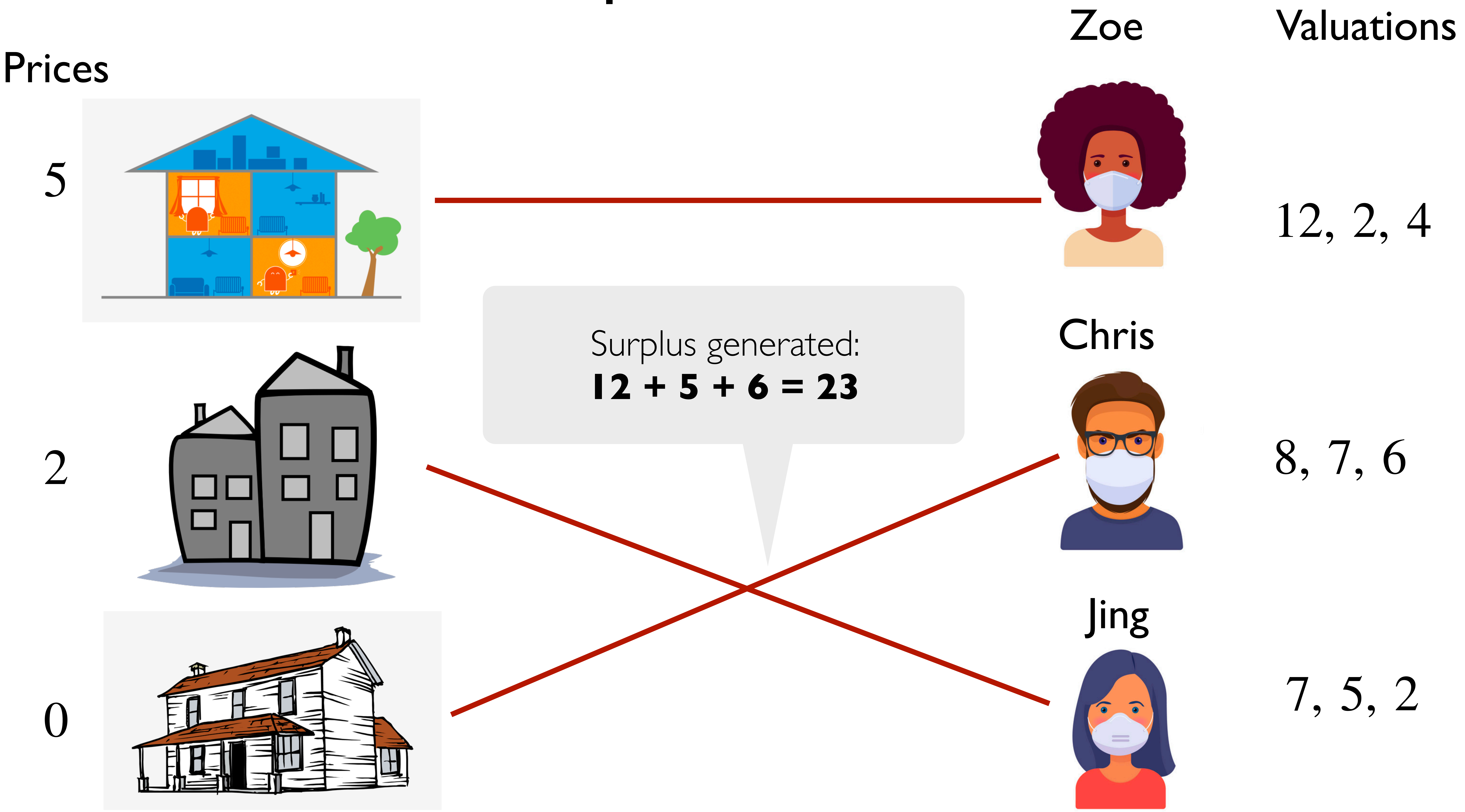
Competitive Equilibrium

- Market-clearing prices $\mathbf{p}$ along with the matching M from buyers to their preferred item is called a **competitive** or **Walrasian equilibrium**
- Requirements of competitive equilibrium are strong
  - Put a price tag p_j on each good
  - Let each buyer i independently pick whichever good they want
- Magically, there are no conflicts and each buyer gets what they want
  - (Allowing ties to be broken in a coordinated way)
- **Question.** Seems too good to be true, does it always exist?
- **Question.** Should we be happy with the outcome of a competitive equilibrium?

Preferred-Item Graph



Preferred-Item Graph



First Welfare Theorem

- Matchings in a competitive equilibrium are exactly the matching with maximum possible value!
- **First Welfare Theorem.** If $(M, \mathbf{p})$ is a competitive equilibrium, then M is a matching with maximum total value, that is,
    - $\sum_{i=1}^{n} v_{iM(i)} \geq \sum_{i=1}^{n} v_{iM'(i)}$ for every matching M'
- In particular, among all possible ways of allocating items such that each buyer is matched to at most one item good and each item is matched to at most one buyer, the allocation achieved at a competitive equilibrium maximizes welfare

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

$$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)} \quad \text{for every bidder } i$$

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Summing up the inequality in blue over all bidders

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

M^* can assign each bidder at most one item

First Welfare Theorem Proof

- **Proof.** Consider some matching M^* with the maximum-possible total value
- What we know: $(M, \mathbf{p})$ is a competitive equilibrium
- Using envy-free condition to compare M and M^* at price $\mathbf{p}$:

  $$v_{iM(i)} - p_{M(i)} \geq v_{iM^*(i)} - p_{M^*(i)} \quad \text{for every bidder } i$$

- Let the sum of prices $\sum_{j=1}^{m} p_j = P$
- Reorganizing this inequality, we get that value of $M \geq$ value of M^* ∎

M^* can assign each bidder at most one item

$$\underbrace{\sum_{i=1}^{n} v_{iM(i)}}_{\text{total value of } M} - \underbrace{\sum_{i=1}^{n} p_{M(i)}}_{= P \text{ by CE property (b)}} \geq \underbrace{\sum_{i=1}^{n} v_{iM^*(i)}}_{\text{total value of } M^*} - \underbrace{\sum_{i=1}^{n} p_{M^*(i)}}_{\leq P},$$

Takeaways

- Competitive equilibrium automatically solves a non-trivial computational problem: **computing a maximum weight matching in a bipartite graph!**
  - Polynomial-time solvable but the algorithm is quite nontrivial
- Individually selfish agents reach a globally efficient outcome
- When economists say "markets are efficient", they are referring to a phenomenon like competitive equilibrium
- **Question.** Given their strong requirements, is a competitive equilibrium even guaranteed to exist?

Competitive Eq: Existence

- **Theorem.** In every market where at most one good is assigned to each buyer, there is at least one competitive equilibrium
  - Equivalently, market-clearing prices are guaranteed to exist
- We prove this constructively through an mechanism that shows how such prices might emerge organically in a market
- Intuition idea behind our **"ascending-price mechanism"**
  - If a set of k items is preferred by more than k buyers at its current price, then the prices of these items should rise
  - Keep identifying such "constricted sets" and increasing prices until the market clears

Ascending-Price Mechanism

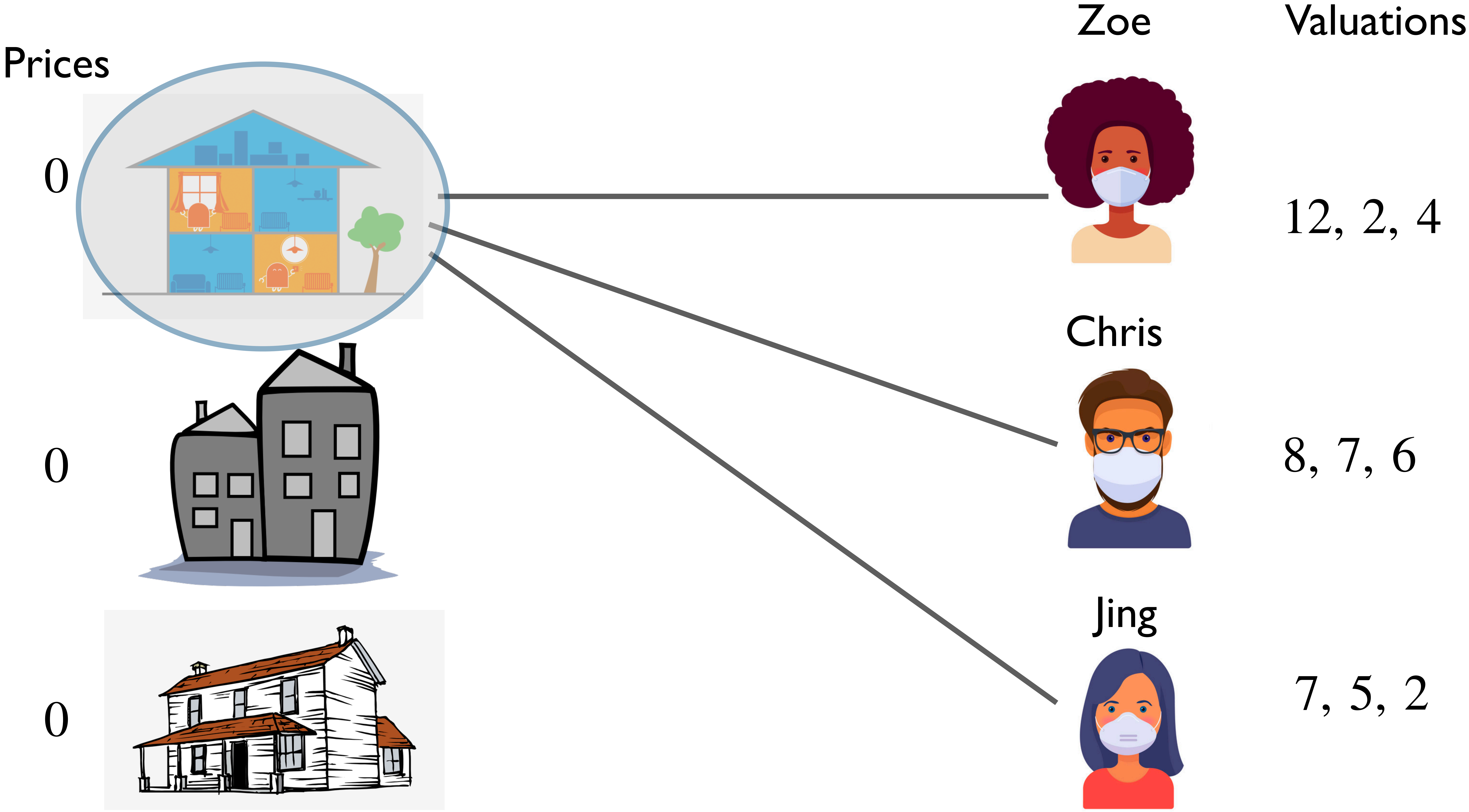
- Start with prices of all items $p_j = 0$
- Assume all valuations are integers $v_{ji} \in \mathbb{Z}$ (simplifying assumption)
- **Step 1.** Check if the current prices are market clearing, if so we are done
  - build the preferred graph, check if there is a buyer-perfect matching
- **Step 2.** Else, there must a constricted set:
  - There exists $S \subseteq \{1,\ldots,n\}$ such that $|S| > |N(S)|$
  - $N(S)$ are items that are **over-demanded**
  - If there are multiple such sets, choose the **minimal set** $N(S)$
    - Increase $p_j \leftarrow p_j + 1$ for all items in the set $j \in N(S)$
  - Go back to **Step 1.**

