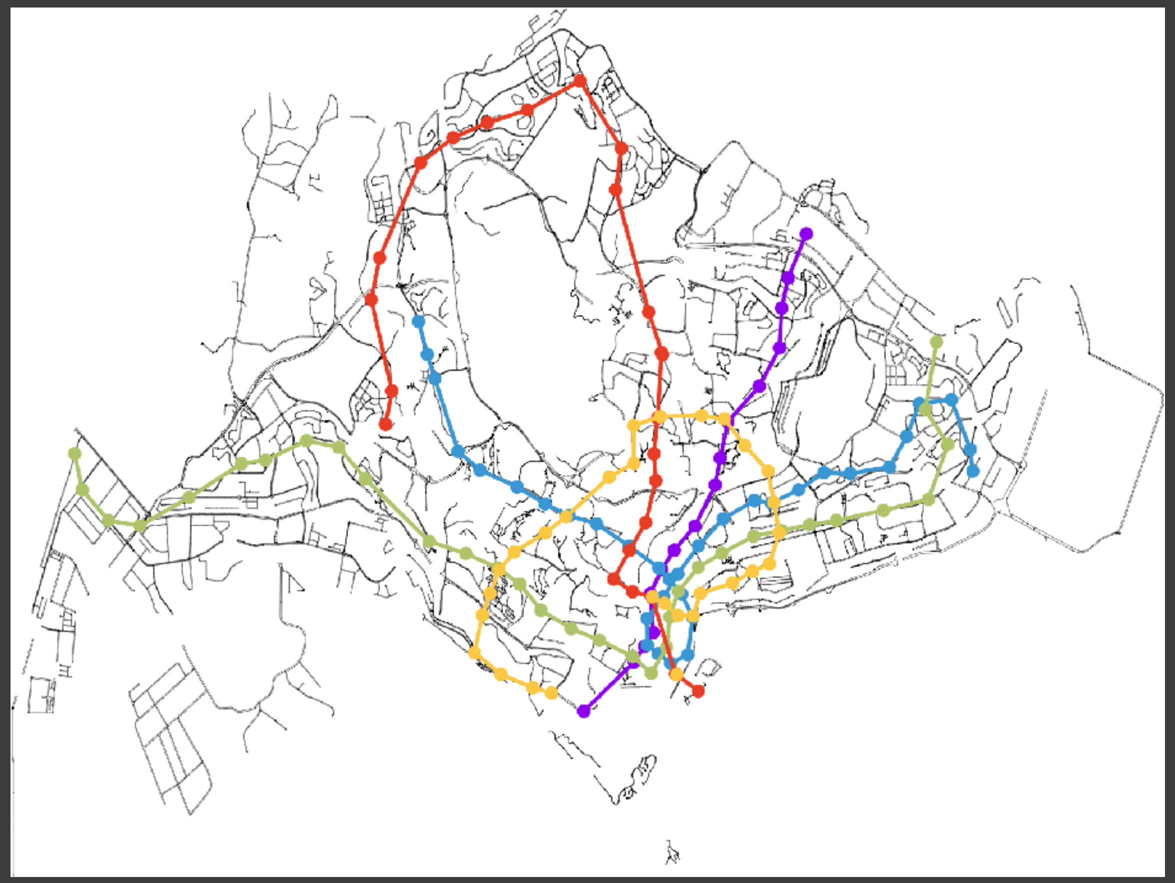
Spring 2022

CSCI 357: Algorithmic Game Theory

Lecture 2: Game Theory I

Shikha Singh



Announcements and Logistics

- **Assignment 0** on GLOW: Join Slack and post an introduction, fill out course survey, sign up for a short Zoom chat (or come to office hours)
- **Assignment 1** is out and due Thursday 10 pm
  - Questions to get practice with equilibrium concepts we will cover today and get started with a few proofs
  - Type in LaTeX using template, submit on Gradescope
  - One very short paper to get you started thinking about how these concepts fit into the landscape
- Office (Zoom) hours tomorrow and Wed **2.30 - 4 pm:** same link

Do you have any questions ?

