Network Flows

Admin

- TA evaluation form!
- Sign up to be a TA! (links in email)
  - due Friday!
- Problem set 7 released tonight
- We'll talk in detail about the midterm on Monday
- Questions?

Problem set 4 discussion

- One problem caused a lot of issues
- Let's quickly talk about it
- Takeaway: off-by-one issues can be very important for recursive algorithms!
  - This is real and makes recursive algorithms very difficult to debug
  - The techniques we talk about in this class can help!

Story So Far

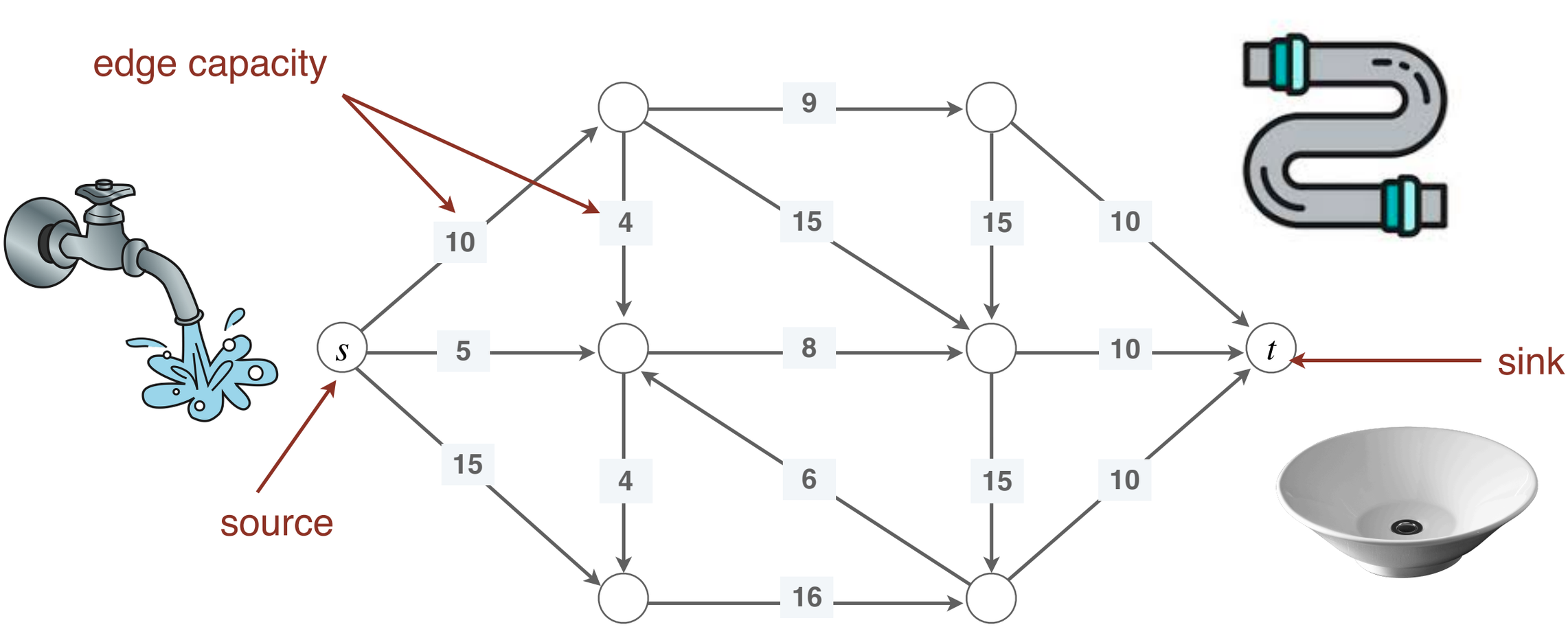
- Algorithmic design paradigms:
    - **Greedy**: simplest to design but works only for certain limited class of optimization problems
        - A good starting point for most problems but rarely optimal
    - **Divide and Conquer**
        - Solving a problem by breaking it down into smaller subproblems and recursing
    - **Dynamic programming**
        - Recursion with memoization: avoiding repeated work
        - Trading off space for time

Network Flows

- Graph-based problem; looks like a lot of what we learned in part 1
- Soon, we'll use what we learn about network flows to solve much more general problems
- Problems where you **revisit*** (and improve) past solutions
- Solve problems that even dynamic programming can't* solve!
- Restricted case of Linear/Convex Programming; "algorithmic power tools"

What's a Flow Network?