Single Item Case

- A single item (labelled 1) for which each buyer has a value $v_i > 0$
- Add $n-1$ dummy items $(2, \ldots, n)$ that everyone values at 0
- At the beginning preferred-item graph has edges from each buyer to item 1
- Thus, $\{1\}$ is our minimal constricted set
- We need to keep raises the price of item 1 until all except one buyer has a preferred edge to at least one item in $\{2, 3, \ldots, n\}$
- At what price does this happen?
    - Exactly when $p_1 =$ second-highest valuation
    - The person with the highest valuation is matched to item 1

We have recreated the **second-price auction** outcome!

Preferred-Item Graph



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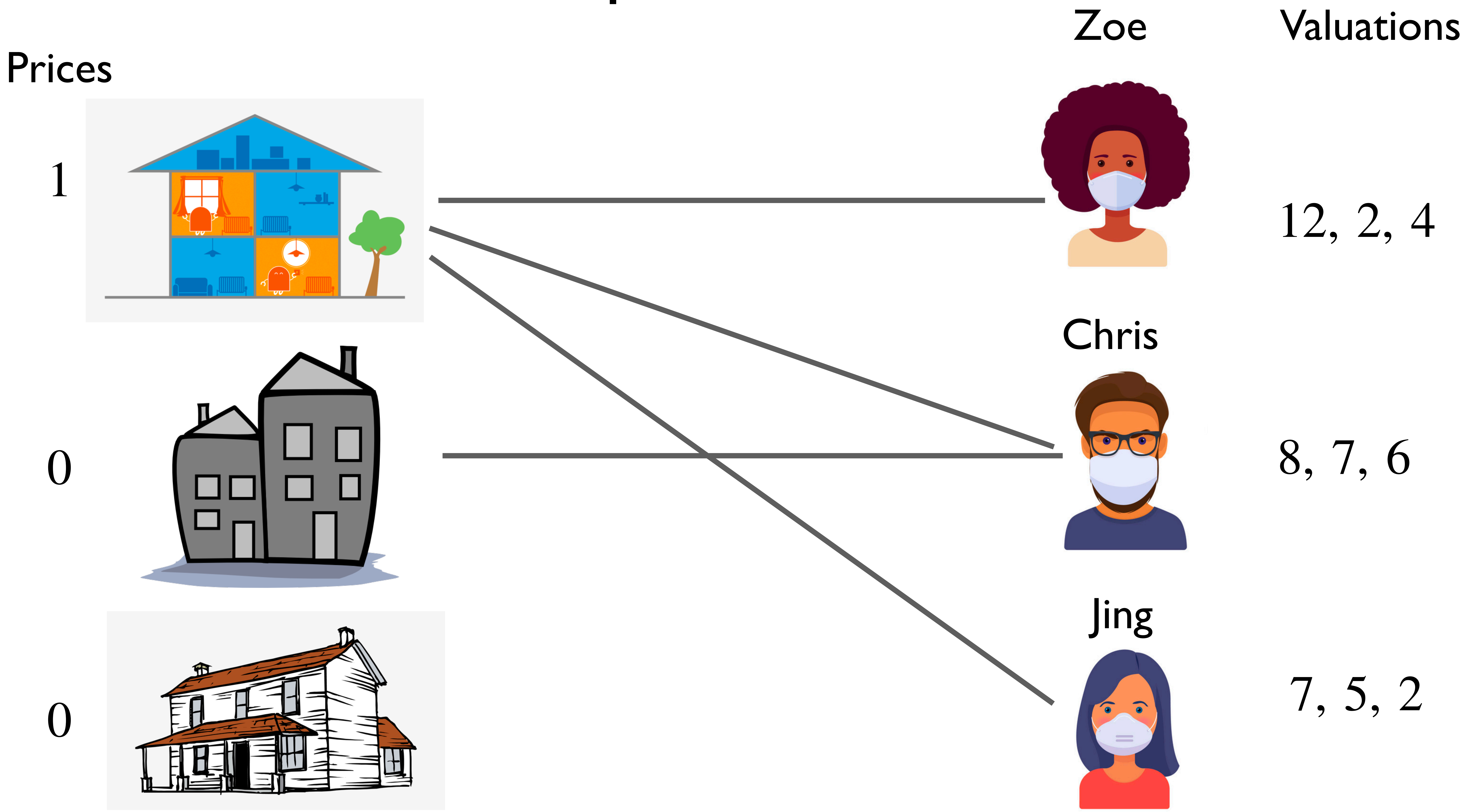

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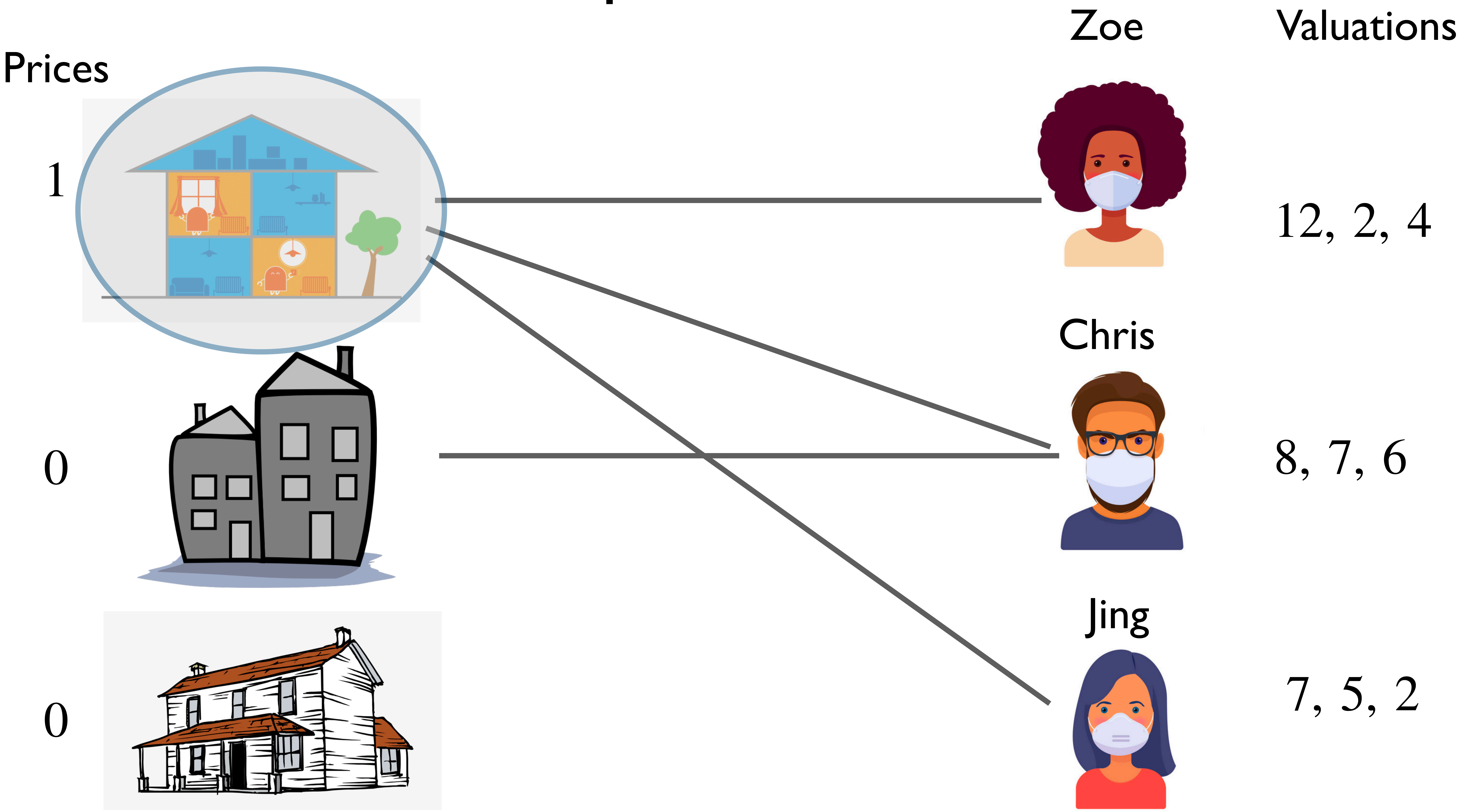

