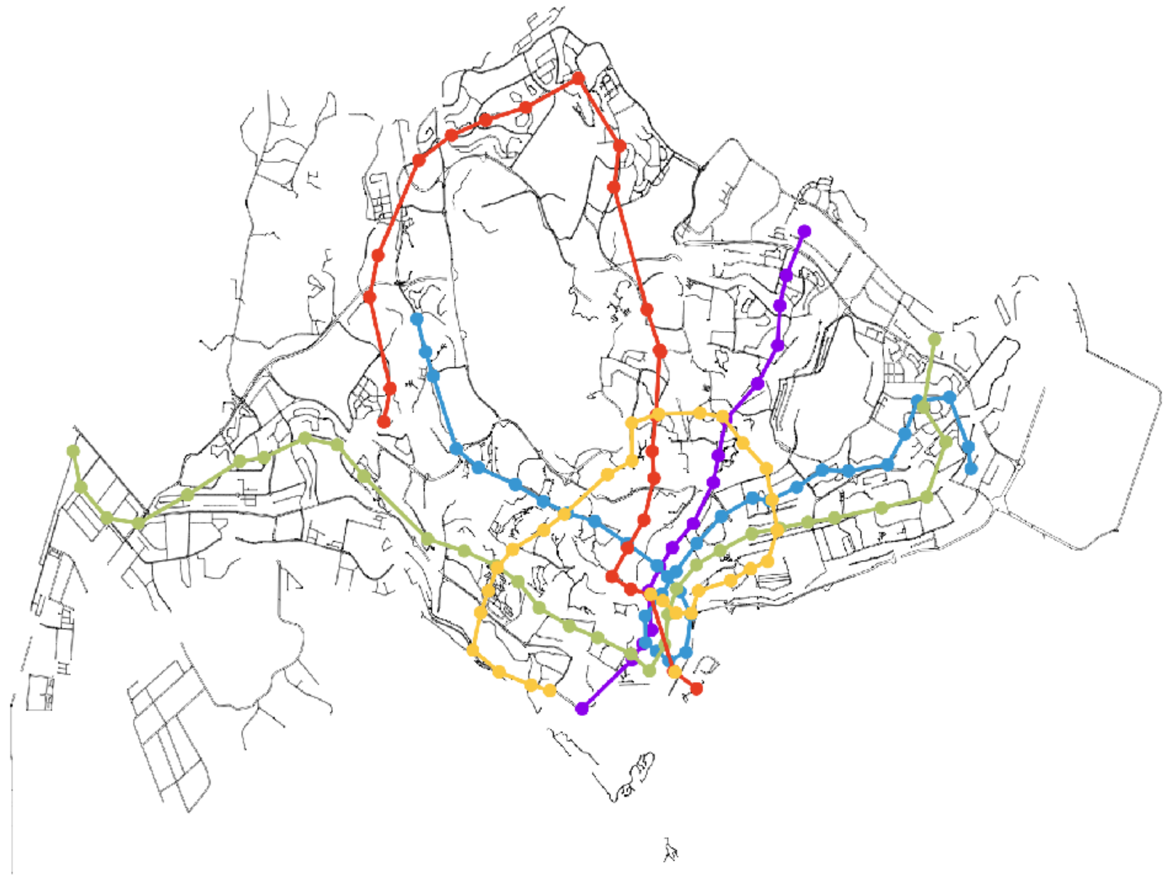
CSCI 357: Algorithmic Game Theory

Lecture 6: Sponsored Ad Markets (Theory vs Practice)

Shikha Singh



Announcements and Logistics

- **Assignment 3** due by **11 pm Thursday**
- **Partner work** guidelines are similar to pair programming: solve problems logically together, one person drives writing with alternation
- 2 questions both about surplus maximization and Myerson's payments
    - Intuition about critical bids/ approximation approach of AGT
  - Change in **Wed office hours** for next two weeks
- If you cannot make it to office hours, just reach out on Slack
- Workload distribution and learning in this class
  - Goal with partner assignments

Questions?

Announcements and Logistics

- **Assignment 2:** feedback will be returned later this week
- **Assignment 4** will also be a **partner assignment**
  - Simulation assignment to understand sponsored ad auctions
  - Need your Github IDs to share starter code: watch out for google form
- Not too many lines of code: but need to understand the simulation infrastructure
  - Goal: simulate simple auctions and do empirical analysis of how revenue/utility of agents
  - How do strategic agents reach eqm in a repeated auction setting?

Questions?

Instructor Masking in Lectures

"Starting next Monday, February 21, instructors may unmask in class if they believe it will enhance instruction and if they maintain a safe distance ."

I would like your feedback: **https://tinyurl.com/357mask** (please fill it before next class)

Midterm 1: Save the Date March 12

- Pencil-paper exam, ~3 ish hours
- Can take it any time on (Sat) **March 12**
  - You have to be on campus
  - Pick up the exam from my office and return there
- If you have a conflict please reach out asap
- Open book, open notes
- Reasoning behind format
  - Goal to not be memory or time constrained

Questions?

HW 2 Rewind:

- Sealed-bid SP vs ascending clock auction (w public/private drop out)
  - In both cases, truthfulness is dominant strategy
  - Pros and cons?
- In public dropout, there is more transparency compared to sealed bid
  - Sealed bid requires everyone to trust the auctioneer
- Ascending auctions do not reveal value of winning bidder to others
- Extra information in public dropout can be used by bidders
  - Can help them learn the price/value of others but that can sometimes lead to undesirable competitive behavior

Last Time

- Myerson's lemma for single parameter settings:
  - Says allocation can be made DSIC iff it is monotone
  - Gives unique DSIC payment rule

$$p_i(z, \mathbf{b_{-i}}) = z \cdot x_i(z, \ \mathbf{b_{-i}}) - \int_0^z x_i(z, \ \mathbf{b_{-i}}) \ dz$$

- We applied this rule to sponsored search auctions and derived the theoretical DSIC payment rule

Recursive definition might help think about it!

