CS 357: Algorithmic Game Theory **Spring 2022**

Assignment 6 (due 04/07/2022)

Instructor: Shikha Singh *Link to* *Solution Template*

Note. This is a single-person assignment. Points will be awarded for clarity, correctness and completeness of the answers. The assignment is due **at 11 pm EST** on the due date.

Matching Markets with Money

Problem 1. Consider a single-item market with $n \geq 2$ buyers. Each buyer i has a valuation v_i for the item. Add $n-1$ dummy items that all buyers value at 0. Give necessary and sufficient conditions on a market-clearing price vector $\mathbf{p} = (p_1, \ldots, p_n)$ of such a market (based on the valuations of the bidders).

Problem 2. Consider a matching market with n buyers and n items where each buyer only wants a single item. Let M^* be a matching that maximizes the surplus $\sum_{i=1}^{n} v_{iM^*(i)} \geq \sum_{i=1}^{n} v_{iM(i)}$ for any matching M. Let $\mathbf{p}$ be any market clearing price of this market and E be the edges in the preferred-item graph under $\mathbf{p}$. Show that $M^* \subseteq E$.

Matching Markets without Money

Problem 3. (One-Sided Matching) In class we discussed the serial dictatorship mechanism for the one-sided matching market. If you were not impressed by that mechanism, let us look at a bad alternative to it which, unfortunately, was used to assign kids to elementary schools in a number of major cities for many years.[1]

- Each students submits a complete ranked list of their preferences.
- The students are ordered in some way (e.g., by lottery numbers)
- *Phase 1.* The students are considered in this order. When student i is considered, if her top-ranked school is still available, then she is assigned to that school. Otherwise, she is not assigned in this phase.
- *Phase 2.* The unassigned students are considered in the same order as before. When student i is considered, if her second-ranked school is still available, then she is assigned to that school. Otherwise, she is not assigned in this phase.
- And we continue similarly with Phase 3 considered third-choices of unassigned students, ... Phase i considering ith choices, etc. until all students are assigned.

Show that this mechanism is not strategyproof (DSIC) by giving an explicit counter-example. Explain what type of strategic behavior you would expect to see from participants.

[1] This is also called the *Boston mechanism*, as it was used in Boston high schools until 2005.

Problem 4 (Parkes and Sueken 12.3)**.** In 2003, Abdulkadiroglu and Sonmez [1] formulated school choice as a two-sided matching problem. One side of the market is straightforward—students have preferences over schools. On the other side of the market, schools do not have preferences, but have *priorities*. School priorities are often determined by the school zone policies, and factor things like where students live, where their siblings attend school, etc. [2]. Because priorities are not preferences, emphasis if often given to preferences of students when comparing the outcomes of these mechanisms.

Abdulkadiroglu and Sonmez proposed two mechanisms that are strategyproof on the student side and allow schools to set a priority structure: the deferred acceptance (DA) mechanism and the top-trading cycle (TTC) mechanism. We will explore the trade-offs between these mechanisms when it comes to the welfare of the students.

Consider the following preferences orders for students s_1, s_2, s_3, and priority orders for schools t_1, t_2, t_3. Assume that each school has capacity one.

Table 1: Preferences of students

s_1	t_2	t_1	t_3
s_2	t_1	t_2	t_3
s_3	t_1	t_2	t_3

Table 2: Priorities of schools

t_1	s_1	s_3	s_2
t_2	s_2	s_1	s_3
t_3	s_2	s_1	s_3

(a) Interpreting priority orders as preference orders, use student-proposing DA to find the student-optimal stable matching. If we ignore school priorities, is there a matching that Pareto dominates this matching for students?

(b) To generalize TTC mechanism for this problem, we modify the mechanism as follows:

In each step, each school with remaining capacity points to the unmatched student with most priority, and each unmatched student points to the most preferred school with remaining capacity. Paths alternate between students and schools, and "trading on a cycle" corresponds to each student on the cycle being matched with its requested school.

Run the generalized TTC mechanism on the above example. Is the outcome Pareto optimal for students? Is the outcome stable? Justify your answers.

Remark. The TTC mechanism guarantees better welfare for students, but is not stable. Arguably, stability is not a major concern in this application (many policies prohibit schools from admitting students on an ad hoc basis).

Despite this, the DA mechanism has been adopted by many school districts [2]. According to the authors, the main barriers of TTC being adopted in practice is the difficulty of describing the mechanisms to school boards and parents.

Problem 5. In this question, we will show that no mechanism for two-sided matchings can be both stable and strategyproof. To simplify the proof, we are going to extend our model slightly. Let the preference of each student be a complete ordering over the set $H \cup \emptyset$ where H is the set of hospitals and $\emptyset$ additionally allows students to declare some hospitals as *unacceptable*. For example, in an example with two students and two hospitals, a student's preference list can be $h_1, \emptyset, h_2$, which says they want to only match with h_1 and would rather be unmatched than match with h_2. Similarly, the hospitals' preferences are a complete ordering over the set $S \cup \emptyset$, where S is the set of students. Everything we discussed in class carries over to this setting.

Using this model, prove that there exists no mechanism for two-sided matching that is both stable and strategy-proof. (*Hint.* Consider an instance where in a truthful ordering s_1 prefers h_1 over h_2, s_2 prefers h_2 over h_1, h_1 prefers s_2 over s_1 and h_2 prefers s_1 over s_2.)

References

[1] Atila Abdulkadiroğlu, Nikhil Agarwal, and Parag A Pathak. The welfare effects of coordinated assignment: Evidence from the nyc hs match. Technical report, National Bureau of Economic Research, 2015.

[2] Jacob Leshno and Irene Lo. The cutoff structure of top trading cycles in school choice. *Available at SSRN 2954870*, 2017.