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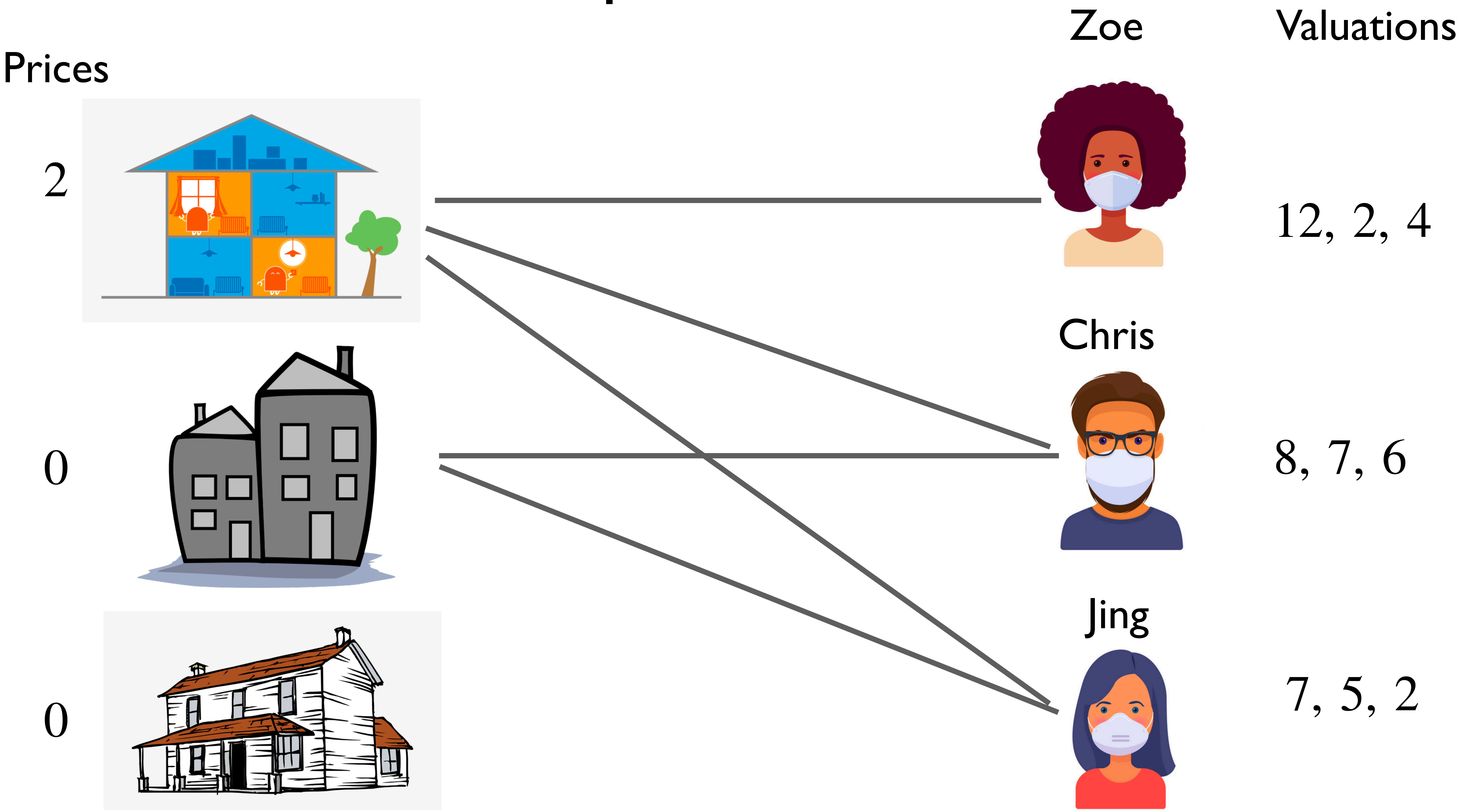

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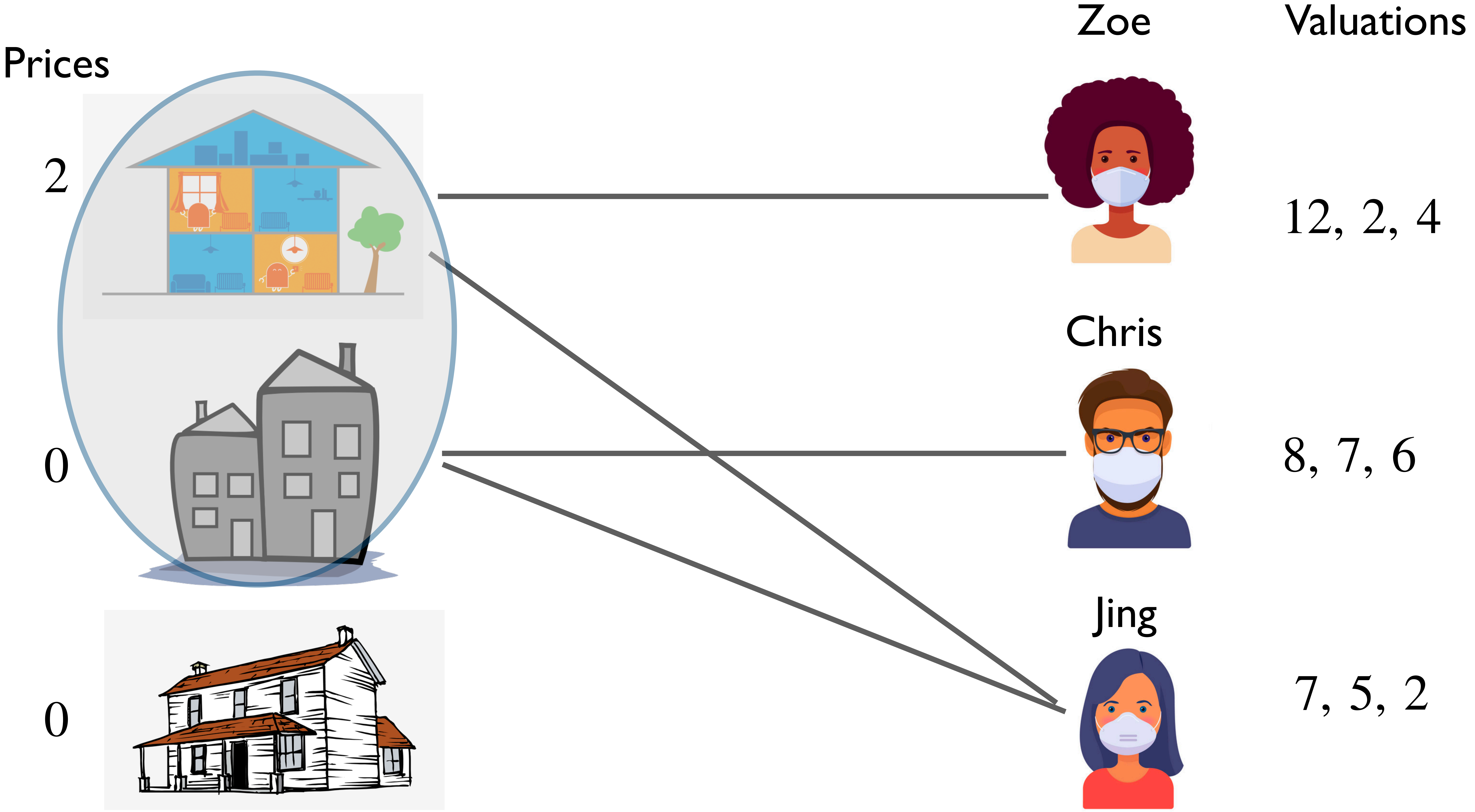

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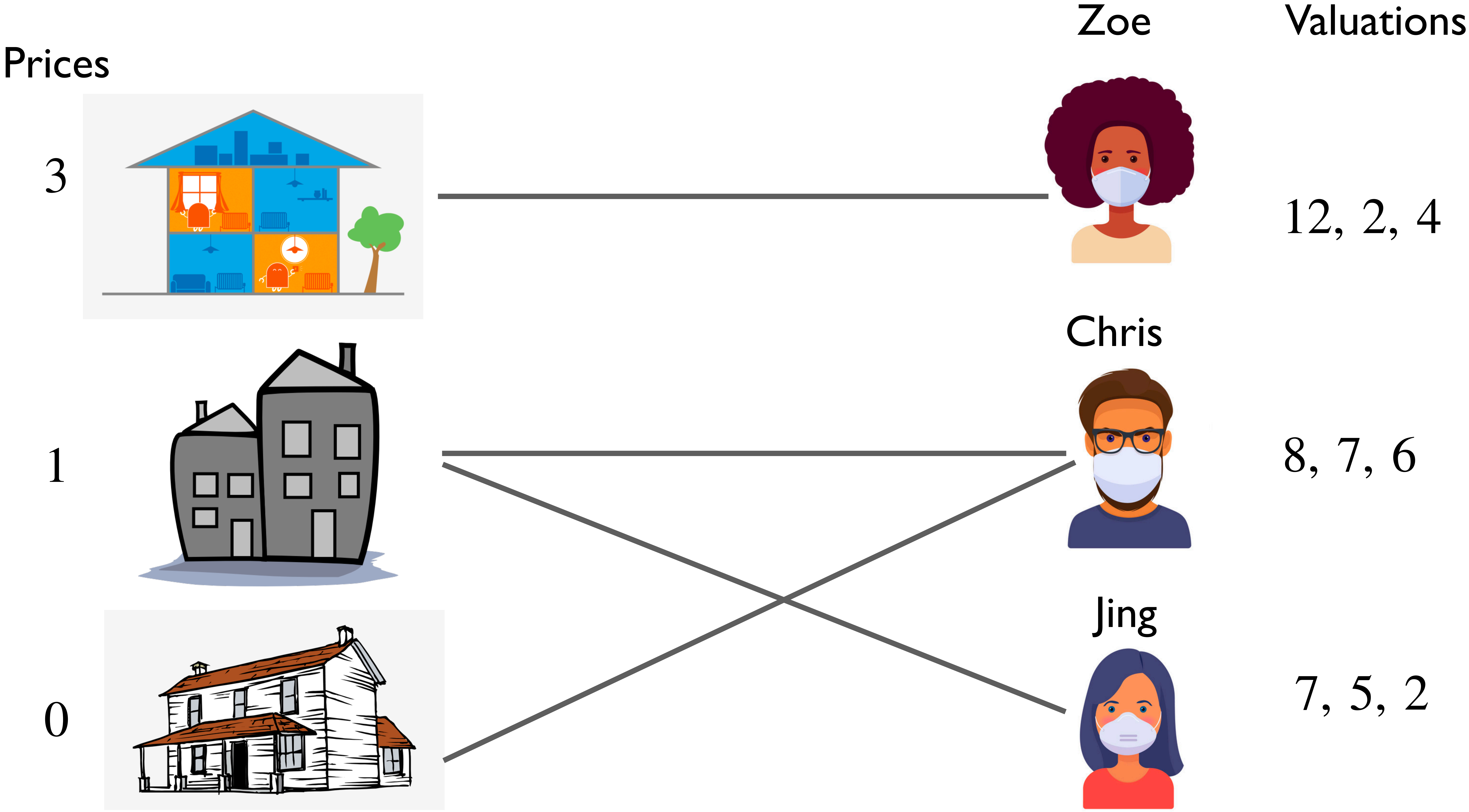