Total payment: $$p_i(\mathbf{b}) = \sum_{j=i}^{k} \left(b_{j+1} \cdot (\alpha_j - \alpha_{j+1}) \right) = b_{i+1}(\alpha_i - \alpha_{i+1}) + p_{i+1}(\mathbf{b})$$

Today: Theory vs Practice

- What happens in the practice of sponsored search auctions
    - Rich history
    - Theory has often predicted behavior in practice pretty well
- Downsides of DSIC payment rules given by Myerson?
    - Can be complicated and hard to explain
    - Can be computationally expensive (as you will see in HW 3)
- Do we even need them? Does strategic bidding actually take place?
    - If so, how bad is it?
- Today: what happens in sponsored ad auctions in practice

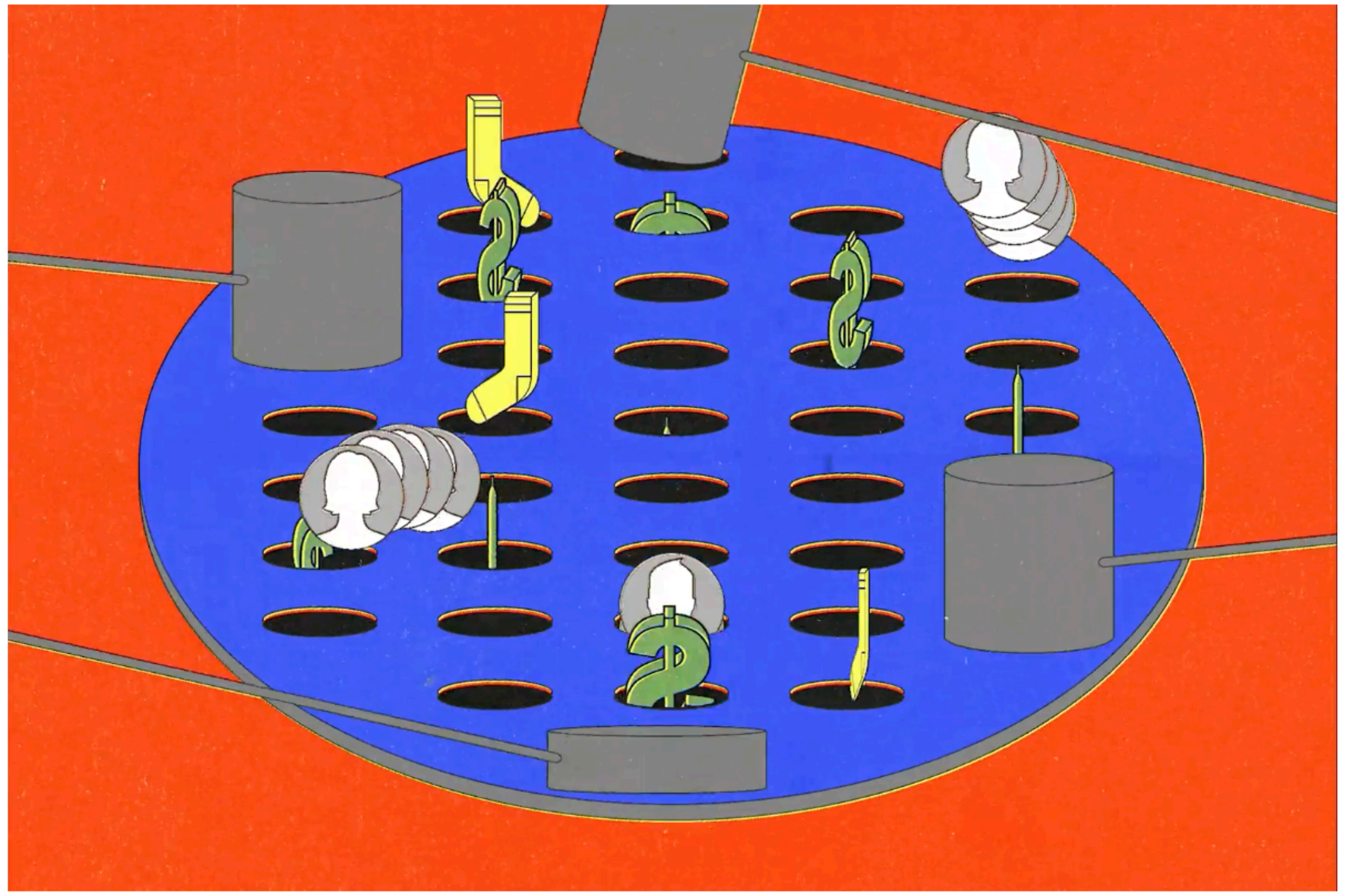
Why Study Digital Ad Markets

The New York Times

THE ON TECH NEWSLETTER

Google and Facebook's Ad Empires

The tech giants talk a lot about the "metaverse" and cloud computing. What really powers them is selling us socks.



By Rapapawn

Google, Facebook, and Amazon to account for 64% of US digital ad spending this year

Article by Sara Lebow | Nov 3, 2021

"AdSense counts more than ***2 million content publishers*** *as customers. Approved publishers can enter their Google code onto their sites or videos, and advertisers bid to show up* ***in those ad slots in auctions****. If a publisher's content displays an ad through AdSense, that publisher receives 68% of the revenue recognized by Google in connection with that service."*

https://www.cnbc.com/2021/05/18/how-does-google-make-money-advertising-business-breakdown-.html

Sponsored Search Auction

- We have k ad slots and n bidders (advertisers)
- Each add slot i has a **"click through rate"** α_i that models the number of clicks generated by the slot
- We assume that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$
- Bidders have a **private value** v_i per click
    - Bidder i's value for being assigned slot j is $v_i \cdot \alpha_j$
- Each bidder submits a "bid-per-click" (amount they are willing to pay per click)
- Auction collects the bid profile $\mathbf{b} = (b_1, \ldots, b_n)$ and outputs an allocation rule $\mathbf{x}(\mathbf{b})$ (who gets what slot) and payment rule $\mathbf{p}(\mathbf{b})$ (who pays what)

Truthful Mechanism (VCG)

