CS 357: Algorithmic Game Theory **Spring 2022**

Assignment 3 (due 02/24/2022)

Instructor: Shikha Singh *Solution Template*

Instructions. This is a partner assignment: up to two students can work together and submit a joint write up and receive the same grade. You can invite your partner to jointly edit the same Overleaf document by using the "share" feature. One student in the group should submit the joint PDF on Gradescope, and add the other's name under "Group": see this tutorial video for steps on how to do this. I strongly encourage collaboration during as it is a great way to learn from each other and build community within the classroom.

Topics and Readings.

- Single-parameter mechanism design and Myerson's lemma: R Ch 3.
- Knapsack auctions are discussed in R Ch 4.

Welfare Maximization and Externality Pricing

Problem 1. (Nisan, Roughgarden and Tardos)

(a) Consider an arbitrary single-parameter environment, with feasible allocation $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Given bids $\mathbf{b} = (b_1, \ldots, b_n)$, the surplus-maximizing allocation rule is $\mathbf{x}(\mathbf{b}) = \text{argmax}_{(x_1,\ldots,x_n)\in X} \sum_{i=1}^{n} b_i x_i$. Prove that this allocation rule is monotone.[1] (*Hint.* Consider a player i, and fix $\mathbf{b}_{-i}$. Increase i's bid from b_i to b_i', where $b_i' = b_i + \delta$ and $\delta > 0$, and show that the i's allocation cannot get worse.)

(b) Continuing part (a), we consider a surplus maximizing allocation rule $\mathbf{x}(\mathbf{b})$ and now restrict to feasible allocations X that contain only 0-1 vectors—that is, each bidder either wins or loses.

In this case, Myerson's payment rule can be written as:

$$p(b_i, \mathbf{b}_{-i}) = \begin{cases} 0 & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 0 \\ b_i^*(\mathbf{b}_{-i}) & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 1 \end{cases}$$

where $b_i^*(\mathbf{b}_{-i})$ is the bidder i's *critical bid*, that is, the lowest bid at which i gets a non-zero allocation.

Given feasible allocations containing 0-1 vectors, we can identify each feasible allocation with a "winning set" of bidders (the set of bidders i with $x_i = 1$ in that allocation).

Prove that, when S^* is the set of winning bidders and $i \in S^*$, then i's critical bid $b_i^*(\mathbf{b}_i)$ equals the difference between

[1] Assume that ties are broken in a deterministic and consistent way, such as lexicographically.

(i) the maximum surplus of a feasible allocation that excludes i[2]—that is, the maximum surplus that can be generated if i was not present

(ii) the surplus $\sum_{j \in S^* \setminus \{i\}} b_j$ generated by the winners (other than i) in the chosen outcome S^*—that is, the surplus that is generated (by others) given i wins

Hint. Write down the surplus-maximizing allocation for the two cases when (a) $x_i = 1$ ($i \in S^*$) and (b) when $x_i = 0$ ($i \notin S^*$). The optimal allocation will choose whether or not to allocate to i (that is, when to switch from (b) to (a)) at a bid b_i where the surplus generated by case (a) is at least as good as the surplus generated by case (b).

Remark. In other words, a winning bidder pays their "externality"—the surplus loss they impose on others.

(c) Finally, consider a 0-1 single-parameter environment. Suppose that you are given a black box algorithm that can compute the surplus maximizing allocation rule $\mathbf{x}(\mathbf{b})$ for any bid profile $\mathbf{b} = (b_1, \ldots, b_n)$ in time $O(f(n))$.

Give an algorithm that implements the payments (identified in part (b)) by invoking this black box along with the Big Oh in terms of $f(n)$ and n.

Knapsack Auctions

Problem 2. (Roughgarden) Knapsack auctions are another widely applicable example of single-parameter mechanisms.

In a knapsack auction, each bidder i has a **publicly known** size w_i and a private valuation v_i. The seller has a capacity W. A feasible allocation X is defined as the 0-1 vectors $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} w_i \cdot x_i \leq W$. (As usual, $x_i = 1$ indicates that i is a winning bidder.) The goal is to design allocation and payment rules as to (a) maximize surplus $\sum_{i=1}^{n} x_i v_i$ subject to the capacity constraints, and (b) elicit truthful bids (DSIC).

Knapsack auctions come up whenever there is a shared resource with limited capacity. For example, each bidder's size could represent the duration of an advertisement, the valuation their willingness-to-pay for its ad being shown during the Super Bowl, and the seller capacity the length of a commercial break.

Unfortunately, problem of finding an allocation that maximizes surplus (the well-known Knapsack problem) is NP-hard. Thus, we resort to approximation algorithms.

Consider the following greedy scheme:

- Sort and relabel bidders such that:

$$\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \ldots \geq \frac{b_n}{w_n}$$

- Allocate items in that order until there is no room left in the Knapsack.

(a) Show that this allocation scheme is not a 2-approximation by producing an example where it fails to achieve 50% of the optimal surplus.

[2] You should assume that there is at least one such feasible allocation.

(b) Consider the following modified greedy algorithm:

- Let $i_{\max}$ be the single job with the highest bid $b_{\max}$.
- Compute the allocation generated by the above greedy scheme, let Q be the bidders it would allocate to, and let $S_1 = \sum_{i \in Q} b_i$ be the social welfare generated by this allocation.
- If $S_1 > b_{\max}$, then allocate to bidders in Q, otherwise allocate to $i_{\max}$.

Show that this modified scheme is a 2-approximation: that is, if OPT is the maximum surplus possible by any feasible allocation, then this greedy allocation scheme generates surplus at least $1/2 \cdot \text{OPT}$. (*Hint.* Show that $S_1 + b_{\max} \geq \text{OPT}$.)

(c) We would like to use the 2-approximation algorithm to design a DSIC mechanism. Myerson's lemma tells us that a allocation can be made DSIC iff it is monotone. Argue that the greedy 2-approximation scheme proposed in (b) is monotone.

(d) Finally, we would like to apply Myerson's lemma to compute payments that result in a DSIC mechanism. We do so through examples. Compute the greedy allocation (using the 2-approximation algorithm) and the payment of each bidder equal to their "critical bid" for the following two cases:

- Consider four bidders, capacity $W = 5$, and with the following (bid, size): $(\$5, 2), (\$6, 1), (\$6, 3), (\$12, 5)$. What is the allocation and payments?
- Now consider the input from above and suppose bidder 2 changes their bid to 8, that is, $(\$5, 2), (\$8, 1), (\$6, 3), (\$12, 5)$. Assuming the same capacity $W = 5$, what is the new allocation and corresponding payments.

(e) (*Extra-credit; optional*) Can you give a general expression/method to compute Myerson payments for the Knapsack auction based on the greedy 2-approximation.

Optional Feedback Question

Getting a pulse of the class: choose all that apply (or supply your own option):

(a) I am regretting taking a theory course

(b) Analyzing strategic behavior is cool!

(c) Challenging but manageable

(d) Other (please specify)

Acknowledgment. Remember to cite your collaborators and resources in the acknowledgment section of the solution template.