Prisoner's Dilemma

- Two alleged criminals questioned in separate rooms
- Each player has two actions:
    - **Cooperate (C):** stay silent and not admit to anything
    - **Defect (D)**: testify agains the other person
- If both stay silent **(C, C)**, each serves 1 year in prison for minor offense
- If one confesses against the other **(C, D)** or **(D, C)**, confessor goes free while other person gets a long prison sentence
- If both confess **(D, D)**, they each serve 3 years in prison
- We can write their preferences as an ordering

	C	D
C	a, a	b, c
D	c, b	d, d

$$c > a > d > b$$

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- But more commonly, we use numbers to denote their utility

	C	D
C	$-1, -1$	$-5, 0$
D	$0, -5$	$-3, -3$

Prisoner's Dilemma

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- But more commonly, we use numbers to denote their utility

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Reasoning About Play

- Suppose you are player 1 (row)
- If player 2 (column) plays C, what should you play?

	C	*D*
C	4, 4	0, 5
D	5, 0	2, 2

Reasoning About Play

- Suppose you are player 1 (row)
- If player 2 (column) plays C, what should you play?
- If player 2 (column) plays D, what should you play?

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Reasoning About Play

- Suppose you are player 1 (row)
- If player 2 (column) plays C, what should you play?
- If player 2 (column) plays D, what should you play?
- Regardless of what player 2 does, it is best to play D
- How about player 2?
  - Can reason similarly
  - Regardless of what player 1 does, it is best for player 2 to also play D

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Dominant Strategy Equilibrium

- Strongest guarantee a game can have of player behavior
- **Definition idea:** regardless of other players actions, each player has a dominant strategy
- **Notation.** Suppose other players use actions a_{-i}, where $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ (standard notation for everyone's actions except i)
- Overall action profile is $a = (a_i, a_{-i})$
- **Domination:** For a player i, an action a_i (weakly) dominates action $a_i' \in A$ if it is always beneficial to play a over a'
  - That is, for all $a_{-i} \in A_{-i}$: $u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$ and the inequality is strict for some $a_{-i} \in A_{-i}$

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Dominant Strategy Equilibrium

- What actions are dominated in Prisoner's dilemma?
  - C is dominated by D for both players
- An action $a \in A_i$ is **dominant** for player i if it weakly dominates all actions $a_i' \in A_i, \quad a_i' \neq a_i$
- **Dominant strategy equilibrium (DSE):** An equilibrium where each player plays their dominant action, that is,
  - Action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a dominant- strategy equilibrium of a simultaneous-move game (N, A, u) if, and only if, we have

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

for all $a_{-i} \in A_{-i}$, all $a_i \in A_i$ and all agents $i \in N$

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Dominant Strategy Equilibrium

- DSE is a strong guarantee on player behavior
  - No brainer for players to play DSE
- In mechanism design, goal is to shoot for mechanisms that admit a DSE
- What is the DSE in prisoner's dilemma?
  - (D, D): both players defect
- **Question**. Will a DSE always exist in a game?
  - We will see an example soon

	C	D
C	4, 4	0, 5
D	5, 0	2, 2

Pareto Optimality

- Is (D, D) a good outcome overall?
- **Question.** Which outcome is better as a whole (to a neutral observer)?
- An outcome o is at least as good for every player, as another outcome o', and there is some agent who strictly prefers o to o'
  - Reasonable to say o is better than o'
  - We say o **Pareto dominates** o'
- Definition (**Pareto Optimality**).
  - An outcome o^* is Pareto-optimal if there is no other outcome that Pareto dominates it
- **Question.** Can you point out Pareto optimal outcomes in Prisoner's dilemma?

	C	D
C	4, 4	0, 5
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All but (D, D), the unique DSE, are Pareto optimal!

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- **Definition** (**Pareto Optimality**).
  - An outcome o^* is Pareto-optimal if there is no other outcome that Pareto dominates it
- **Question.** Can you point out Pareto optimal outcomes in Prisoner's dilemma?
  - All except (D, D) are Pareto optimal: the dilemma!
  - What is good for one, not good for the group!

All but (D, D), the unique DSE, are Pareto optimal!

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Pareto Optimality

- An outcome o is at least as good for every player, as another outcome o', and there is some agent who strictly prefers o to o'
  - Reasonable to say o is better than o'
  - We say o Pareto dominates o'
- **Definition** (**Pareto Optimality**).
  - An outcome o^* is Pareto-optimal if there is no other outcome that Pareto dominates it
- **Question.** Does every game have at least one Pareto optimal outcome?
  - Yes, since there can be no outcome that is dominated: every outcome is Pareto optimal

Does DSE Always Exist?