- Also called the Vickrey-Charles-Grove or VCG mechanism
- **Surplus maximizing allocation:**
  - Rank bidders by the bid value $b_1 \geq b_2 \geq \ldots \geq b_n$
  - Assign slot i to bidder i, where $1 \leq i \leq k$
- **Payment rule**:
  - We have derived this using Myerson's lemma
  - Total payment of bidder i who is assigned slot i is $\sum_{j=i}^{k} \left(b_{j+1} \cdot (\alpha_j - \alpha_{j+1}) \right)$
  - Per click payment of bidder i in slot i is thus $\sum_{j=i}^{k} \left(\frac{b_{j+1}}{\alpha_i} \cdot (\alpha_j - \alpha_{j+1}) \right)$

VCG Payments Example

$$\frac{1}{\alpha_i}\sum_{j=i}^{k}\left(b_{j+1}\cdot(\alpha_j-\alpha_{j+1})\right)$$

- Suppose **bidders bid truthfully,** what should they pay?
- Would bidder 1 have an incentive to underbid and obtain the second slot at a lower price?

	Value per click v_i	Payment per click p_i	$\alpha_i(v_i - p_i)$
$\alpha_1 = 0.1$	$v_1 = 10$	$p_1 = 6.5$	$u_1 = 0.35$
$\alpha_2 = 0.05$	$v_2 = 9$	$p_2 = 4$	$u_2 = 0.25$
	$v_3 = 4$	$p_3 = 0$	$u_3 = 0$

VCG Payments Example

$$\frac{1}{\alpha_i}\sum_{j=i}^{k}\left(b_{j+1}\cdot(\alpha_j-\alpha_{j+1})\right)$$

- $p_1 = \frac{1}{0.1}\left(9(0.1 - 0.05) + 4(0.05)\right)$
- $p_2 = \frac{1}{0.05}\left(4(0.05 - 0)\right)$

	Value per click v_i	Payment per click p_i	$\alpha_i(v_i - p_i)$
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VCG Payments Example

- $u_1 = 10 \cdot 0.1 - (9(0.1 - 0.05) + 4(0.05)) \quad = 0.35$
- Would bidder 1 have an incentive to underbid and obtain the second slot at a lower price?

	Value per click v_i	Payment per click p_i	$\alpha_i(v_i - p_i)$
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VCG Payments Example

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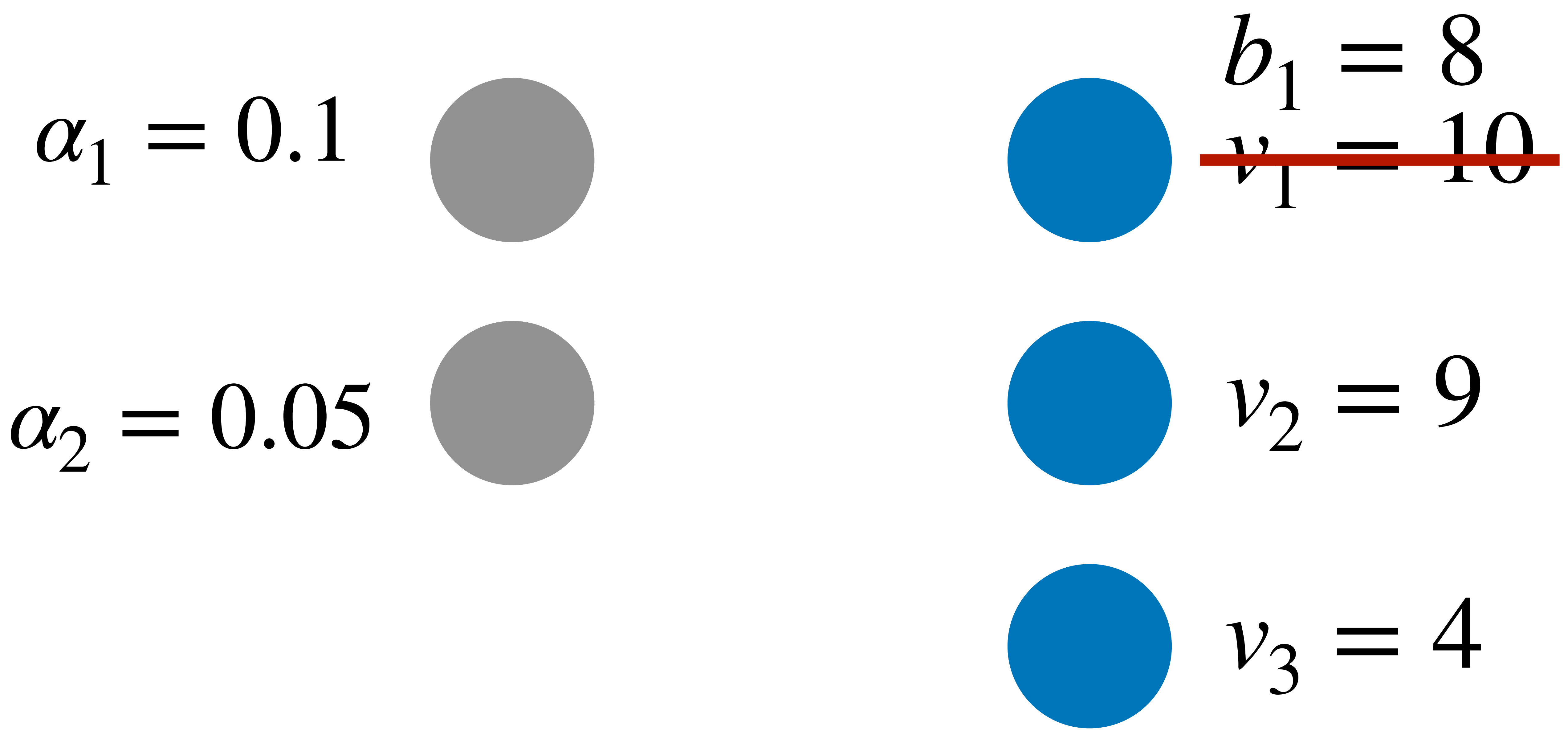
VCG is Strategyproof