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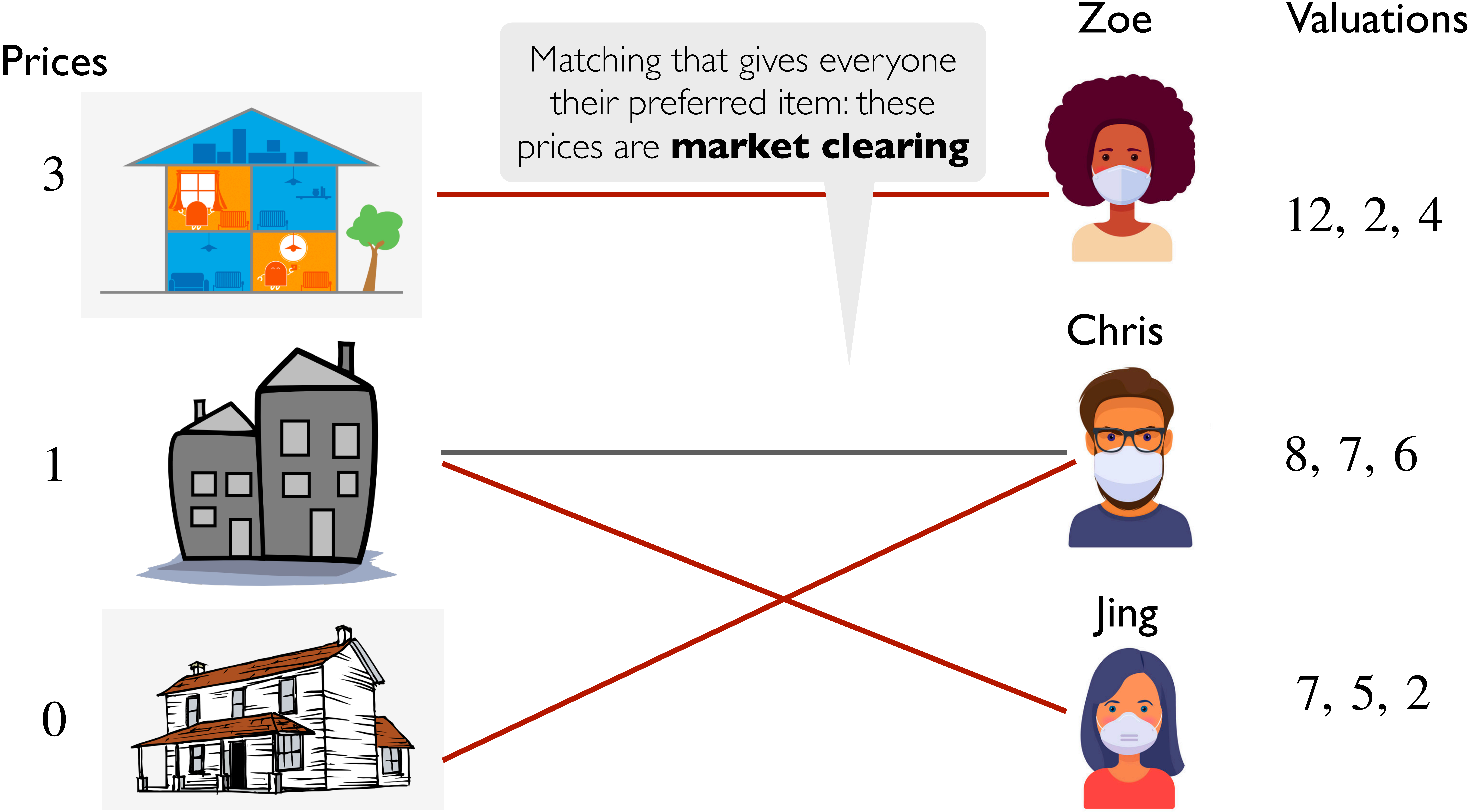


Towards a Proof

- Does this auction ever end?
  - If it ends, we know we have reached market clearing prices
- Can the prices keep rising forever?
  - At some point, the prices are too high for everyone
- **Proof idea**.
  - Maintain invariant: items with nonzero prices are always tentatively matched
  - Show that every buyer has at least one preferred item
  - Show that the price rising process must eventually come to an end by analyzing the "potential energy of the auction"

Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
    - Initially M is empty and all prices are zero: invariant satisfied

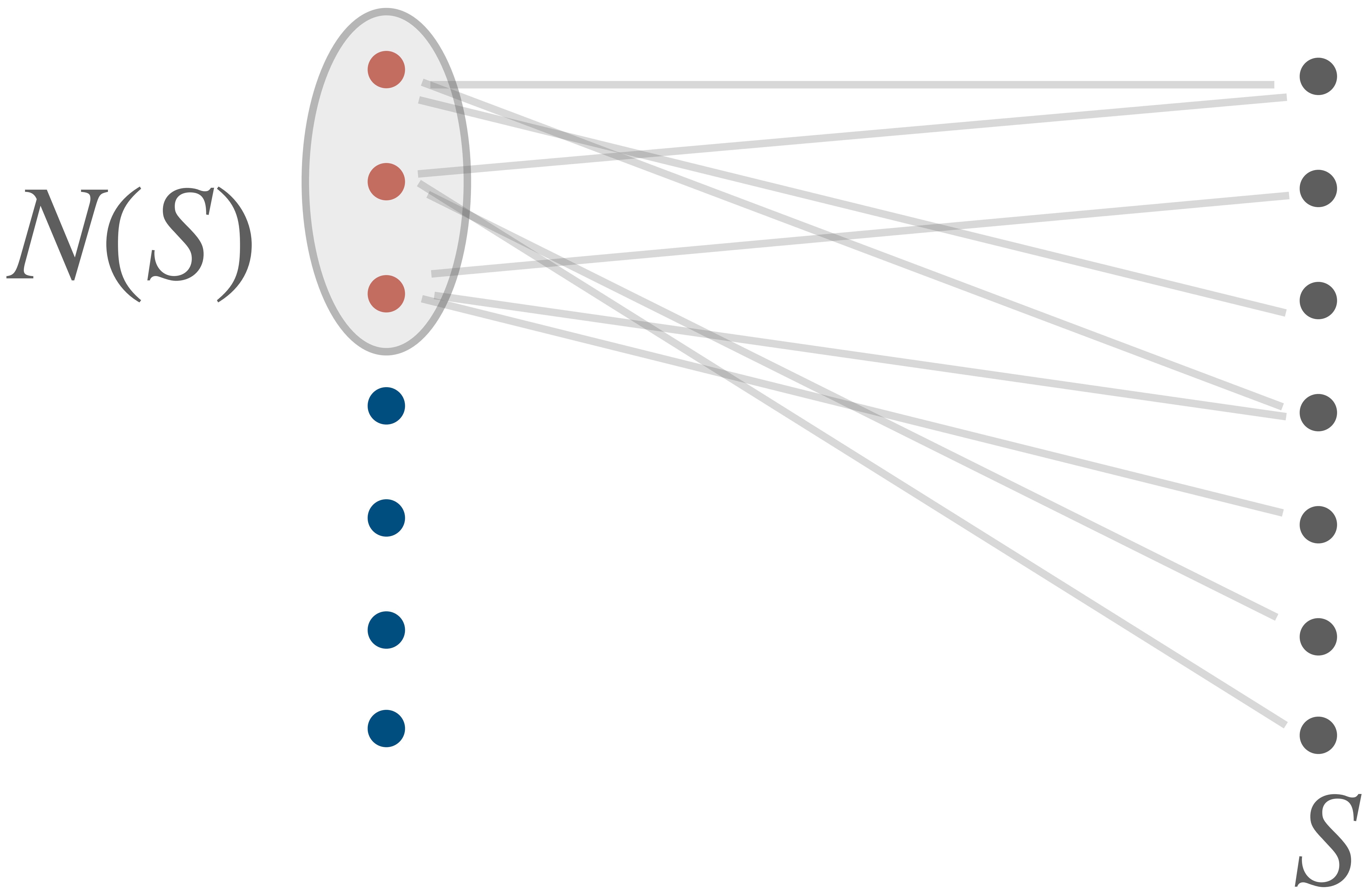
Analyzing Our Auction

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- At the new price, all edges between S to $N(S)$ still exist (buyers in S may have more edges to items outside that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, remove those edges from the matching)
  - Why is this matching possible?
  - We use Hall's theorem on items in $T = N(S)$