- Dominant strategy equilibria is a very strong guarantee, but may not exist for most games
- Does the given game have a DSE? Why?

	L	*M*	*R*
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Does DSE Always Exist?

- Dominant strategy equilibria is a very strong guarantee, but may not exist for most games
- Does the given game have a DSE? Why?
  - Player 1 should play T if player 2 plays L or M
  - Player 1 should play B if player 2 plays R
  - No single dominant action!

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Eliminating Dominated Strategies

- Even when DSE does not exist, there still may exist a **dominated strategy** for some player
- Does player 2 have a dominated strategy?

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Eliminating Dominated Strategies

- Even when DSE does not exist, there still may exist a **dominated strategy** for some player
- Does player 2 have a dominated strategy?
    - R is strictly dominated by M
    - Can be eliminated
- In the reduced game, does player 1 have a dominated strategy?
    - B is strictly dominated by T
- Finally, L is dominated by M

Only outcome that survives (T, M)

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Rationalizability

- **Question.** Why is it okay to eliminate "dominated" actions like we did?
- Does it ever make sense to play a "dominated" action?
- **Idea:** A dominated action can never be a best response

Only outcome that survives (T, M)

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Rationalizability

- **Question.** Why is it okay to eliminate "dominated" actions like we did?
- Does it ever make sense to play a "dominated" action?
- **Idea:** A dominated action can never be a best response
- **Towards Nash:** In a "stable" outcome, each rational players must play their best response to others
  - If they can change their action to improve their utility, they would: outcome is not stable
- **Our goal:** to find out when are there stable solutions

Only outcome that survives (T, M)

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Best Response

- **Best response definition**: Let $\text{BR}(a_{-i})$ denote the set of actions that form i's best response given a_{-i} then,

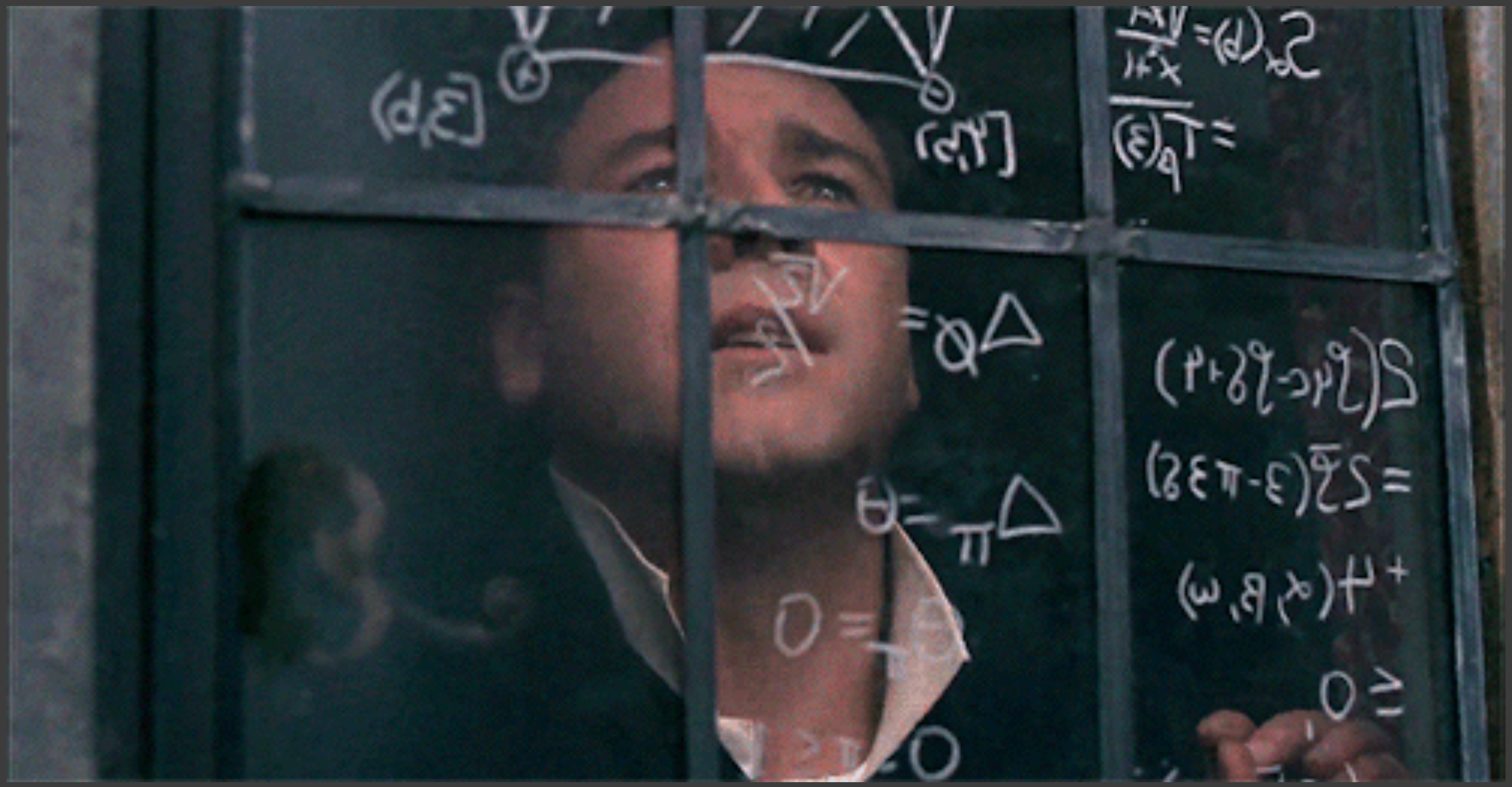
  $a_i^* \in \text{BR}(a_{-i})$ if only if
  $u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}) \ \forall a_i \in A_i$