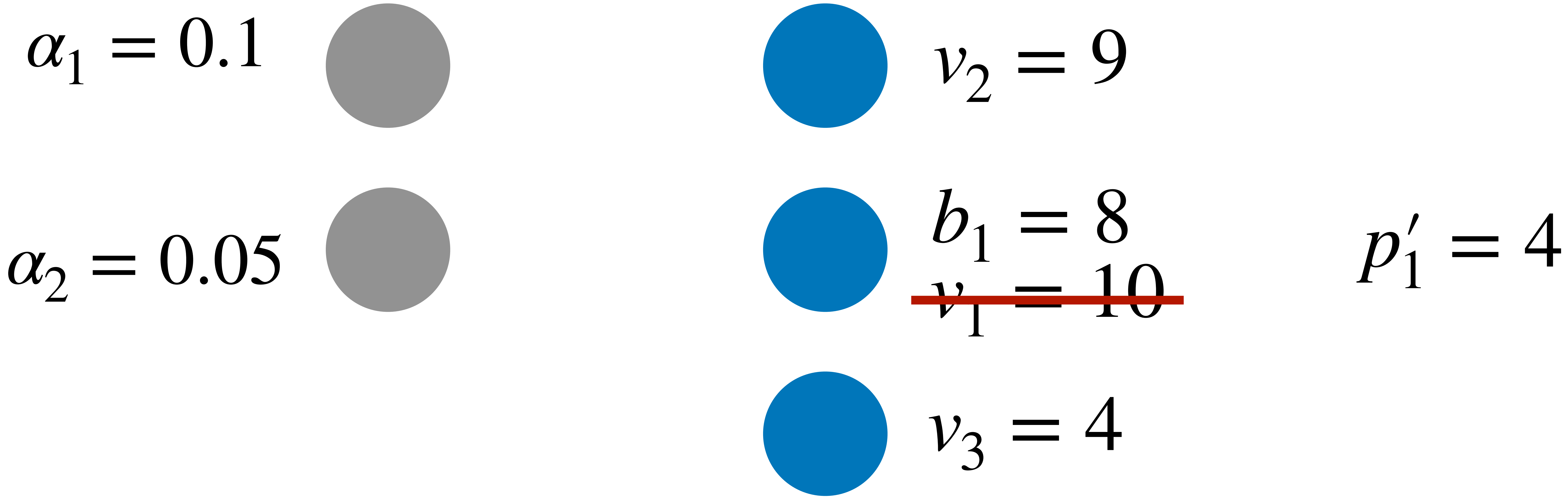
- $u_1 = 10 \cdot 0.1 - (9(0.1 - 0.05) + 4(0.05)) \;\; = 0.35$
- Would bidder 1 have an incentive to underbid and obtain the second slot at a lower price?



VCG is Strategyproof

- $p_1' = \frac{1}{0.05}\big(4(0.05)\big) = 4$
- $u_1' = 10 \cdot 0.05 - 4(0.05)$
- $u_1 - u_1' = 10(0.1 - 0.05) - 9(0.1 - 0.05) \geq 0$

Generalizes to $(v_1 - v_2)(\alpha_1 - \alpha_2)$ which is always positive: intuition why VCG mechanism is DSIC



Sponsored Search Practice

- Auction is run every second (rather than once)
- Bidders can adjust their bids based on history, time of the day etc

Sponsored Search: Practice

- Auction is run every second (rather than once)
- Bidders can adjust their bids based on history, time of the day etc
- In late 90s, Yahoo was the main search engine with ads
    - Used first-price auctions
    - We will analyze first-price auctions in next lecture
    - But what do you think bidders would do in such a "repeated auction"

Strategic bidder behavior in sponsored search auctions

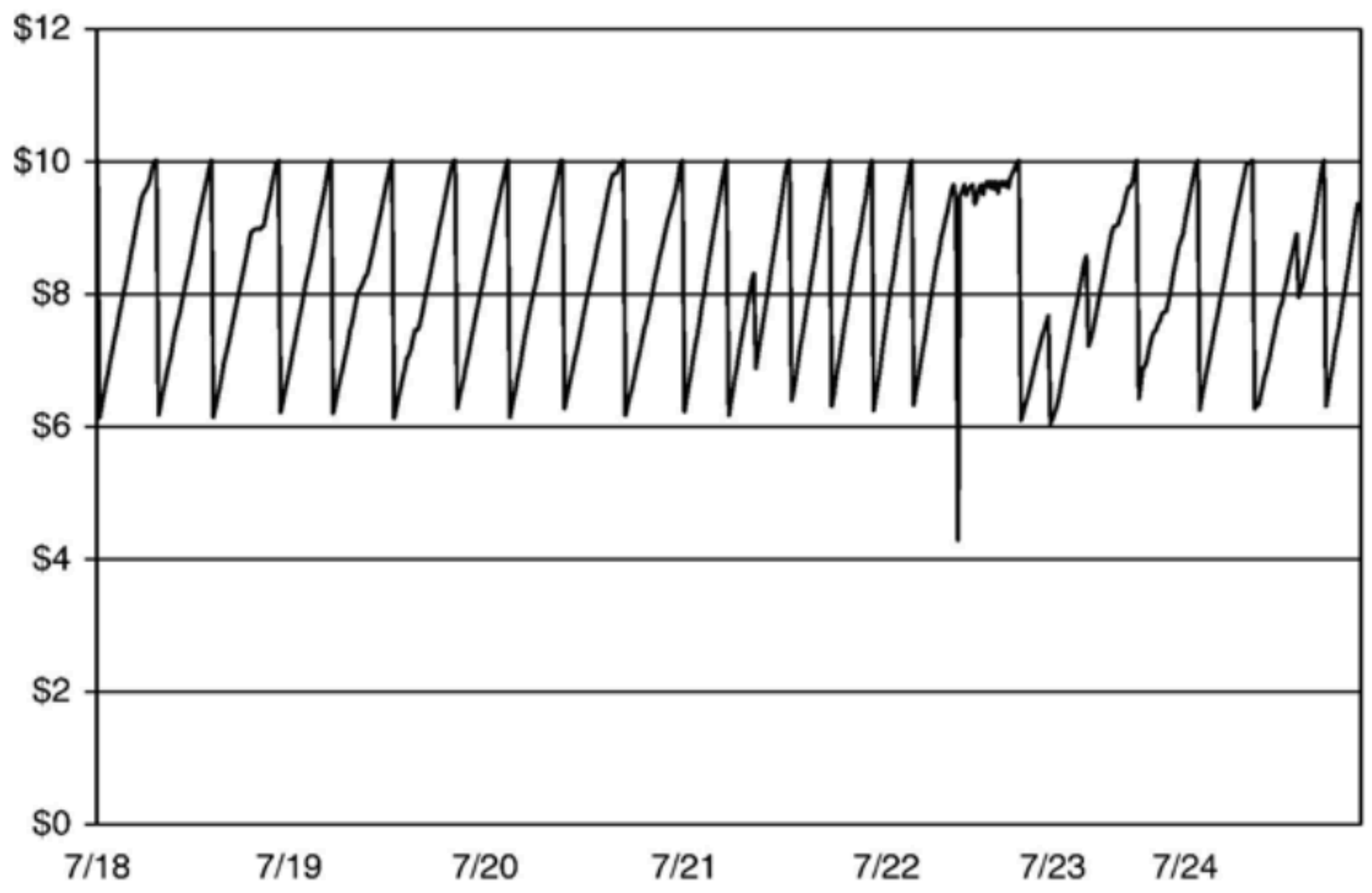
Benjamin Edelman [a], Michael Ostrovsky [b,*]

