CSCI 334: Principles of Programming Languages

Lecture 9: Computability, part 2

Instructor: Dan Barowy

Williams

Topics

Garbage collection

Halting problem

Reduction proofs

Your to-dos

1. Lab 5, **due Sunday 10/16** (partner lab)
2. Review quiz solutions if you haven't already…

Announcements

- **Field trip to WCMA**, Tuesday, Oct 18.
- **Midterm exam**, in class, Thursday, Nov 3.
- Colloquium: **What I Did Last Summer (Research)**, 2:35pm in Wege Auditorium with **cookies**.

Mountain Day, whenever that is…

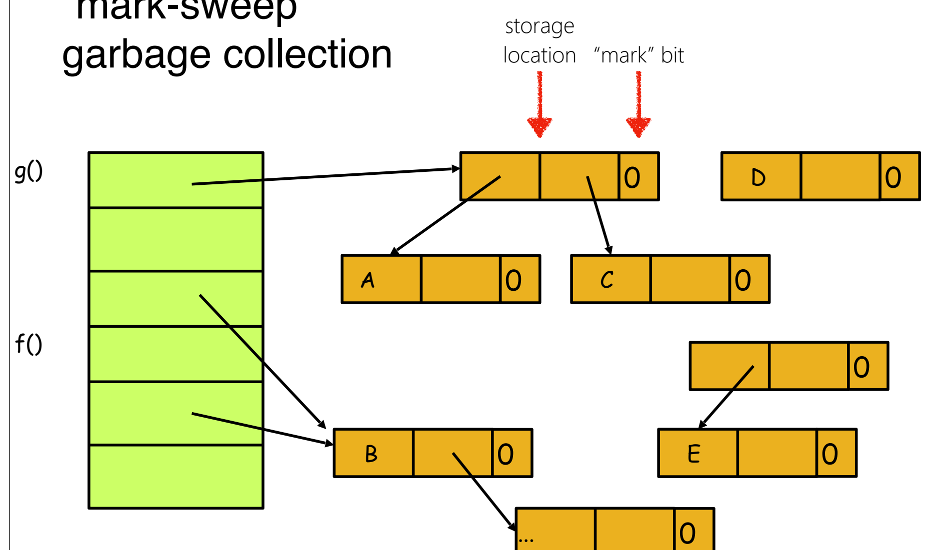
- Rescheduled office hours (faculty "retreat")

Garbage collection

A **garbage collection algorithm** is an algorithm that determines whether the storage, occupied by a value used in a program, **can be reclaimed for future use**. Garbage collection algorithms are often tightly integrated into a programming language **runtime**.

John McCarthy

"mark-sweep" garbage collection



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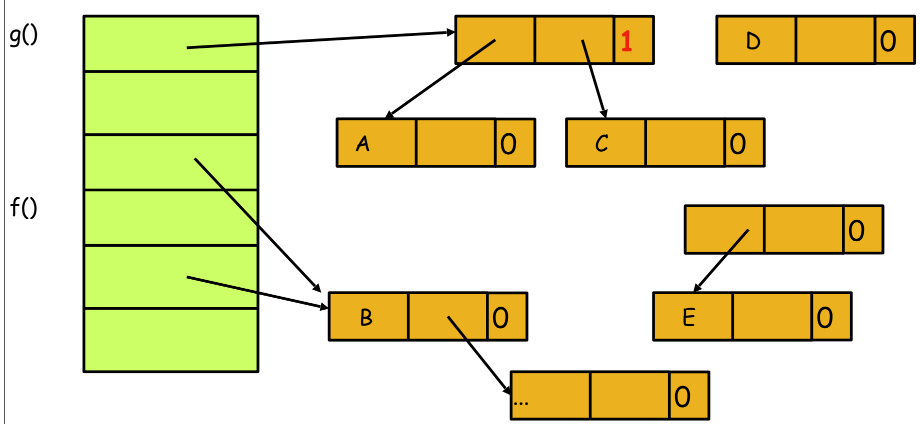