- A dominant action is always a best response (regardless of others actions)

- For games that do not have a DSE, we fix action of others a_{-i} and then reason what i should do

    - Each player reasons this way...

Nash Equilibrium



Nash Equilibrium

- In a Nash equilibrium, every agent plays a best response to the actions of others
- **Pure-strategy Nash equilibrium**: Action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a pure-strategy Nash equilibrium of a simultaneous-move game (N, A, u) if, and only if, for each player i and for all actions $a_i' \in A_i$:

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})$$

- Whenever we need to verify an action profile is a Nash: check if anyone has an incentive to **unilaterally deviate**
- What is a pure Nash equilibrium in the example?

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Nash Equilibrium

- In a Nash equilibrium, every agent plays a best response to the actions of others
- **Pure-strategy Nash equilibrium**: Action profile $a^* = (a_1^*, \ldots, a_n^*)$ is a pure-strategy Nash equilibrium of a simultaneous-move game (N, A, u) if, and only if, for each player i and for all actions $a_i' \in A_i$:

$$u_i(a_i,\ a_{-i})\ \geq\ u_i(a_i',\ a_{-i})$$

- Whenever we need to verify an action profile is a Nash: check if anyone has an incentive to **unilaterally deviate**
- What is a pure Nash equilibrium in the example?
    - (T, M)

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

BoS Game

- Classic coordination game called **Battle of the Sexes** or the less problematic variant BoS (Bach or Stravinsky):
  - Two people wish to go out together to a concert of music by either Bach or Stravinsky
  - They both want to go out together, but one person prefers Bach and the other person prefers Stravinsky
  - Assume no cellphones to coordinate ahead of time
- What are the pure Nash equilibria of this game?