First Price vs Second Price?

- What are some pros/cons of each format?
- What is the benefit of first-price auction?
- What is a downside of second price?
    - Even if it is public ascending clock?

Sponsored Search: Practice

- Authors looked at data and observed sawtooth bidding pattern



(b) 1 week

Strategic bidder behavior in sponsored search auctions

Benjamin Edelman [a], Michael Ostrovsky [b,*]

Generalized Second Price Auction

- Developed by Google in 2002 in response to the bidding wars
- Attempt to enforce truthful behavior by generalizing Vickrey (second price auctions)
- Collect bids-per-click $\mathbf{b} = (b_1, \ldots, b_n)$
- GSP allocation rule: same as VCG
  - Order and reindex such that $b_1 \geq b_2 \geq \ldots \geq b_n$
  - Assign slot i to bidder i, where $1 \leq i \leq k$
- The price-per-click for bidder who is assigned slot i:
  - **"Critical bid":** the minimum amount they could have bid to obtain slot i
  - This is just b_{i+1} the bid of the person below i
- For a single slot this is truthful, but what if we have more than one slot?

GSP is Not Strategyproof

- In a GSP auction, it is not the dominant strategy of bidders to bid their true value-per-click
- **Idea.** Incentive to underbid to acquire fewer clicks at a reduced price.
- Consider the following example
    - Say bidders 2 & 3 bid truthful, does bidder 1 have a useful deviation?

$\alpha_1 = 0.1$

$\alpha_2 = 0.05$

$v_1 = 10$

$v_2 = 9$

$v_3 = 4$

GSP is Not Strategyproof

- In a GSP auction, it is not the dominant strategy of bidders to bid their true value-per-click
- Suppose bidder 2 and 3 are truthful
- What are the prices of each bidder?
- What is the current utility of bidder 1?

$\alpha_1 = 0.1$ $\quad v_1 = 10$ $\quad p_1 = 9$ $\quad u_1 = 0.1$

$\alpha_2 = 0.05$ $\quad v_2 = 9$ $\quad p_2 = 4$

$v_3 = 4$ $\quad p_3 = 0$

GSP is Not Strategyproof

- In a GSP auction, it is not the dominant strategy of bidders to bid their true value-per-click
- Suppose bidder 2 and 3 are truthful, does bidder 1 have a useful deviation?

$\alpha_1 = 0.1$

$\alpha_2 = 0.05$

$b_1 = 8$

~~$v_1 = 10$~~

$v_2 = 9$

$v_3 = 4$

GSP is Not Strategyproof

- In a GSP auction, it is not the dominant strategy of bidders to bid their true value-per-click
- Suppose bidder 2 and 3 are truthful, does bidder 1 have a useful deviation?

Being untruthful improves utility!

$\alpha_1 = 0.1$ | $v_2 = 9$ | $p_2 = 8$

$\alpha_2 = 0.05$ | $b_1 = 8$ ~~$v_1 = 10$~~ | $p_1' = 4$ | $u_1' = 0.05(10 - 4) = 0.3$

$v_3 = 4$ | $p_3 = 0$

Analyzing GSP's Equilibrium

- Not being strategyproof, makes GSP more difficult to analyze
- Its properties have been extensively studied computer scientists and economists (Varian, Edelman, Ostrovsky, and Schwarz)
- They formulate the auction as a complete-information game and make the following simplifying assumption:
  - Bidders know each other's value **(Complete-information game)**
  - Argument in favor: reasonable if bidders can observe patterns of bidding behavior of market competitors
  - Search engines often provide "market data", price points, etc. sc reasonable for advertisers to learn the market
- Easier to analyze using **Nash equilibrium**

ALL MODELS ARE WRONG, BUT SOME ARE USEFUL

GEORGE E P BOX

GSP Nash Equilibrium

GSP's Nash Equilibrium

- Turns out even with simplifying assumptions, still many challenges:
  - Turns out GSP has **many** Nash equilibria
  - Some of which aren't "good": that is, not reasonable to assume that players will play such an equilibrium
  - How do we choose between the various Nash?
- Formally, the utility of bidders is
  - $u_i(\mathbf{b}) = \alpha_j(v_i - p_j)$ if bidder i receives slot $j \neq \emptyset$ where p_j is the price-per-click of slot j
  - Utility is 0 is bidder receives no slot
- At a Nash equilibrium, bidders must not be increase their utility by unilaterally deviating to a different bid b_i, keeping b_{-i} fixed