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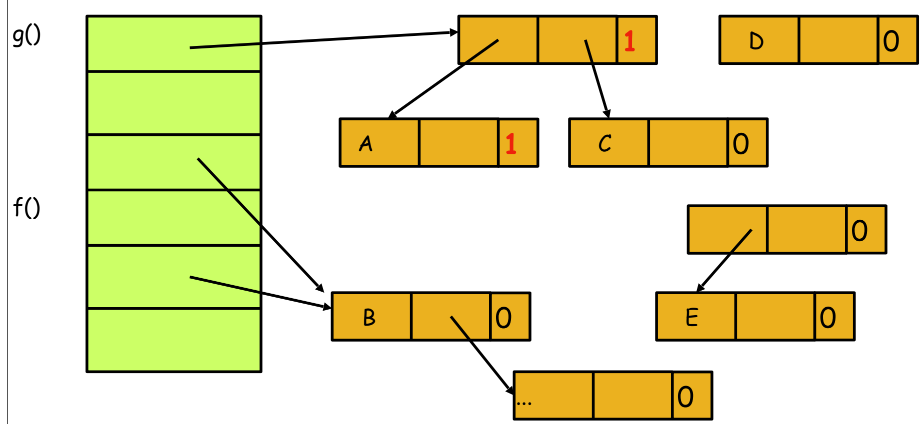

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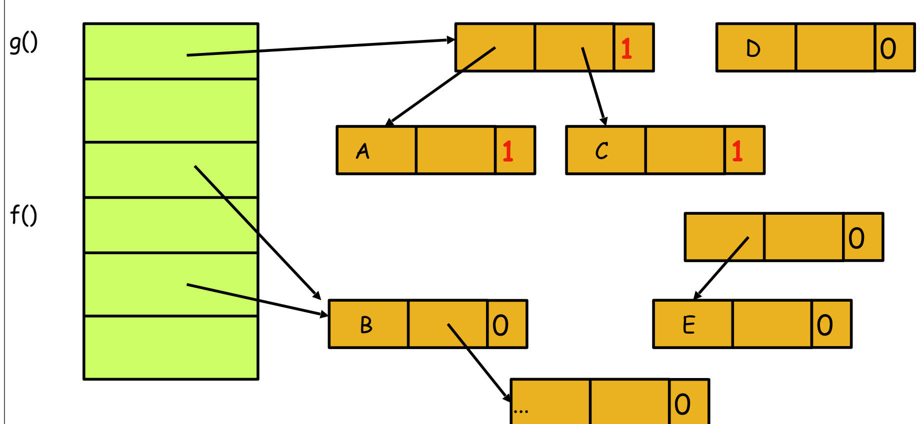

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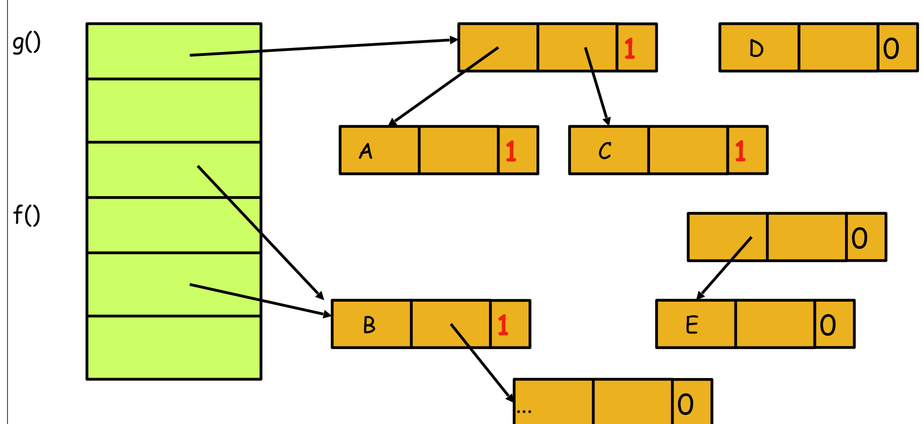