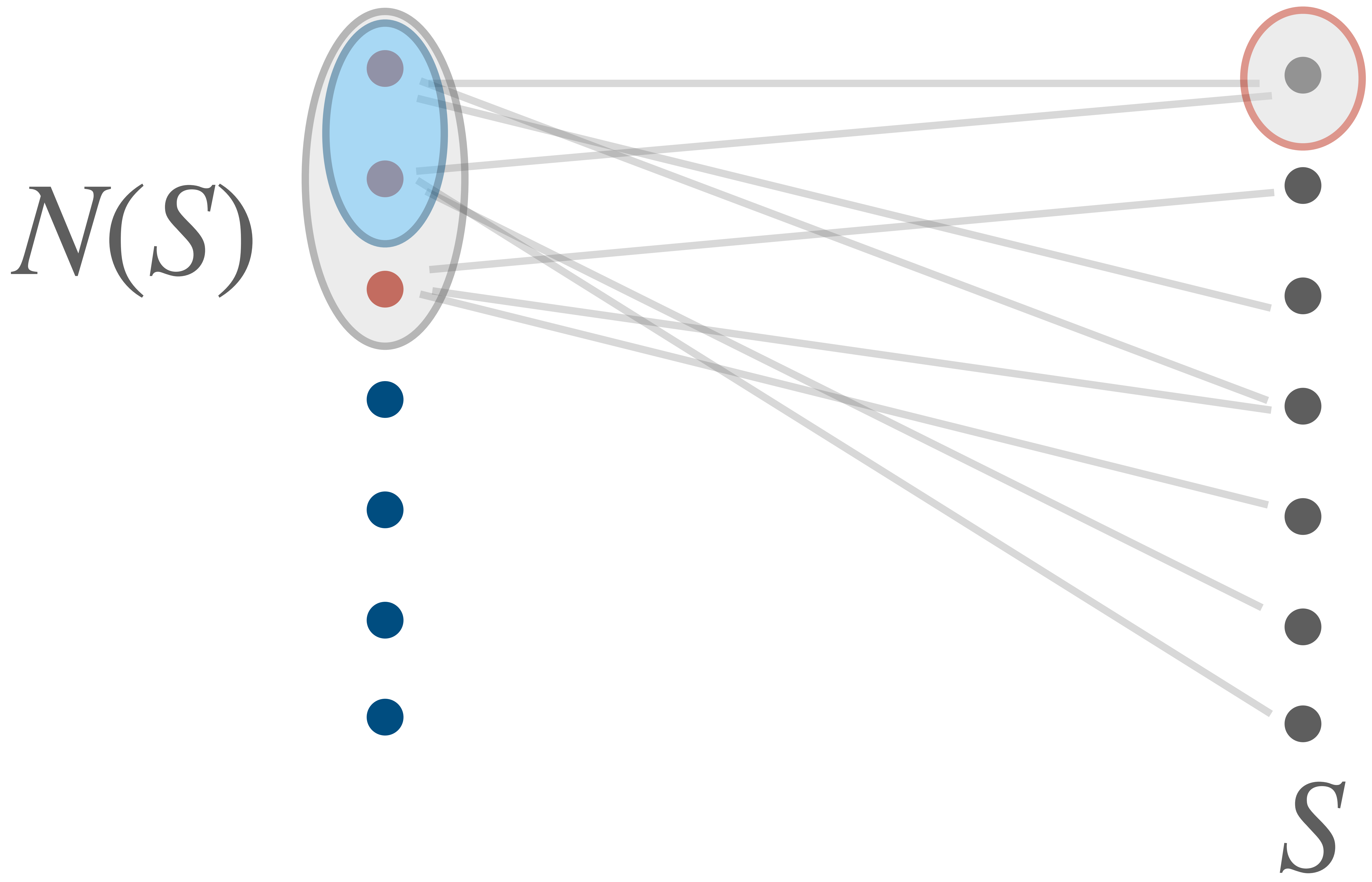
Why Such a Matching Exists

- Let $T = N(S)$ be the minimal constricted set at this step
  - That is no other constricted set is a subset of T
- Hall's theorem says we can match all items in $T = N(S)$ to some buyers in S, as long as there is no subset $T' \subseteq T$ such that
  - $|N(T')| < |T'|$
- We can show that such a subset cannot exist if T is a minimal constricted set
  - By contradiction, suppose such a subset exists
  - Can remove T' from T and $N(T')$ from S and end up with a constricted set that is a subset of T $\Rightarrow \Leftarrow$

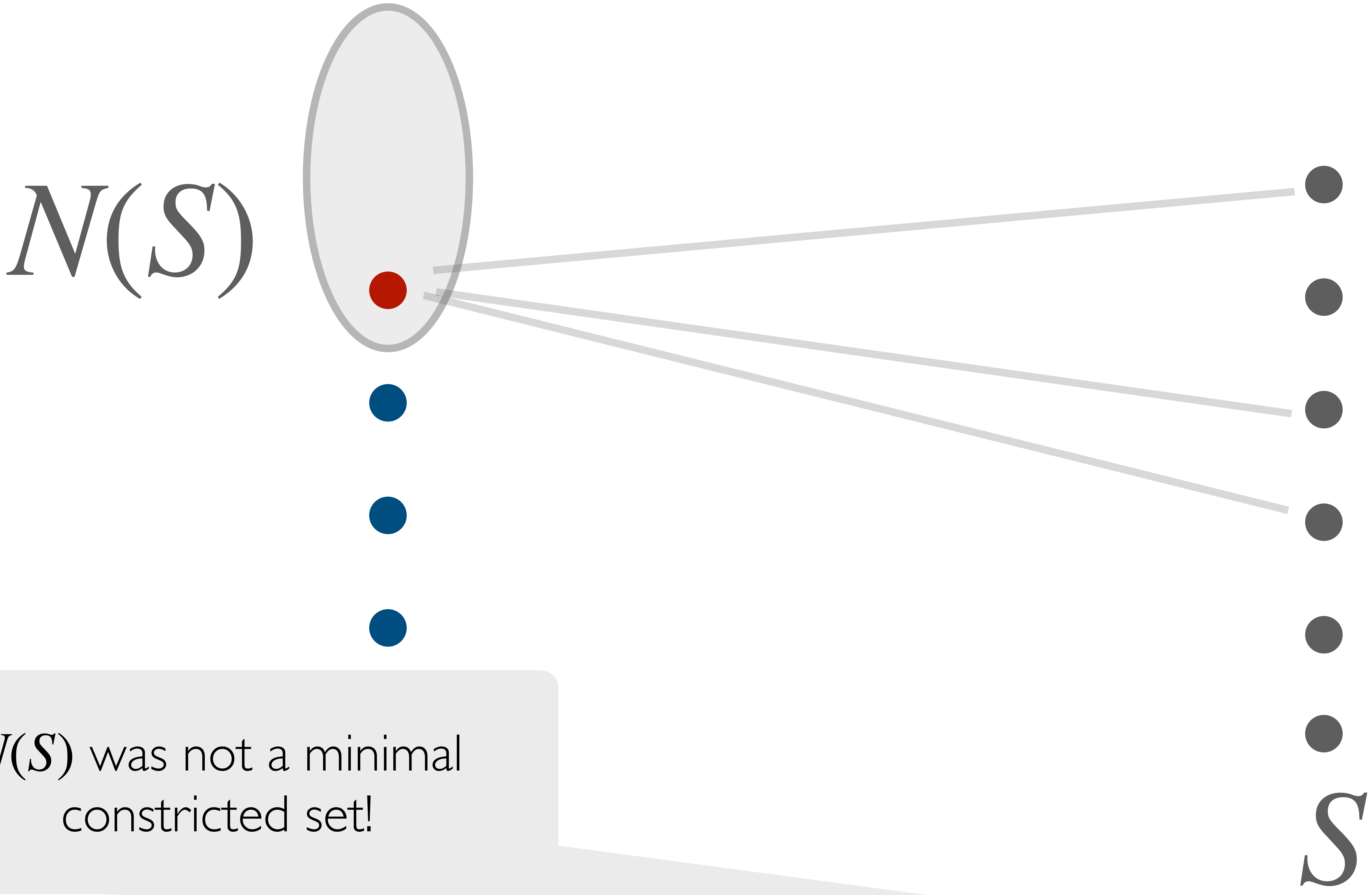
Why Such a Matching Exists



Why Such a Matching Exists



Why Such a Matching Exists



$N(S)$ was not a minimal constricted set!

Maintaining a Matching

- **Maintain invariant:** if an item has non-zero cost, that item is tentatively matched to some buyer: $p_j > 0 \implies \exists i : (j, i) \in M$
- Suppose until step t you have invariant maintained and we identify minimal constricted set $N(S)$ whose prices increase by 1 in this step
- Notice that at this price, all edges between S to $N(S)$ still exist (buyers in S may not have more edges to items outside of edge that are now just as good)
- Tentatively match items in $N(S)$ to buyers in S (if these items were matched to other buyers, or buyers to other items, drop those edges)
- Notice that items outside of $N(S)$ must still be tentatively matched to buyers outside of S (since all neighbors of S are in $N(S)$)
- Thus the invariant is maintained at time t as well ∎

Proving Our Algorithm Terminates

- **Theorem.** The ascending price auction terminates.
- **Proof.** We show that the auction starts with a certain amount of **"potential energy"**, which keeps going down as the auction proceeds, and thus at some point the auction must end when it runs out of potential.
- Let the **potential of an item** j be its price p_j at any round
- Let the **potential of a buyer** i be the maximum utility u_i^* it can obtain from any item at the given prices
- Total potential: $$\Phi = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

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- Total potential: $$\Phi = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$
- Notice that total potential is **always nonnegative** because $p_j \geq 0$ for all j and $u_i^* \geq 0$ for all i

Proving Our Algorithm Terminates

- **Theorem.** The ascending price auction terminates.
- Proof.
- At the the beginning, all prices are zero and $u_i^* = \max_j v_{ij}$
- Thus, before the auctions starts $\Phi_0 = \sum_i \max_j v_{ij}$
- We show that at every time we execute step 2 (raise prices of any constricted set), the potential goes down by at least 1
  - As Φ is always nonnegative and goes down by one each time, means **step 2** can only be executed a most Φ_0 times

Helpful Property

- **Property.** Each buyer has degree at least 1 in the preferred graph at all times
  - If there is always at least one item with zero price then this holds
- **Lemma.** At most $n-1$ items will have strictly positive price
- **Proof.** Say the auction is about to raise the nth item price to >0
- Means that we already have $n-1$ items with nonzero price
- By our invariant these items have been tentatively matched to buyers
  - Thus $n-1$ buyers have already been matched
- nth item can just be matched to last remaining buyer at price 0
- No need for auction to raise its price. ∎

Proving Our Algorithm Terminates

- **Claim.** The total potential Φ goes down by at least one each time we raise prices of a constricted set $N(S)$
- Since each buyer has at least one outgoing edge, $|N(S)| \geq 1$
- When we raise the price of all items in $N(S)$, how much does it increase the potential?
  - $|N(S)|$
- However, the value of u_i^* goes down by one for each node in S, how much does this decrease the potential?
  - $|S|$
- Since $|N(S)| < |S|$, then potential decreases by at least 1
- Thus, the algorithm must terminate in Φ_0 steps ∎

$$\Phi = \sum_{\text{items } j} p_j + \sum_{\text{buyers } i} u_j^*$$

Recap

- Defined market-clearing prices for matching markets:
  - Prices at which we can find a matching between each buyer and their preferred item and such a matching “clears the markets” (every non-zero priced item is sold)
- Called competitive or Walrasian equilibrium
- First welfare theorem says competitive equilibria are always efficient (maximize social welfare)
- Proved that a competitive equilibrium is always guaranteed to exist for the single-item case through an ascending-price auction that terminates in market-clearing prices

Competitive Equilibrium: Caveats

Caveats

- Requires coordination among buyers for tie-breaks

Do Prices Coordinate Markets?