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Multiple Nash Equilibrium

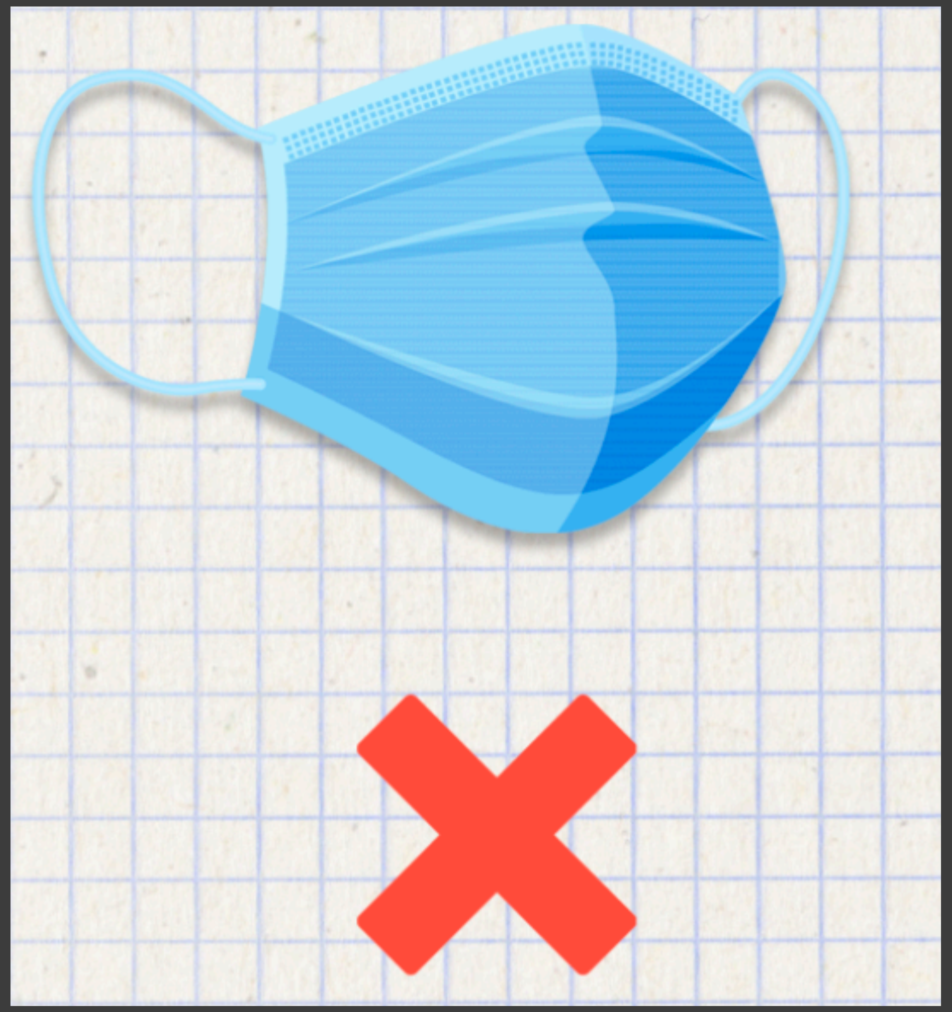
- Classic coordination game called **Battle of the Sexes** or the less problematic variant BoS (Bach or Stravinsky):
  - Two people wish to go out together to a concert of music by either Bach or Stravinsky
  - They both want to go out together, but one person prefers Bach and the other person prefers Stravinsky
  - Assume no cellphones to coordinate ahead of time
- What are the Nash equilibria of this game?
  - (B, S) and (S, B) are both Nash equilibria!

	Bach	*Stravinsky*
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Mask or No Mask Game

- Two players, symmetric game
- Actions are to wear or not wear a mask
- When both wear masks, both are protected; if neither wear masks, neither are protected
- If only one wears masks, the other person is protected
- Pareto optimal outcome? Nash outcomes?