Formalizing Nash Eq Conditions

- Suppose bidder i who current has slot i deviates to b_i' to obtain a higher (better) slot $j < i$, then
  - How big should b_i' be to win?
  - Beat out b_j but below b_{j-1} that is, $b_j < b_i' < b_{j-1}$
  - What is the payment-per-click it has to make for slot j?
    - $p_j = b_j$
  - Expected utility from this deviation:
    - $\alpha_j(v_i - b_j)$
- To be in a Nash equilibrium, this deviation must not be profitable:
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j)$ for every higher slot $j < i$

Formalizing Nash Eq Conditions

- Suppose bidder i who current has slot i deviates to b_i' to obtain a lower (worse) slot $j > i$, then
  - How big should b_i' be to win slot j?
    - Just above b_{j+1} but below b_j, that is, $b_{j+1} < b_i' < b_j$
  - What is the payment-per-click it has to make for slot j?
    - $p_j = b_{j+1}$
  - Expected utility from this deviation:
    - $\alpha_j(v_i - b_{j+1})$
- To be in a Nash equilibrium, this deviation must not be profitable:
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$ for every lower slot $j > i$

Summary: Nash Equilibrium

- For an assignment between bidders and slots to be a Nash equilibrium, the following two conditions must hold

  $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j)$ for every higher slot $j < i$

  $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$ for every lower slot $j > i$

GSP's Equilibrium Example

- We can verify best response for bidder 2:
  - Deviate up to slot 1 at price \$4 or deviate down to slot 3 at price \$1

$\alpha_1 = 0.2$ | $v_1 = 4$ | $b_1 = 4$ | $p_1 = 2.1$

$\alpha_2 = 0.18$ | $v_2 = 10$ | $b_2 = 2.1$ | $p_2 = 2$

$\alpha_3 = 0.1$ | $v_3 = 2$ | $b_3 = 2$ | $p_3 = 1$

$v_3 = 1$ | $b_4 = 1$

$u_2 = 0.18(10 - 2) = 1.44$

GSP's Equilibrium Example

- Suppose bidder 2 targets and wins slot 1, whats the price?
  - $p_1 = 4$
- Utility goes down, no incentive to deviate

$\alpha_1 = 0.2$	$v_1 = 4$	$b_1 = 4$	$p_1 = 2.1$
$\alpha_2 = 0.18$	$v_2 = 10$	$b_2 = 2.1$	$p_2 = 2$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 2$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

$$u_2' = 0.2(10 - 4) = 1.2$$

GSP's Equilibrium Example

- Suppose bidder 2 targets and wins slot 3, whats the price?
  - $p_3 = 1$
- Utility goes down, no incentive to deviate

$\alpha_1 = 0.2$	$v_1 = 4$	$b_1 = 4$	$p_1 = 2.1$
$\alpha_2 = 0.18$	$v_2 = 10$	$b_2 = 2.1$	$p_2 = 2$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 2$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

$u_2' = 0.1(10 - 1) = 0.9$

GSP's Equilibrium Example

- Can verify similarly for other bidders
- This bid profile is a **Nash equilibrium**, but is it a good one?
    - Does not maximize surplus! Economically inefficient outcome

$\alpha_1 = 0.2$	$v_1 = 4$	$b_1 = 4$	$p_1 = 2.1$
$\alpha_2 = 0.18$	$v_2 = 10$	$b_2 = 2.1$	$p_2 = 2$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 2$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

Envy-Free Nash Equilibrium

Envy-Free Outcome

- The challenge in analyzing GSP is that there can be multiple equilibria
  - How do bidders select---depends on which equilibria is more plausible and reached by a straightforward bidding strategy
- **Envy-free outcome.** We say that a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$ where $b_1 \geq b_2 \geq \ldots \geq b_n$ is **envy-free** if for every bidder i

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$$

- **Interpretation:** (current price-per-click of slot j is $p_j = b_{j+1}$)
  - *each bidder i is as happy getting its current slot at its current price as it would be getting any other slot at that slot's current price*

Envy-Free Outcome

- **Envy-free outcome.** We say that a bid profile $\mathbf{b} = (b_1, \ldots, b_n)$ where $b_1 \geq b_2 \geq \ldots \geq b_n$ is **envy-free** if for every bidder i

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$$

- **Exercise.** Envy-free outcome $\Longrightarrow$ bids in GSP are value ordered, that is, $v_1 \geq v_2 \geq \ldots \geq v_n$ for bids $b_1 \geq b_2 \geq \ldots \geq b_n$

- We want to show: envy-free outcomes are a subset of Nash equilibrium
- Remember Nash equilibrium conditions:
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j)$ for every higher slot $j < i$
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$ for every lower slot $j > i$

Envy-Free Outcome is a Nash Eq

- **Lemma.** An envy-free outcome of the GSP auction must be a Nash equilibrium
- **Proof.** We showed the GSP outcome is a Nash equilibrium if the following conditions hold for each bidder i:
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j)$ for every higher slot $j < i$
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$ for every lower slot $j > i$
- At an envy-free outcome we have for each bidder i
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1})$
- Since $b_{j+1} < b_j$ we have that
  - $\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}) \ \geq \alpha_j(v_i - b_j)$ ∎

GSP's Equilibrium Example

- Is this equilibrium envy free?
  - Consider bidder 2, does he envy slot 1 at its current price?
  - Utility from slot 1's current price: $0.2(10 - 2.1) = 1.58$

Greater than current utility of **1.44** (Not envy free)

$\alpha_1 = 0.2$	$v_1 = 4$	$b_1 = 4$	$p_1 = 2.1$
$\alpha_2 = 0.18$	$v_2 = 10$	$b_2 = 2.1$	$p_2 = 2$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 2$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

$$u_2 = 0.18(10 - 2) = 1.44$$

GSP: An Envy-Free Equilibrium

- Verify that the following outcome is envy free
  - No bidder envies a different slot at its **current price**