Justin Hsu* Jamie Morgenstern† Ryan Rogers‡ Aaron Roth§ Rakesh Vohra¶

June 23, 2016

Abstract

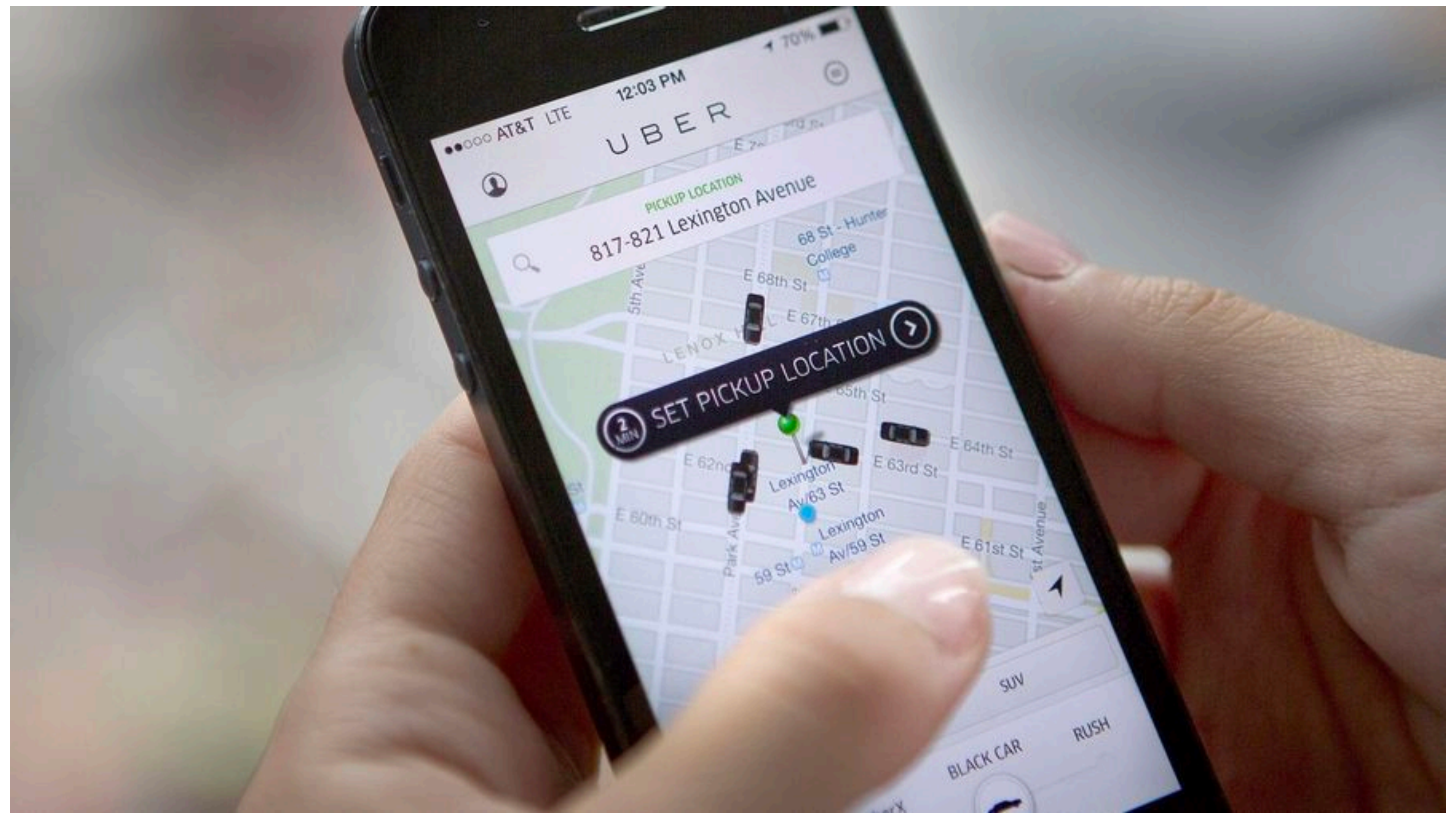
Walrasian equilibrium prices have a remarkable property: they allow each buyer to purchase a bundle of goods that she finds the most desirable, while guaranteeing that the induced allocation over all buyers will globally maximize social welfare. However, this clean story has two caveats: first, the prices may induce indifferences. In fact, the minimal equilibrium prices *necessarily* induce indifferences. Accordingly, buyers may need to coordinate with one another to arrive at a socially optimal outcome—the prices alone are *not* sufficient to coordinate the market; second, although natural procedures converge to Walrasian equilibrium prices on a fixed population, in practice buyers typically observe prices without participating in a price computation process. These prices cannot be perfect Walrasian equilibrium prices, but instead somehow reflect distributional information about the market.

The Myth of the Invisible Auctioneer

- One fundamental assumption when we executed the ascending price mechanism to compute market-clearing prices is:
    - The market does not actually clear until prices have settled at their equilibrium point
- As if an invisible auctioneer is coordinating the prices and lets the market know when the prices have converged and trade can actually take place
- In practice, one might imagine that sales are actually happening concurrently with price adjustment
- Should we still expect prices to converge to a competitive equilibrium?

Fluctuations in Practice

- It turns out, the way buyers and sellers respond to prices in the short-run can dramatically influence market convergence
- **Example**. Surge pricing on ride-sharing platforms can be viewed as an attempt to find market-clearing prices
- However, if passengers and drivers respond to prices myopically, the resulting behavior can be erratic
- These types of price oscillations are typical of scenarios in which sellers respond to price changes
- This is known as the **cobweb model** in economics
    - Prices slowly spiral around the supply & demand curves



General Matching Markets

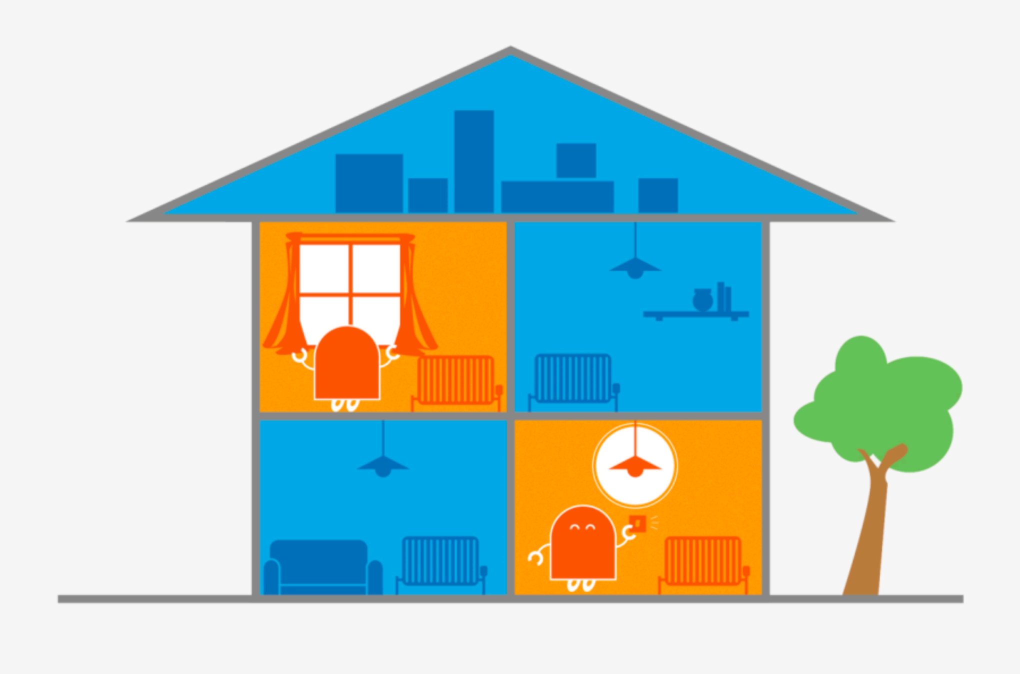
- Market clearing prices **may not exist in general markets**
- **Example**, suppose our market has two items $\{L, R\}$
- Two buyers Alice and Maya
- Alice wants both $v_a(\{L, R\}) = 5$, $v_a(\{L\}) = v_s(\{R\}) = 0$
- Maya wants either, $v_p(\{L\}) = v_p(\{R\}) = v_p(\{L, R\}) = 3$
- What's the welfare-maximizing allocation?
  - Give both to Alice
- What must the price of each be so that Maya doesn't want it?
  - $p(\{L\}) \geq 3, p(\{R\}) \geq 3$
- At a price of ≥ 6 does Alice want it?

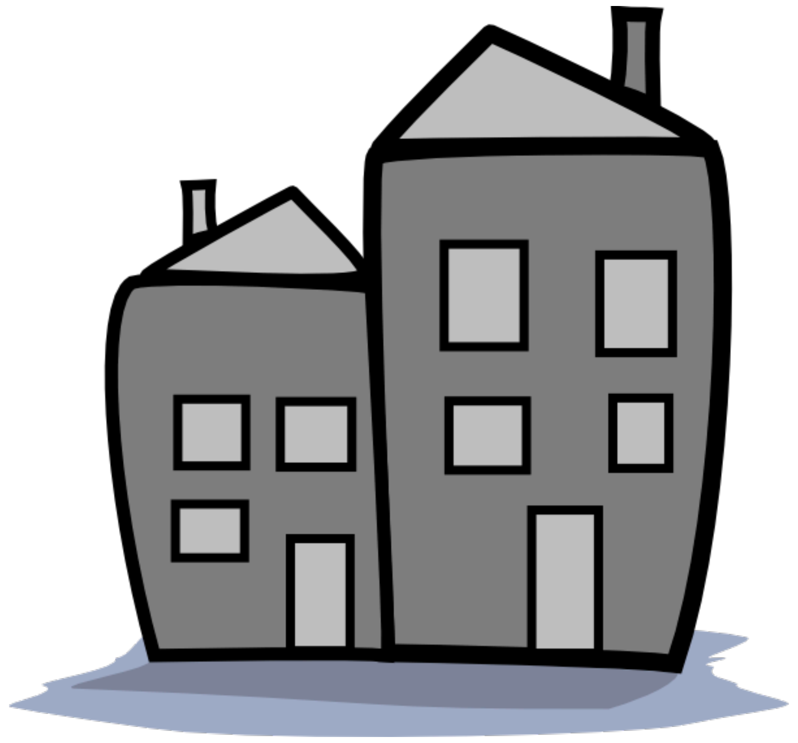
VCG Prices vs Market-Clearing

- VCG prices set **centrally**: ask each buyer to report their valuation and charge each buyer a **"personalized price"** for their allocation
- VCG prices are only set after a matching has been determined (the matching that maximizes total valuation of the buyers)
  - Not just about the item itself, but who gets the item
- Market-clearing prices are **"posted prices"** at which buyers are free to pick whatever item they like
  - prices are chosen first and posted on the item
  - These prices cause certain buyers to select certain items leading to a matching