	M	$\overline{M}$
M	$1,\ 1$	$-1,\ 1$
$\overline{M}$	$1,\ -1$	$-1,\ -1$

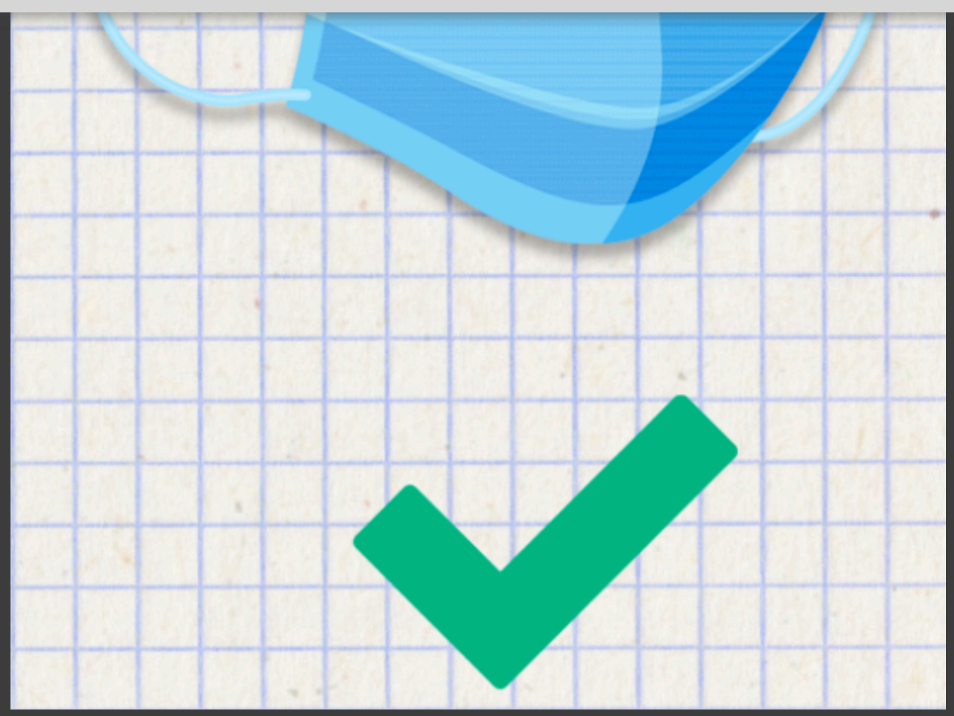


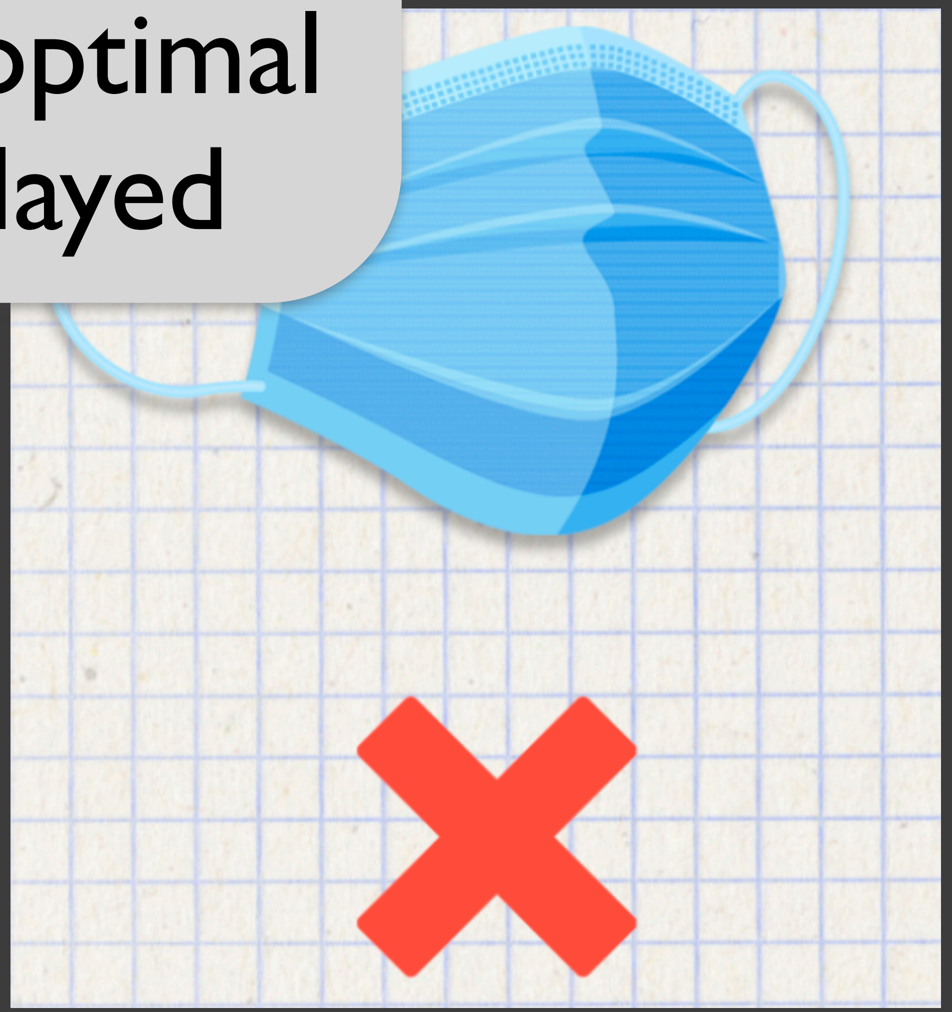
Mask or No Mask Game

- Two players, symmetric game
- Actions are to wear or not wear a mask
- When both wear masks, both are protected; if neither wear masks, neither are protected
- If only one wears masks, the other person is protected
- Pareto optimal outcome? Nash out