- A flow network is a directed graph $G = (V, E)$ with a
  - A **source** is a vertex s with in degree 0
  - A **sink** is a vertex t with out degree 0
  - Each edge $e \in E$ has **edge capacity** $c(e) > 0$



Visualize

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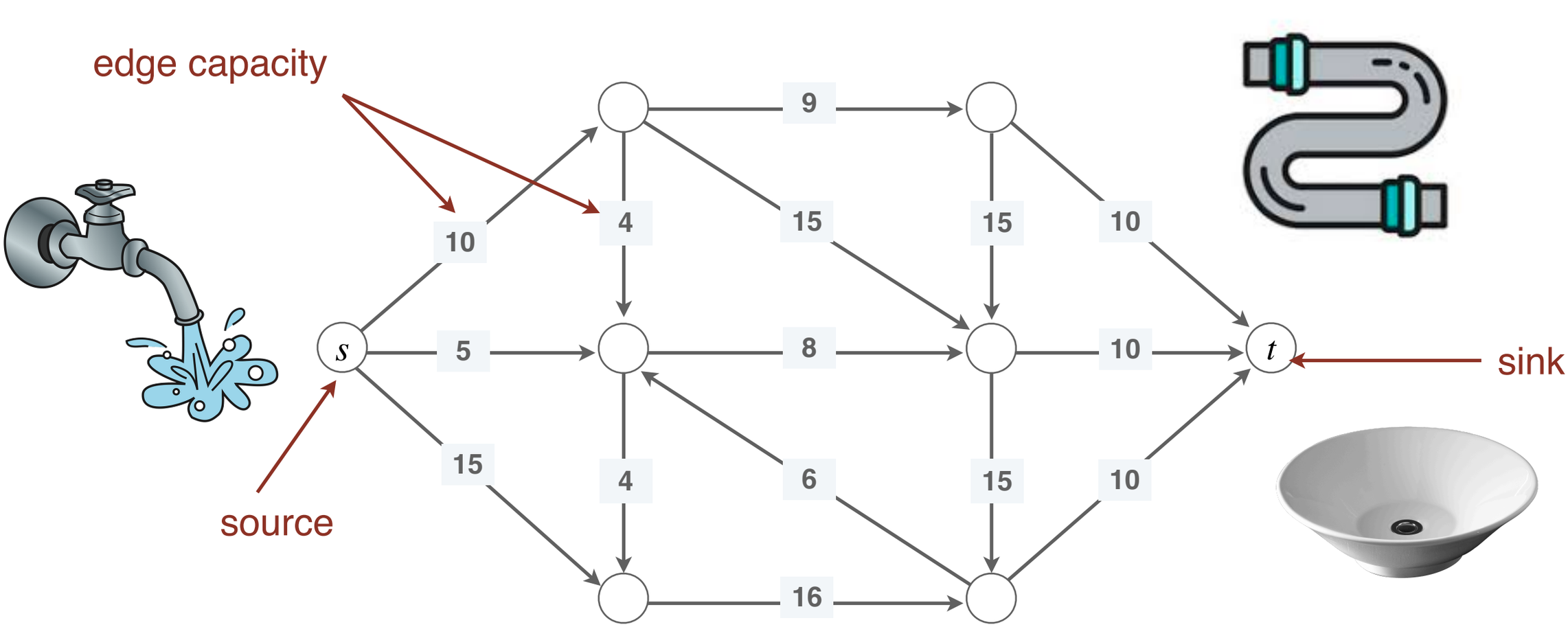


Assumptions

- Assume that each node v is on some s-t path, that is, $s \rightsquigarrow v \rightsquigarrow t$ exists, for any vertex $v \in V$
    - Implies G is connected and $m \geq n - 1$
- Assume **capacities are integers**
    - Will revisit this assumption and what happens if not
- Directed edge (u, v) written as $u \rightarrow v$
- For simplifying expositions, we will sometimes write $c(u \rightarrow v) = 0$ when $(u, v) \notin E$