Preferred-Item Graph

Prices



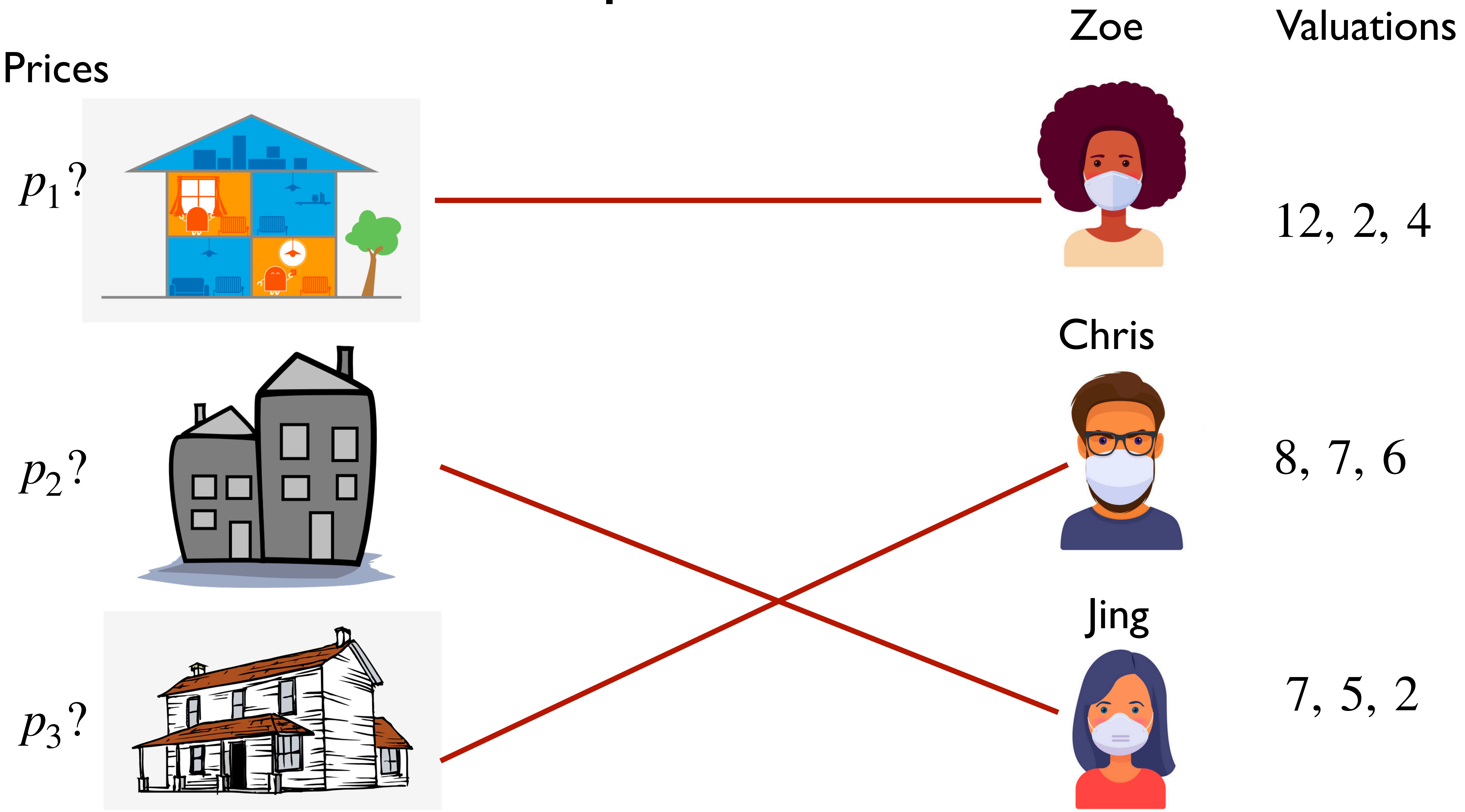


VCG. Need to find surplus maximizing allocation first

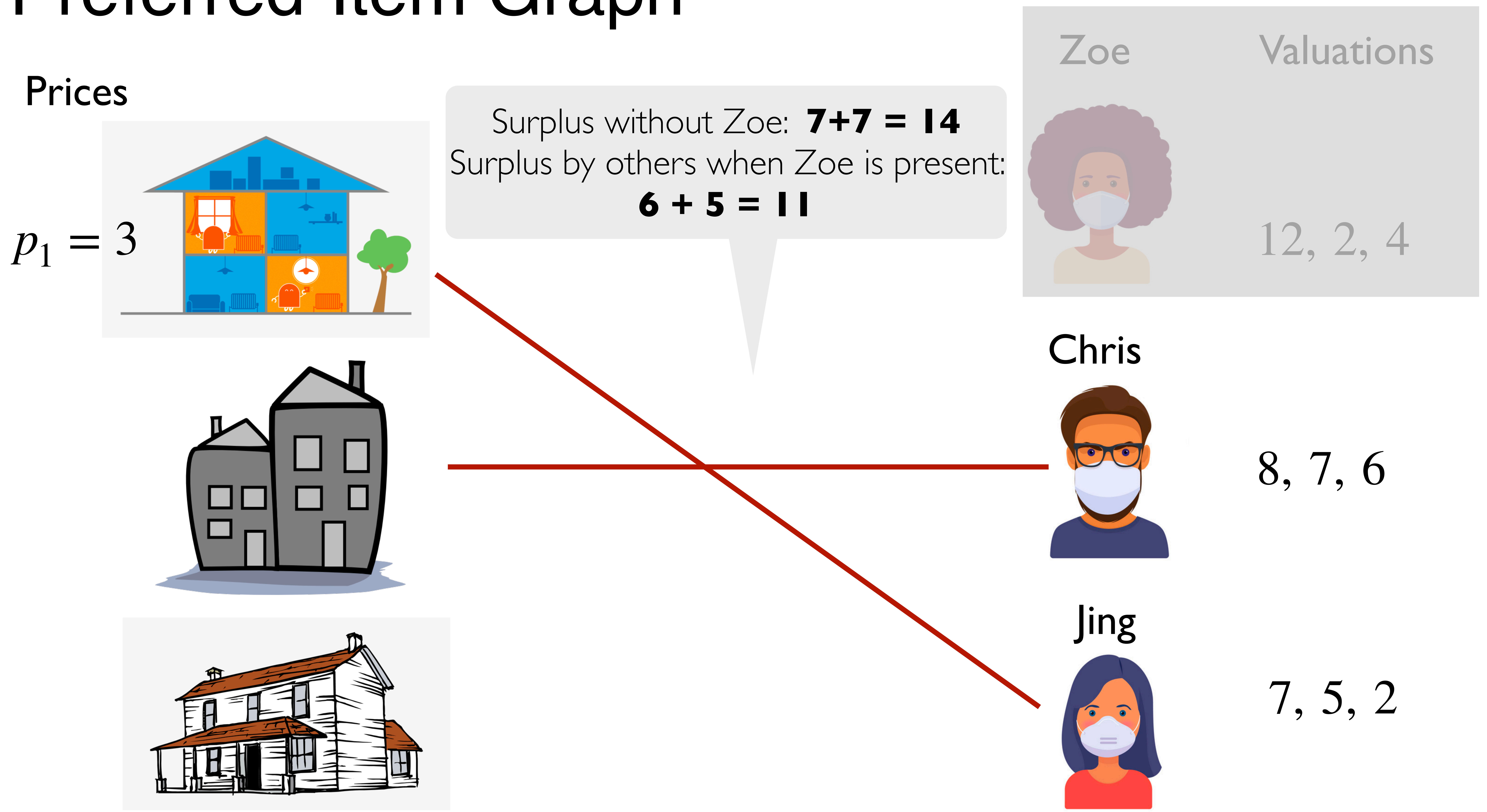
Zoe		12, 2, 4
Chris		8, 7, 6
Jing		7, 5, 2

Valuations

Preferred-Item Graph



Preferred-Item Graph



Preferred-Item Graph

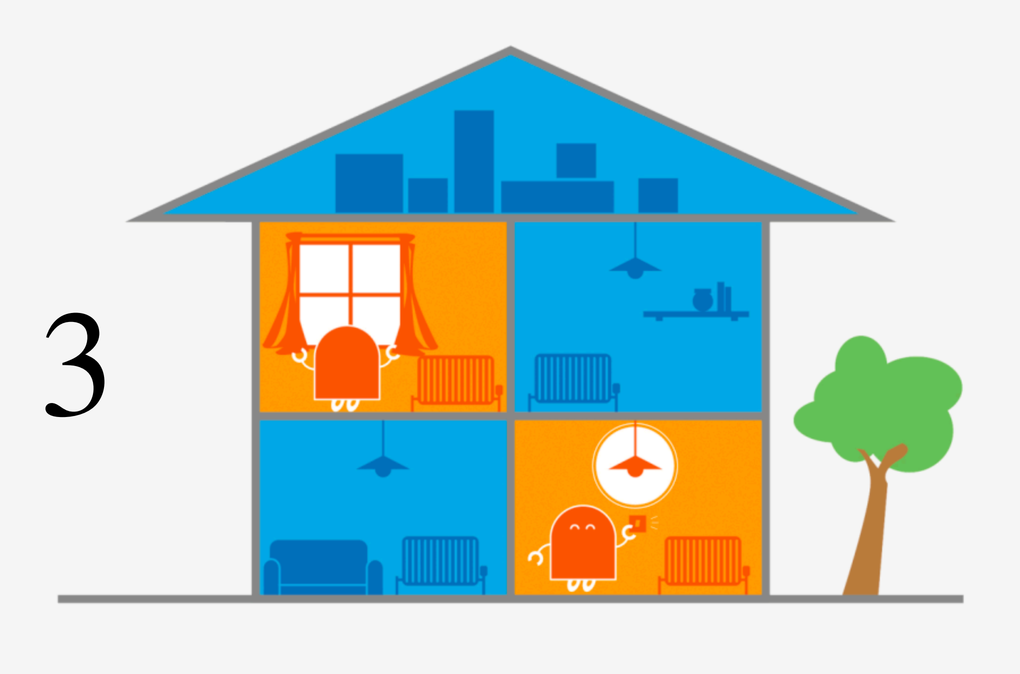
Prices

Zoe

Valuations

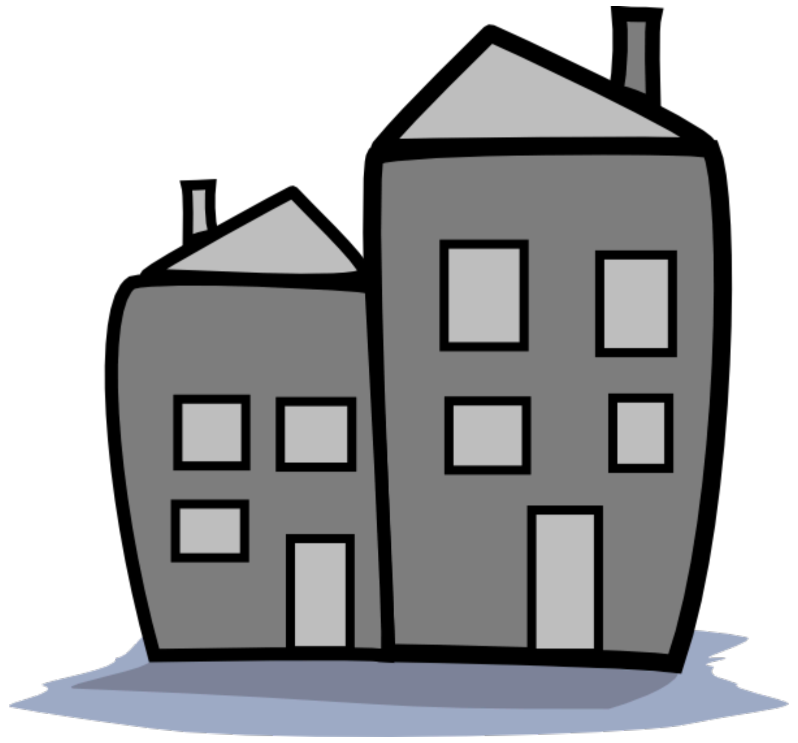
Surplus without Chris: **12+5 = 17**
Surplus by others when Chris is present: **12+5 = 17**