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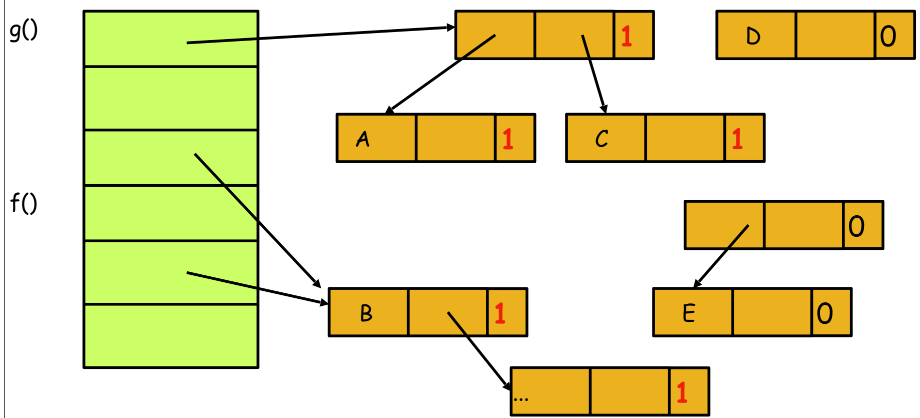
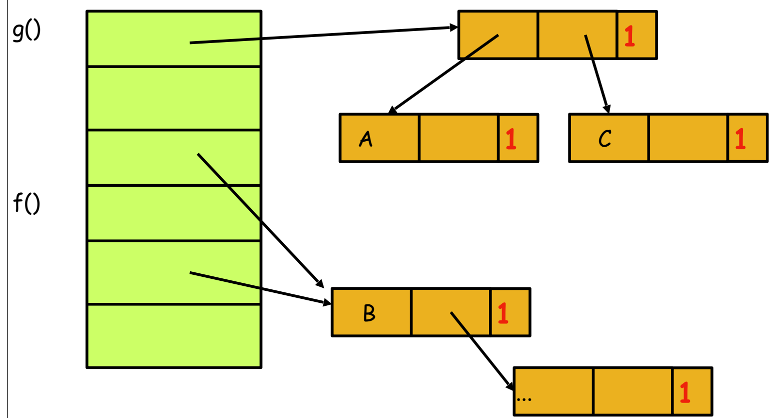
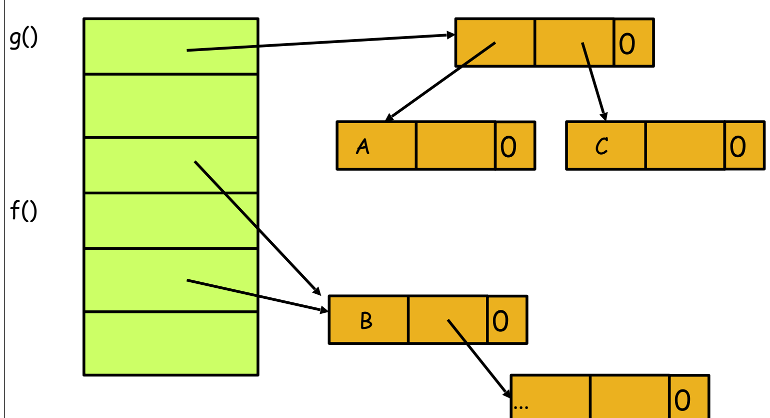


2. Free ("sweep") unreachable cells



3. Clear tags



Decidability Problems

A **decidability problem** is a question with a **yes** or **no** answer about a **particular input**.

"Is x prime?"

In CS, we care about whether there is an **algorithm** for solving decidability problems.

If there is **no algorithm**, then the problem is **undecidable**.

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The Halting Problem

Decide whether program **P** halts on input **x**.

Given program P and input x,

`Halt(P,x)` = { returns `true` if `P(x)` halts; returns `false` otherwise

How might this work?

Clarifications:

`P(x)` is the output of program `P` run on input `x`.

The type of `x` does not matter; assume `string`.

The Halting Problem

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How might this work?

Fact: it is provably impossible to write `Halt`

Notes on the proof

We utilize two key ideas:

- Function **evaluation by substitution**
- **Reductio ad absurdum** (proof form)

Notes on the proof

The *form* of the proof is ***reductio ad absurdum***.

Literally: "reduction to absurdity".

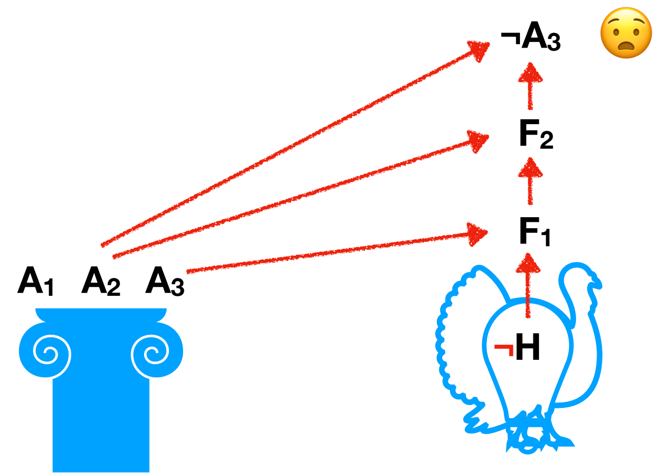
Start with **axioms** and **presuppose the outcome** we want to show.