$\alpha_1 = 0.2$	$v_1 = 10$	$b_1 = 4$	$p_1 = 2$	$u_1 = 1.6$
$\alpha_2 = 0.18$	$v_2 = 4$	$b_2 = 2$	$p_2 = 1.5$	$u_2 = 0.449$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$	$u_3 = 0.1$
	$v_3 = 1$	$b_4 = 1$		

Many Envy-Free Equilibrium

- Can you come up with another bid profile that is envy free?
  - Of course 1 can increase his bid without affecting anything
  - Lets assume wlog $b_1 = v_1$

$\alpha_1 = 0.2$	$v_1 = 10$	$b_1 = 4$	$p_1 = 2$	$u_1 = 1.6$
$\alpha_2 = 0.18$	$v_2 = 4$	$b_2 = 2$	$p_2 = 1.5$	$u_2 = 0.449$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$	$u_3 = 0.1$
	$v_3 = 1$	$b_4 = 1$		

Which Envy Free Eq to Play

- Now how about bidder 2? Can they bid higher? Is that envy-free?

$\alpha_1 = 0.2$	$v_1 = 10$	$b_1 = 10$	$p_1 = 2$	$u_1 = 1.6$
$\alpha_2 = 0.18$	$v_2 = 4$	$b_2 = 2$	$p_2 = 1.5$	$u_2 = 0.449$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$	$u_3 = 0.1$
	$v_3 = 1$	$b_4 = 1$		

Which Envy Free Eq to Play

- Bidder 2 can increase their bid and raise bidder 1's price!

$\alpha_1 = 0.2$ $\quad v_1 = 10 \quad b_1 = 10 \quad p_1 = 2 \quad u_1 = 1.6$

$\alpha_2 = 0.18$ $\quad v_2 = 4 \quad b_2 = 2 \quad p_2 = 1.5 \quad u_2 = 0.449$

$\alpha_3 = 0.1$ $\quad v_3 = 2 \quad b_3 = 1.5 \quad p_3 = 1 \quad u_3 = 0.1$

$v_3 = 1 \quad b_4 = 1$

Which Envy Free Eq to Play

- Why not bid $b_2 = \$9.99$ and increase the price for bidder 1 which can potentially drive that bidder out of future auctions?
- What can go wrong?

$\alpha_1 = 0.2$	$v_1 = 10$	$b_1 = 10$	$p_1 = 9.99$
$\alpha_2 = 0.18$	$v_2 = 4$	$b_2 = 9.99$	$p_2 = 1.5$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

Which Envy Free Eq to Play

- Why not bid $b_2 = \ \$9.99$ and increase the price for bidder 1 which can potentially drive that bidder out of future auctions?
- **Potential concern.** Bidder 1 could retaliate and "jam" bidder 2 by bidding 9.98 which would put bidder 1 in slot 1 at price 9.98

$\alpha_1 = 0.2$	$v_1 = 10$	$b_1 = 10$	$p_1 = 9.99$
$\alpha_2 = 0.18$	$v_2 = 4$	$b_2 = 9.99$	$p_2 = 1.5$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

Which Envy Free Eq to Play

- Why not bid $b_2 =$ $\$9.99$ and increase the price for bidder 1 which can potentially drive that bidder out of future auctions?
- **Potential concern.** Bidder 1 could retaliate and "jam" bidder 2 by bidding 9.98 which would put bidder 1 in slot 1 at price 9.98

$\alpha_1 = 0.2$	$v_2 = 4$	$b_2 = 9.99$	$p_1 = 9.98$
$\alpha_2 = 0.18$	$v_1 = 10$	$b_1 = 9.98$	$p_2 = 1.5$
$\alpha_3 = 0.1$	$v_3 = 2$	$b_3 = 1.5$	$p_3 = 1$
	$v_3 = 1$	$b_4 = 1$	

Which Envy Free Eq to Play

- **Idea.** Bidders will prefer highest bids amongst those that achieve the same position as it drives up the price of their competitors

$\alpha_1 = 0.2$ $v_1 = 10$ $b_1 = 10$

$\alpha_2 = 0.18$ $v_2 = 4$ $b_2 = ?$

$\alpha_3 = 0.1$ $v_3 = 2$ $b_3 = ?$

$v_3 = 1$ $b_4 = 1$

Balanced Bidding

- For bidder i, a balanced bid in slot i (for slots $2,\ldots,k$) is the largest bid b_i for which

$$\underbrace{\alpha_i(v_i - b_{i+1})}_{\text{utility current position}} \geq \underbrace{\alpha_{i-1}(v_i - b_i)}_{\text{utility in case of retaliation}}$$

- Does such a bid b_i always exist?
  - As long as $b_{i+1} \leq v_i$ and $\alpha_i < \alpha_{i-1}$, then yes ($b_{i+1} < b_i \leq v_i$)
- For bidders that have no slot assigned, what is the highest they should bid without any threat of retaliation (and non-negative utility)?
  - Their true value

Credit: Textbook by Parkes and Seuken

Balanced Bidding

- We say a bid profile $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ satisfies the balanced bidding requirement if
  - The following holds for bidder i for $2 \leq i \leq m$

$$\underbrace{\alpha_i(v_i - b_{i+1})}_{\text{utility current position}} = \underbrace{\alpha_{i-1}(v_i - b_i)}_{\text{utility in case of retaliation}}$$

  - Any unassigned bidder bids their true value
- Notice that for value ordered bids, the balanced bidding requirement defines **a unique bid profile** (up to the indifference of the top bidder)

Credit: Textbook by Parkes and Seuken

Balanced Bidding Strategies

- b_i must be the highest bid such that bidder i is indifferent between remaining in slot i and having bidder $i-1$ retaliate

$\alpha_1 = 0.2$ $v_1 = 10$ $b_1 = 10$

$\alpha_2 = 0.18$ $v_2 = 4$ $b_2 = ?$

$\alpha_3 = 0.1$ $v_3 = 2$ $b_3 = ?$

$v_3 = 1$ $b_4 = 1$

Balanced Bidding Strategies

- Bid b_3 must be the highest bid such that bidder 3 is indifferent between remaining in slot 2 and having bidder 1 retaliate
- $0.1(2 - 1) = 0.18(2 - b_3) \implies b_3 = ?$