	M	$\overline{M}$
M	$1, 1$	$-1, 1$
$\overline{M}$	$1, -1$	$-1, -1$

Problem of equilibrium selection:
Reasonable to assume that Pareto-optimal outcomes are more likely to be played





Multiplicity of Equilibria

- When you have multiple equilibria, this creates an equilibrium selection problem
- How do players know which equilibrium to play?
- Reasonable to assume "good" equilibria" are better
  - Social welfare (overall sum of utility is same)
  - Pareto optimal
- When designing our own mechanism, it is important to strive for "good" equilibrium
  - Unique/ Social welfare maximizer, Pareto optimal etc

Existence of Equilibrium

- A DSE is not guaranteed to exist in every game
- What about a pure Nash equilibrium?
  - Does every game have at least one pure Nash equilibrium?
- Any example we have seen of a game where no such equilibrium exists?
- When designing our own mechanisms, we can try to create incentives such that a DSE or pure Nash exists

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Iterated Elimination

Eliminating Dominated Strategies

- Eliminating dominated strategies iteratively is called **iterated elimination of dominated strategies**
  - Turns out this was a pure Nash
  - Is it always the case?
  - **Claim.** If iterated elimination of dominated strategies results in a unique solution, then it is a pure strategy Nash equilibrium (HW 1)
- Is it always possible to find actions to eliminate?
  - HW 1

Only outcome that survives (T, M)

	L	M	R
T	1, 0	1, 2	0, 1
B	0, 3	0, 1	2, 0

Quick Recap

- DSE is the strongest guarantee but may not always hold
- If a game has no DSE, we may be able to find "dominated strategies" of players and iteratively eliminate it
  - If it leads to a single outcome -> pure Nash
- Nash equilibrium is the next best solution concept
  - But "pure" Nash equilibria may not exist
  - Multiple Nash equilibria may exist
- Outcomes are equilibrium may not be a good one globally

Any Questions?

Let's Play A Game

- Each of you must choose an **integer between 1 and 100**
- The player(s) who name the integer closest to **two thirds** of the **average** wins a prize, the other players get nothing
    - If average is X, you want to name an integer close to $2X/3$
- (Ties will lead to multiple winners)
- Click on the google form link on zoom to play:
    - https://tinyurl.com/357numbers

Reasoning About the Game

- What you should do, depends on what other players do
  - What makes game theory interesting and also challenging!
- Suppose you believe that the average play will be X (including your own guess)
- What should do you play in response?
  - Closest integer to $\frac{2X}{3}$
- Some back of envelope calculation
  - $X \leq 100$, that means $\frac{2X}{3} \leq 67$
- That is, number greater than 67 cannot win, regardless of the play