Then, following strict rules of logic, **derive new facts**.

Finally, derive a fact that **contradicts** another fact.

Conclusion: the **presupposition must be false**.

Reductio ad Absurdum



Function Evaluation by Substitution

```
def addone(x):
   return x + 1
```

`addone(1)`	`λx.(+ x 1)1`
`[1/x]x + 1`	`[1/x](+ x 1)`
`1 + 1`	`(+ 1 1)`
`2`	`2`

The Halting Problem

Notes on the proof:

The proof relies on the kind of **substitution** that we've been using to "compute" functions in the lambda calculus.

Remember: **we are looking to produce a contradiction**.

The proof is hard to "understand" because the facts it derives **don't actually make sense**. Don't read too deeply.

The Halting Problem: Proof

Suppose:

`Halt(P,x)` =
- returns `true` if `P(x)` halts
- returns `false` otherwise

`Halt` always halts!

Construct:

`DNH` does not always halt!

`DNH(P)` =
- if `Halt(P,P)` is `true`, `while(1){}`
- returns `false` otherwise

The Halting Problem: Proof

Observations so far:

`DNH(P)` will run forever if `Halt(P,P)` is `true`.
`DNH(P)` will halt if `Halt(P,P)` is `false`.

Rewrite:

`DNH(P)` =
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Rewrite:

`DNH(P)` =
- if `P(P)` halts, run forever
- halt

The Halting Problem: Proof

Observations so far:

`DNH(P)` will run forever if `P(P)` halts.
`DNH(P)` will halt if `P(P)` runs forever.

Rewrite:

`DNH(P)` = { if `P(P)` halts, run forever; halt

The Halting Problem

Isn't `DNH` itself a program?

What happens if we call `DNH(DNH)`?

`P = DNH`

`DNH( P )` will run forever if `P ( P )` halts.
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`P = DNH`

`DNH(DNH)` will run forever if `DNH(DNH)` halts.
`DNH(DNH)` will halt if `DNH(DNH)` runs forever.

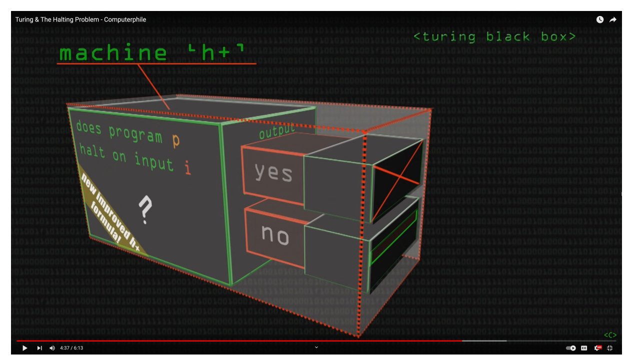
This literally makes no sense. **Contradiction!**

What was our one assumption? `Halt` **exists**.

Therefore, the `Halt` function **cannot exist**.

Need more explanation?

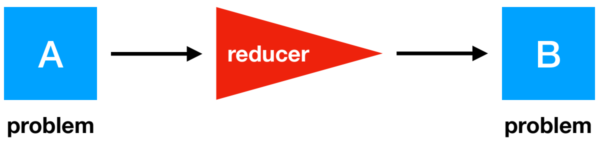
Watch this!



https://youtu.be/macM_MtS_w4

Reductions

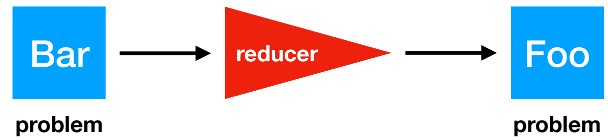
A **reduction** is an **algorithm** that transforms an instance of one problem into an instance of another. Reductions are often **employed to prove something** about a problem given a similar problem.



Reductions

Reductions are often used in a **counterintuitive** way.

For example, if we **want to know whether problem Foo is impossible**, we assume Foo is possible, and then use that fact to show that problem Bar (which **we already know** to be impossible) **appears to be possible**.