$p_1 = 3$



12, 2, 4

Chris



8, 7, 6

Jing

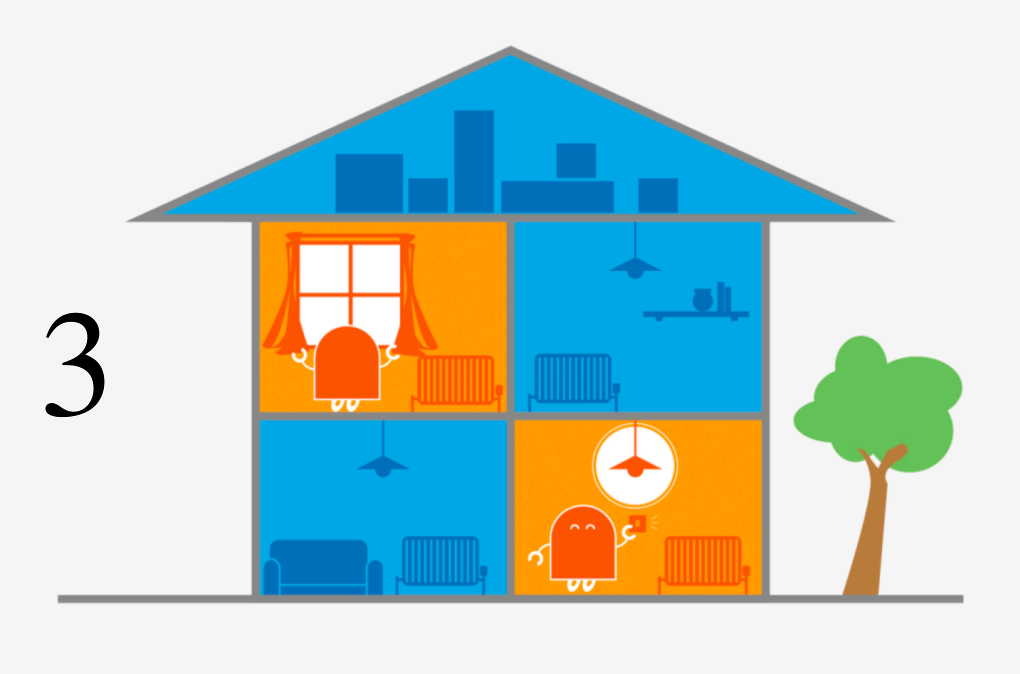
$p_3 = 0$

7, 5, 2

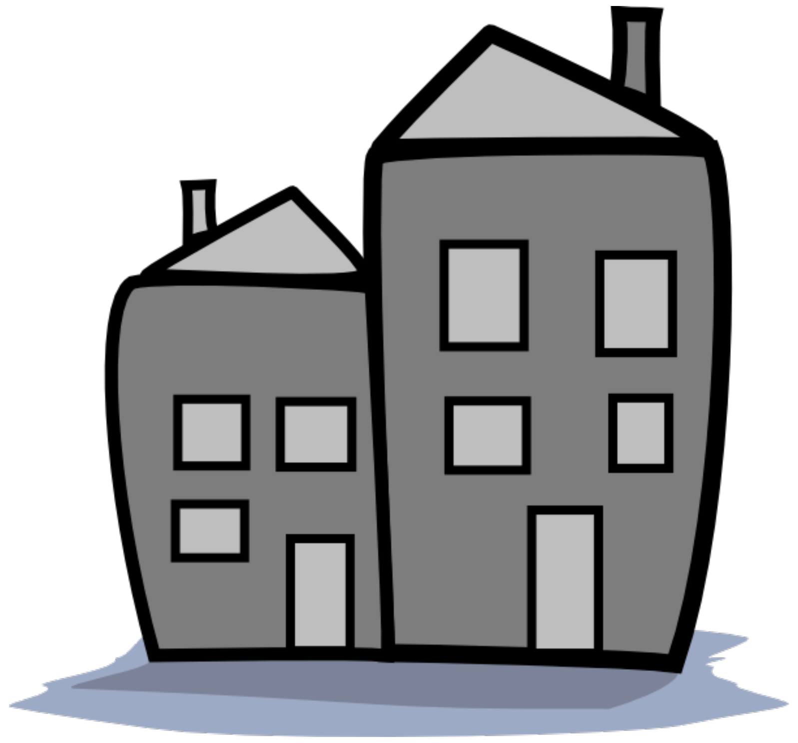
Preferred-Item Graph

Prices

$p_1 = 3$



$p_2 = 1$



$p_3 = 0$

Surplus without Jing: **12+7 = 19**
Surplus by others when Jing is present:
12+6 = 18

Zoe

Valuations

12, 2, 4

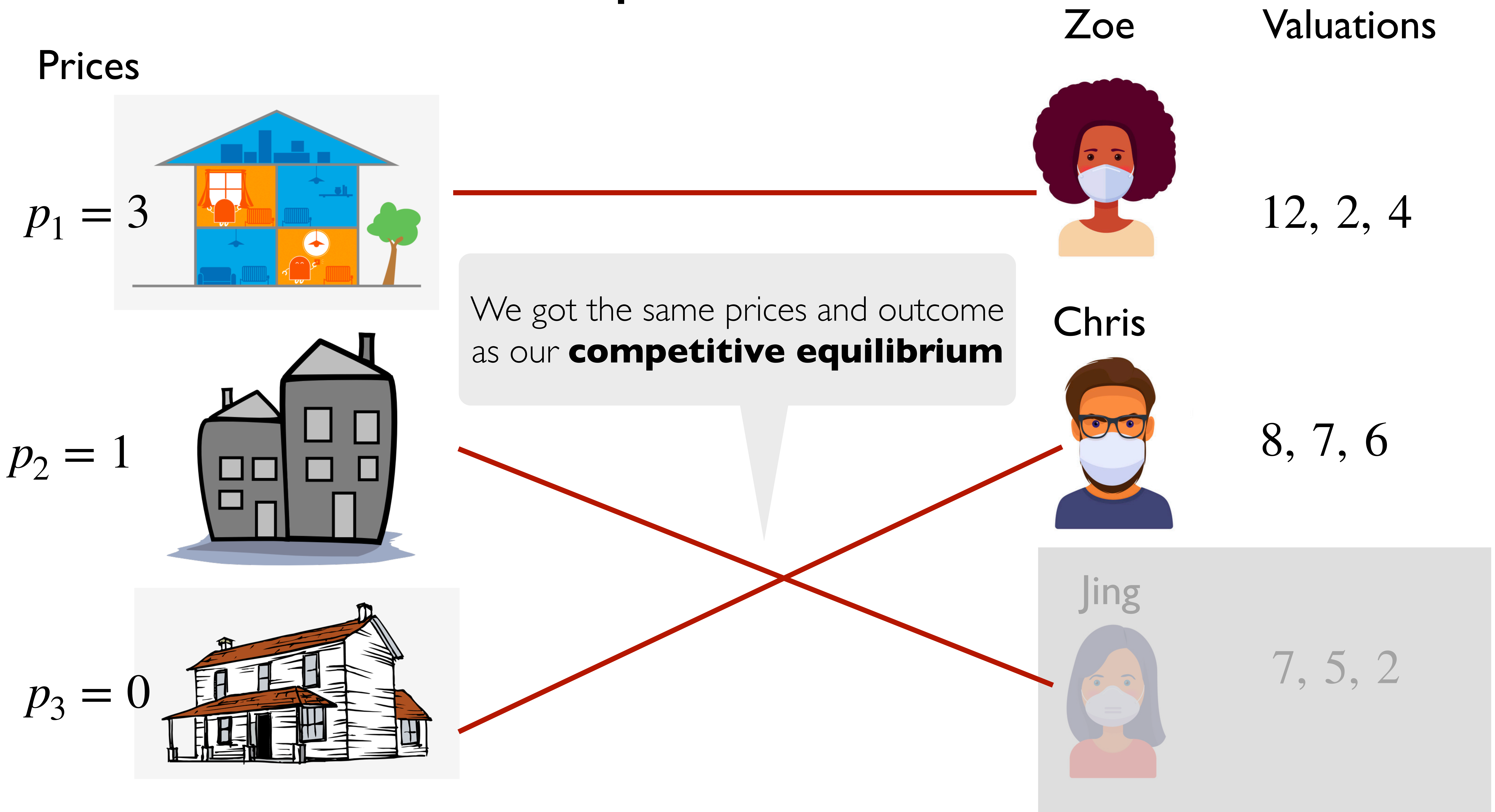
Chris

8, 7, 6

Jing

7, 5, 2

Preferred-Item Graph



VCG Prices are Market Clearing

- Despite their definition as personalized prices, VCG prices are always market clearing (for the case when each buyer wants a single item)
- Suppose we computed VCG prices for a given matching market
- Then, instead of assigning the VCG allocation and charging the price, we post the prices publicly
    - Without requiring buyers to follow the VCG match
- Despite this freedom, each buyer will in fact achieve the highest utility by selecting the item that was allocated by the VCG mechanism!
- **Claim.** In any matching market (where each buyer can receive a single item) the VCG prices form the unique set of **market clearing prices of minimum total sum**.

Since **GSP** prices are identical to VCG at a balanced-bidding envy-free equilibrium, they are also market clearing