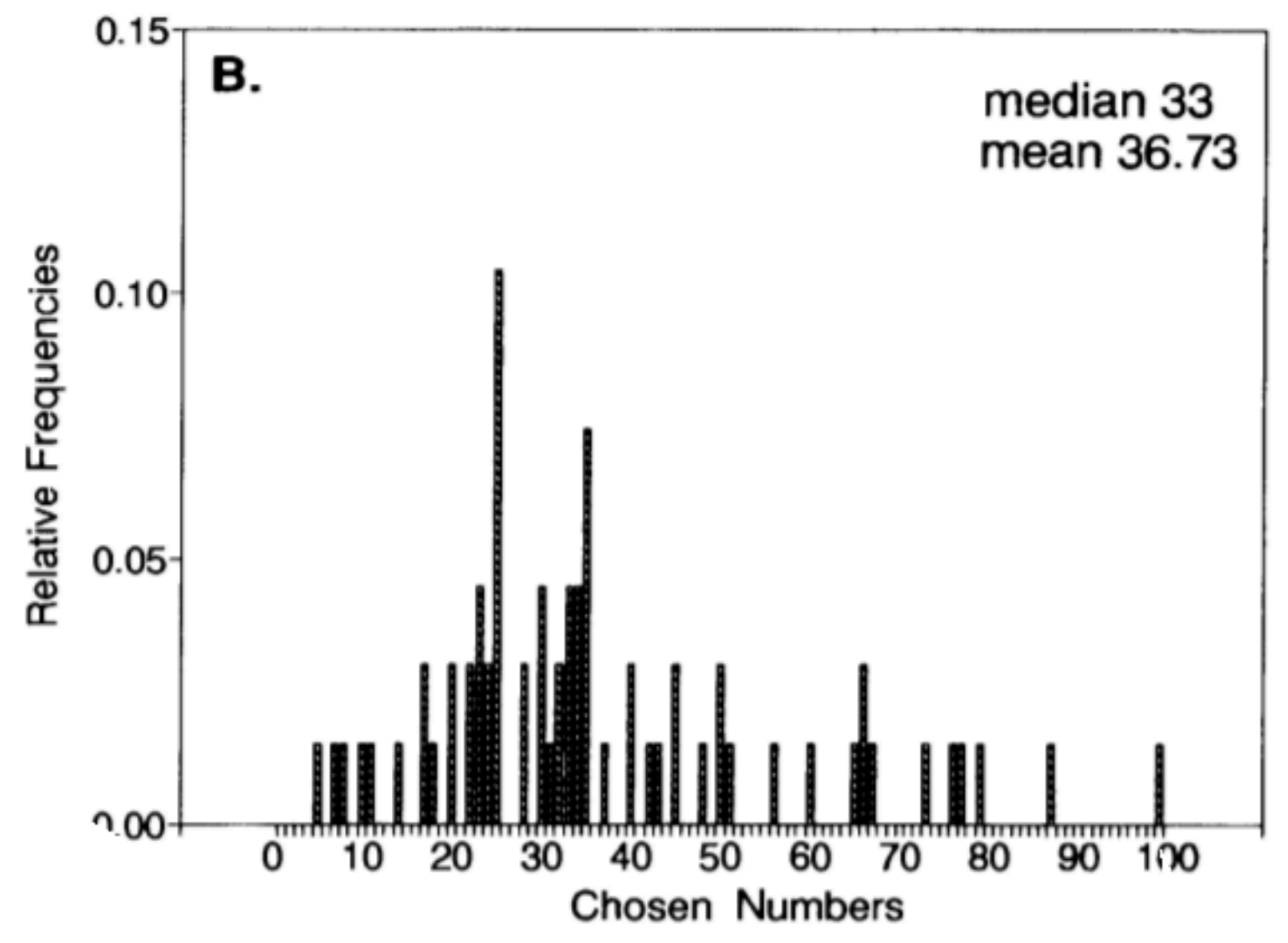
Can We Reason Further?

- So no guess above 67 can win and everyone can reason like this
- If everyone plays $X \leq 67$, what is your best response?
  - Play a number $\leq (2/3) \cdot 67$
- Everyone else is reasoning exactly like you, and thus will play a number $\leq (2/3) \cdot 67$
  - What is your best response?
  - Play a number $\leq (2/3)^2 \cdot 67$
- When does this end?

Stable state: Everyone plays 1; everyone wins: **unique pure Nash equilibrium** of the game

Nash Play: Numbers Game

- If everyone is "perfectly rational" then playing 1 is the unique pure Nash equilibrium of the game
- **Empirical analysis**
    - People may not be perfectly rational; usually peaks around 10/20
- Does that mean Nash is not a good predictor of behavior in practice?
- What happens if we play the game again?
    - Players learn and "converge" to a Nash



Rosemarie Nagel. Unraveling in guessing games: An experimental study (1995)

Keynes Beauty Contest

- **John M. Keynes** came up with the concept of a "newspaper beauty contest" to explain price fluctuations in stock markets
- Relation to stock market:
  - *"People pricing shares not based on what they think their fundamental value is, but rather on what they think everyone else thinks their value is, or what everybody else would predict the average assessment of value to be."*
- **Stylized version**: Guess the 2/3rd of average
  - **21.6** was the winning value in a large online competition organized by the Danish newspaper ***Politiken***.
  - 19,196 people participated and the prize was 5000 Danish kroner
- Difference between **rationality** and **common knowledge of rationality**

Common Knowledge

- Why is it okay to reason like this ("iterated elimination")?
- Common knowledge of rationality means
  - a players knows that each player is rational
  - knows that each player knows that each player is rational
  - knows that each player knows that each player knows that each player is rational
  - and so on, ad infinitum