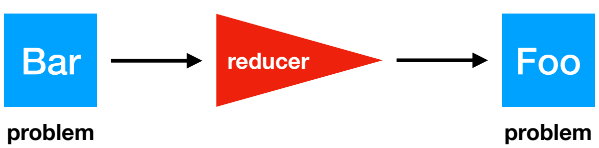


The above is a **contradiction**, meaning that **Foo is not possible**.

Reductions

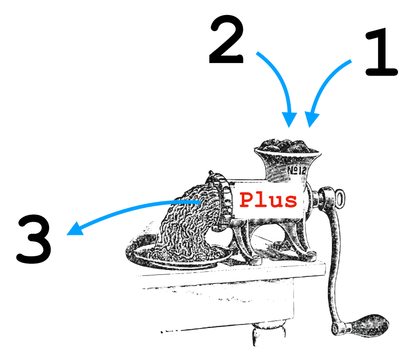
An important part of a reduction is that the reducer be an **ordinary algorithm**.

The reducer **should not solve the problem**. A reducer just converts problems from one form to another.



You will get **a lot** more exposure to reductions in CSCI 361.

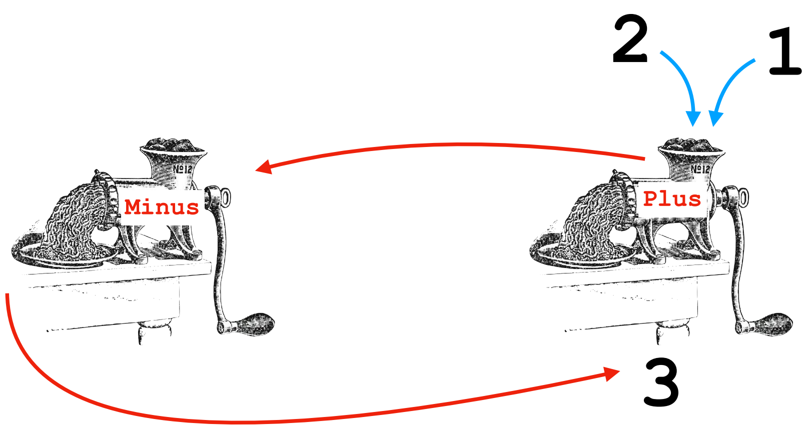
Reductions



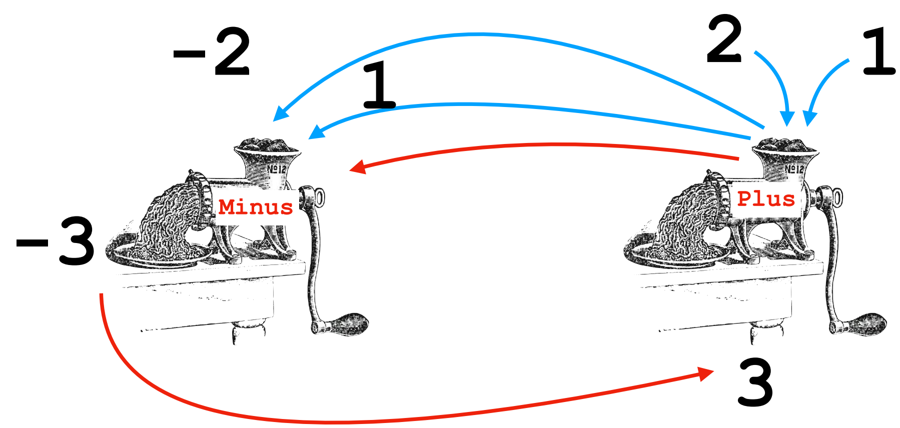
The humble algorithm.

(sorry, vegetarians)

Reductions

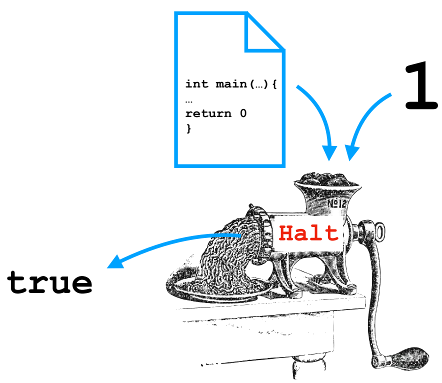


Reductions



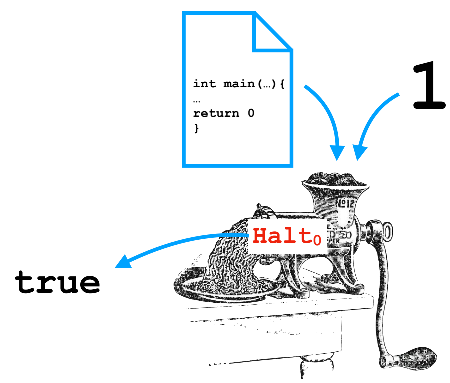

```
let reducer(x: int)(y: int) = -(-x-y)
```

Reductions



We **know** that `Halt` is not computable.

