CS 357: Algorithmic Game Theory **Spring 2022**

Assignment 2 (due 02/17/2022)

Instructor: Shikha Singh *Solution Template*

Instructions. Points will be awarded for clarity, correctness and completeness of the answers. Reasoning must be provided with every answer, i.e., please show your work. All assignments are due **at 11 pm EST** on the day of the deadline.[1]

Topics and Readings.

- Mixed-Nash equilibrium and best-response correspondence: SP Section 2.4. For a lighter read, see EK Section 6.7.
- Mechanism design basics and auctions: R Ch 2. For a ligher read, see EK Section 9.4.

Mixed-strategy Nash equilibrium

Problem 1. Jean-Jacques Rousseau describes a social dilemma game that can arise during a stag hunt. The dilemma is described as follows: a group of hunters have tracked a large stag, and are waiting for it to re-appear. It could be a long wait—hours may go by without the stag reappearing. In the meantime, a hare appears. A single hunter can kill and eat a hare. Eating a hare only feeds one, destroys the stag-trap and the others go hungry. Thus, each hunter faces a dilemma: should they continue to wait for the stag, thereby cooperating with the others, or should they be selfish by killing and eating the hare on their own?

In the two-player version, Ali (row player) and Beth (column player) are hunters planning a joint expedition. Each must decide independently whether to hunt the stag or the hare. Neither can kill the stag alone, but both prefer to hunt the stag if the other does. Alternatively, either can kill the hare alone. If one hunts the stag and the other the hare, then the one who hunts the stag goes home hungry, while the other enjoys the hare. The payoff matrix is shown below.

	Stag	Hare
Stag	$4,4$	$0,3$
Hare	$3,0$	$3,3$

(a) Compute the mixed Nash equilibrium of this game, and draw the best response correspondences for each player on the same graph, similar to the BoS example in Lecture 3. (You may attach a clear picture of a hand-drawn graph; see instructions on including images on Overleaf.) Label all Nash equilibria of the game that involve pure and mixed strategies.

[1] I moved the deadline from 10pm to 11pm due to TA room scheduling issues.

(b) Which of the pure-strategy Nash equilibria are Pareto-optimal among pure strategy profiles?

Sealed-Bid Auctions

Problem 2. An auction is *dominant-strategy incentive compatible* (DSIC) if truthful bidding is always a dominant strategy for every bidder.

Consider a sealed-bid single-item auction with at least three bidders. Prove that awarding the item to the highest bidder, at a price equal to the third-highest bid, yields an auction that is not DSIC.

Problem 3. (Collusion) In class, we showed that in the second-price auction, no bidder has an incentive to deviate from bidding truthfully, regardless of the other bidders' behaviors. But the auction is susceptible to *collusion*, that is, a situation where a two or more bidders submit their bids in a coordinated fashion.

Consider a single-item second-price auction with $N = \{1, 2, \ldots, n\}$ bidders in which all but a subset $S \subseteq N$ of the bidders bid truthfully. The members of S attempt to collude to increase their total utility $\sum_{j \in S} u_j$.

State and prove necessary and sufficient conditions on the valuations of the bidders in S (relative to the others) such that they can increase their total utility by bidding untruthfully.

Multi-Roung Auctions

Many traditional auction-houses, as well as eBay, run multi-round auctions that allow bidders to respond to bids placed by others. Such auctions provide bidders with new kinds of strategically relevant information—they may be able to observe the bids of others and use it to revise their own.

Two common formats for such auctions are: *ascending clock* auctions, where the ask price increases across rounds, and *descending clock* auctions, where the ask price decreases across rounds. Consider the following **ascending-clock auction**: the ask price starts low and increases continuously. Each bidder can drop out at any price and the auction closes when only one bidder remains. The bidder wins, and pays the ask price at which the last competitor dropped out.

This auction seems very similar to the sealed-bid second-price (SBSP) auction. However, how the ascending-clock auction is implemented—in particular, what information is visible to other bidders is crucial in formally comparing them. Recall that in a SBSP auction player values are their private information, and thus each player's strategy s_i only depends on their own value v_i. On the other hand, in any multi-round auction, their strategy can map any information available to them to their action.

We say that two auctions are **strategically equivalent** if, for any strategy profile in one auction, there exists a strategy profile in the other auction such that the outcomes (allocations and payments) are the same, for all value profiles (and vice versa).

Problem 4. Below, we compare the multi-round auctions to second-price auctions from the point of view of how bidders should participate in them.

(a) First, consider the ascending-clock auction where bidders can drop out privately (unobserved by other bidders). Show that the SPSB auction is strategically equivalent to the ascending-clock auction described above.

(b) Now consider an ascending-clock auction where a bidder can observe the drop-out points of other bidders (e.g., each bidder who is still in the game has their hand raised). Is this variant also strategically equivalent to a SBSP auction?

To help understand this, we will establish a property about **winner determination** in SBSP auctions. Fix a strategy profile s of the bidders. Let $\overline{v_1} = (v_i, v_{-i})$ and $\overline{v_2} = (v'_i, v_{-i})$ be two valuation profiles that only differ in bidder i's value ($v_i \neq v'_i$). Let w_1 be the winner of SBSP when bidders bid under $\overline{v_1}$ and w_2 when bidders bid under $\overline{v_2}$ (using the same strategy profile s). Show that in a SBSP auction, a change in a bidders value can only affect their own allocation. In particular, show that either the winner is unaffected: $w_1 = w_2$ or i is be the winner in one of them: $w_1 = i$ or $w_2 = i$.

(c) Using part (b), show that the ascending-clock auction where a bidder can observe the bids of others is **not strategically** equivalent to a SBSP auction. (*Hint.* Give a strategy profile for which the allocation cannot be achieved in any strategy in the SPSB auction.)

(d) (*Extra credit; optional; open ended*) There might be an alternate direct method to prove part (c): that is, to show that the two auctions are not strategically equivalent without using part (b). If you come up with one, describe it here.