$\alpha_1 = 0.2$ | $v_1 = 10$ | $b_1 = 10$

$\alpha_2 = 0.18$ | $v_2 = 4$ | $b_2 = ?$

$\alpha_3 = 0.1$ | $v_3 = 2$ | $b_3 = 13/9$

$v_3 = 1$ | $b_4 = 1$

Balanced Bidding Strategies

- Bid b_2 must be the highest bid such that bidder 2 is indifferent between remaining in slot 2 and having bidder 1 retaliate:

$$0.18(4 - 13/9) = 0.2(4 - b_2) \implies b_2 = 17/10$$

$\alpha_1 = 0.2$ $v_1 = 10$ $b_1 = 10$ $p_1 = 17/10$

$\alpha_2 = 0.18$ $v_2 = 4$ $b_2 = 17/10$ $p_2 = 13/9$

$\alpha_3 = 0.1$ $v_3 = 2$ $b_3 = 13/9$ $p_3 = 1$

$v_3 = 1$ $b_4 = 1$

Compare to VCG

- **Exercise:** Compute the VCG payments for this example

These are exactly the VCG payments!!!

$\alpha_1 = 0.2$ $v_1 = 10$ $b_1 = 10$ $p_1 = 17/10$

$\alpha_2 = 0.18$ $v_2 = 4$ $b_2 = 17/10$ $p_2 = 13/9$

$\alpha_3 = 0.1$ $v_3 = 2$ $b_3 = 13/9$ $p_3 = 1$

$v_3 = 1$ $b_4 = 1$

Bigger Picture

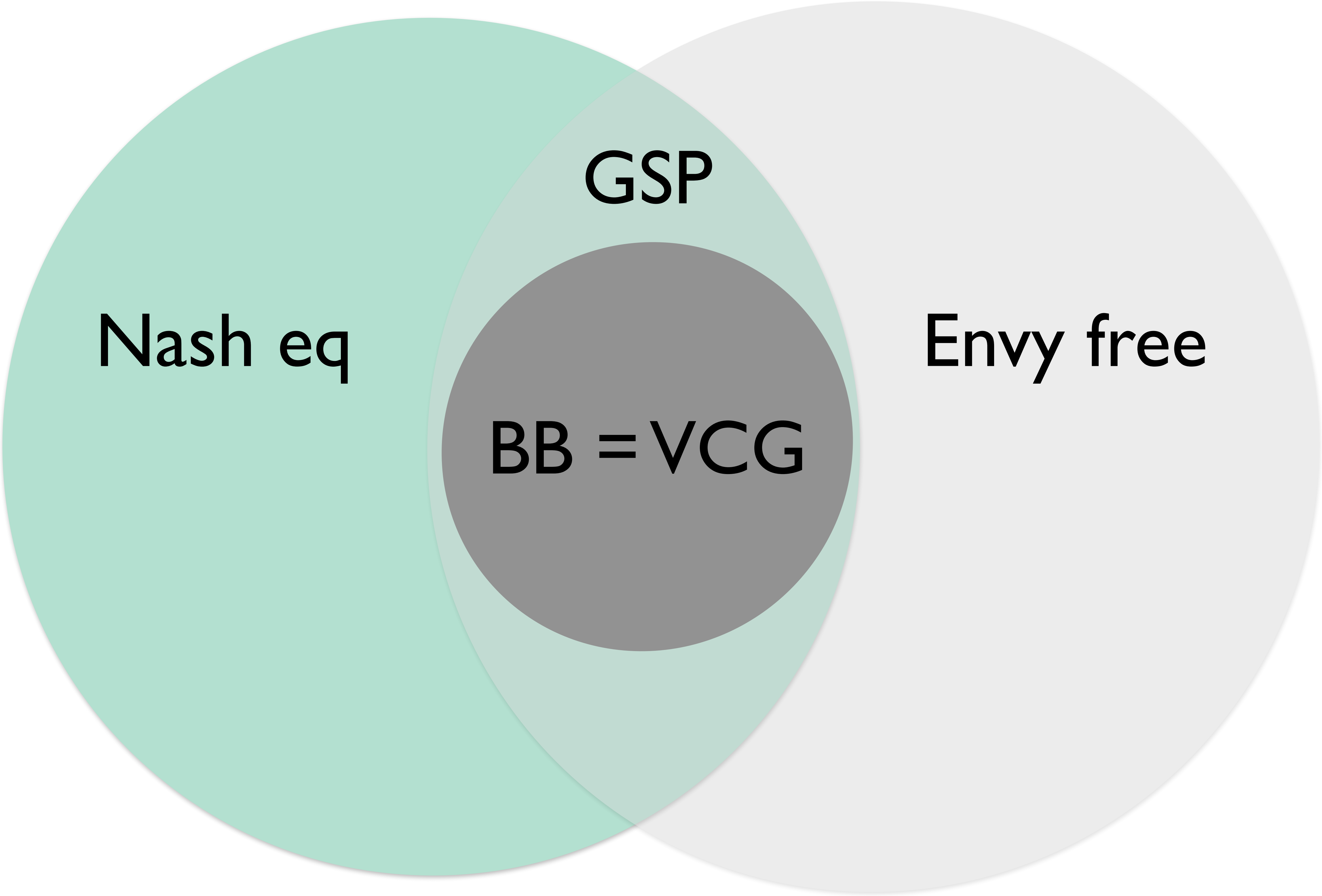


Figure adapted from Textbook by Parkes and Seuken

Bigger Picture

- Lots of Nash equilibrium, some are inefficient and seem unlikely
- Envy-free (solution concept) in GSP $\Longrightarrow$ Nash equilibrium in GSP
- Lots of envy-free Nash still! Which ones are likely to be played?
  - The ones that emerge out of a reasonable "best response dynamics"
- Balanced bidding: **locally** envy free (no one wants to swap with one above)
- What we need to show:
  - Balanced bidding is in fact envy free
- GSP with balanced bidding gets exactly the same outcome (allocation, payments) as VCG with truthful biding!