Reductions



A function `f(i)` **halts not** if and only if `f` **does not halt** on input `i`.

Is $\texttt{Halt}_0$ computable?

Reductions

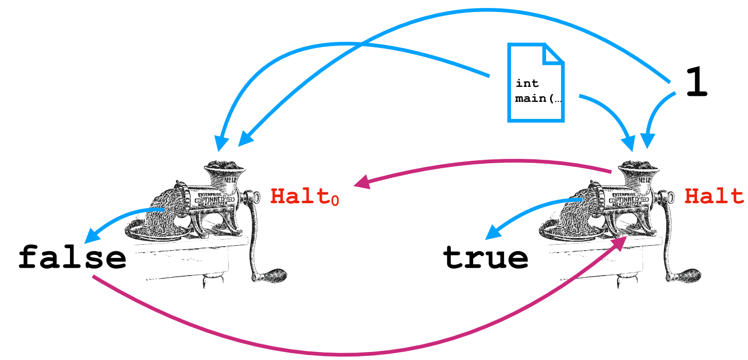
A function `f(i)` **halts not** if and only if `f` **does not halt** on input `i`.

If $\texttt{Halt}_0$ is computable, couldn't we do this?

Assume that $\texttt{Halt}_0$ is computable.
(e.g., it's in your standard library)

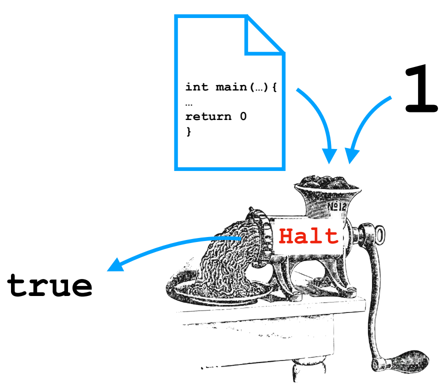
```
def halt(f, i):
  return not halt_0(f, i);
```

Reductions



Reduction: **Construct** `Halt` using $\texttt{Halt}_0$.

Reductions



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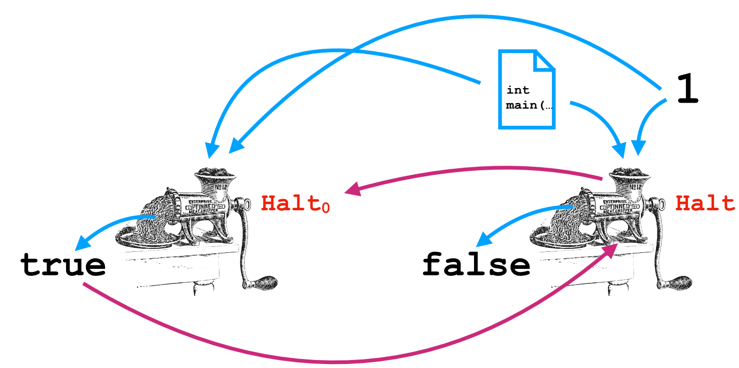
Reductions



If we can build this new machine,
what does that mean for $\texttt{Halt}_0$?

$\texttt{Halt}_0$ is **not computable**.

Reductions

We can use the Halting Problem to show that other problems cannot be solved **by reduction** to the Halting Problem.

We cannot tell, in general…

… if a program will **run forever**.

… if a program will **eventually produce an error**.

… if a program **is done using a variable**.

… if a program **is a virus**!

Generality

```
def myprog(x):
  return 0
```

```
def Halt(f,i):
  if(f = "def myprog(x):\n\treturn 0"):
    return true
  else
    return false
```

The Halting Problem is about **an arbitrary program**.

Recap & Next Class

Today:

Halting problem

Reduction proofs

Next class:

WCMA