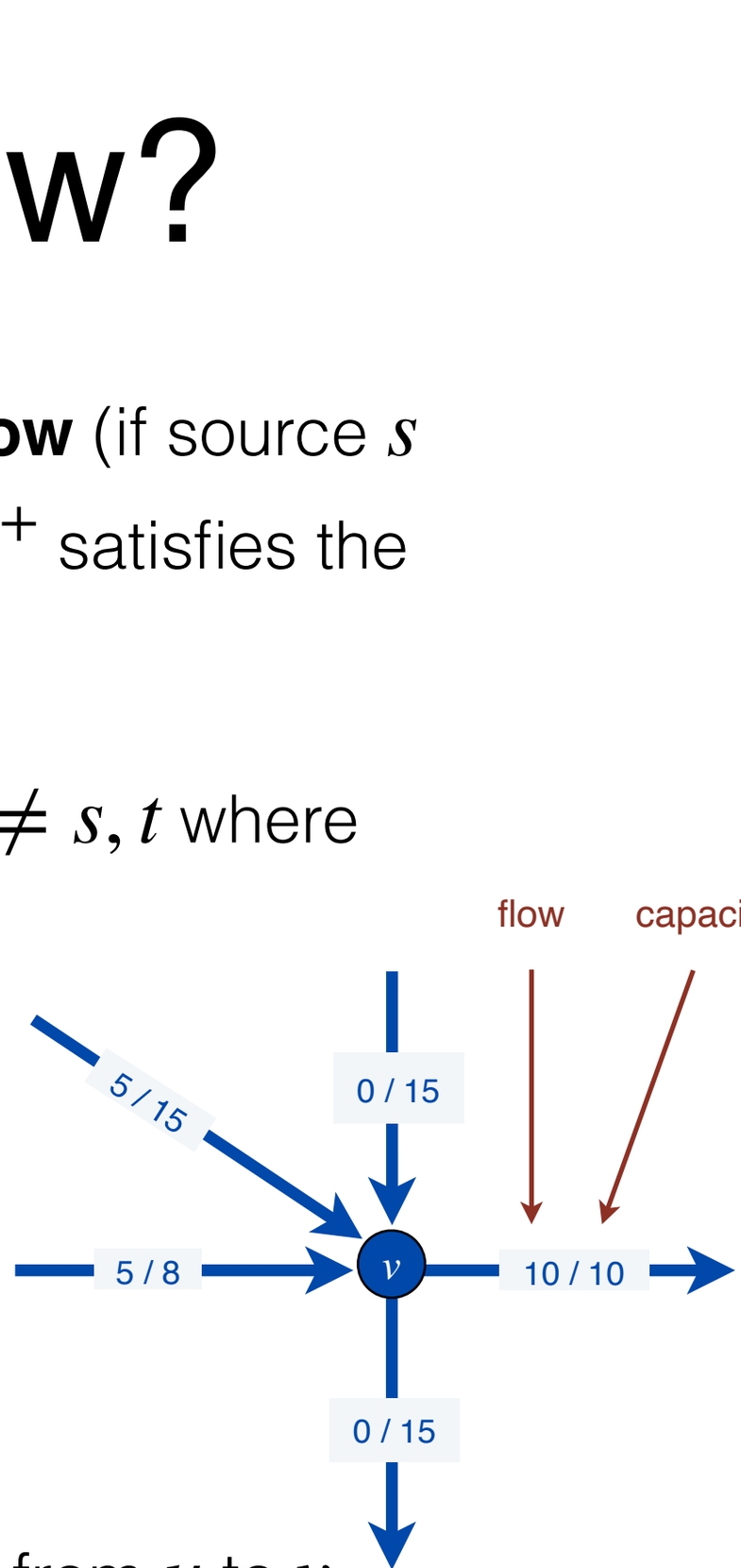
What's a Flow?

- Given a flow network, an (s, t)**-flow** or just **flow** (if source s and sink t are clear from context) $f : E \to \mathbb{Z}^+$ satisfies the following two constraints:

- **[Flow conservation]** $f_{in}(v) = f_{out}(v)$, for $v \neq s, t$ where

$$f_{in}(v) = \sum_{u} f(u \to v)$$

$$f_{out}(v) = \sum_{w} f(v \to w)$$

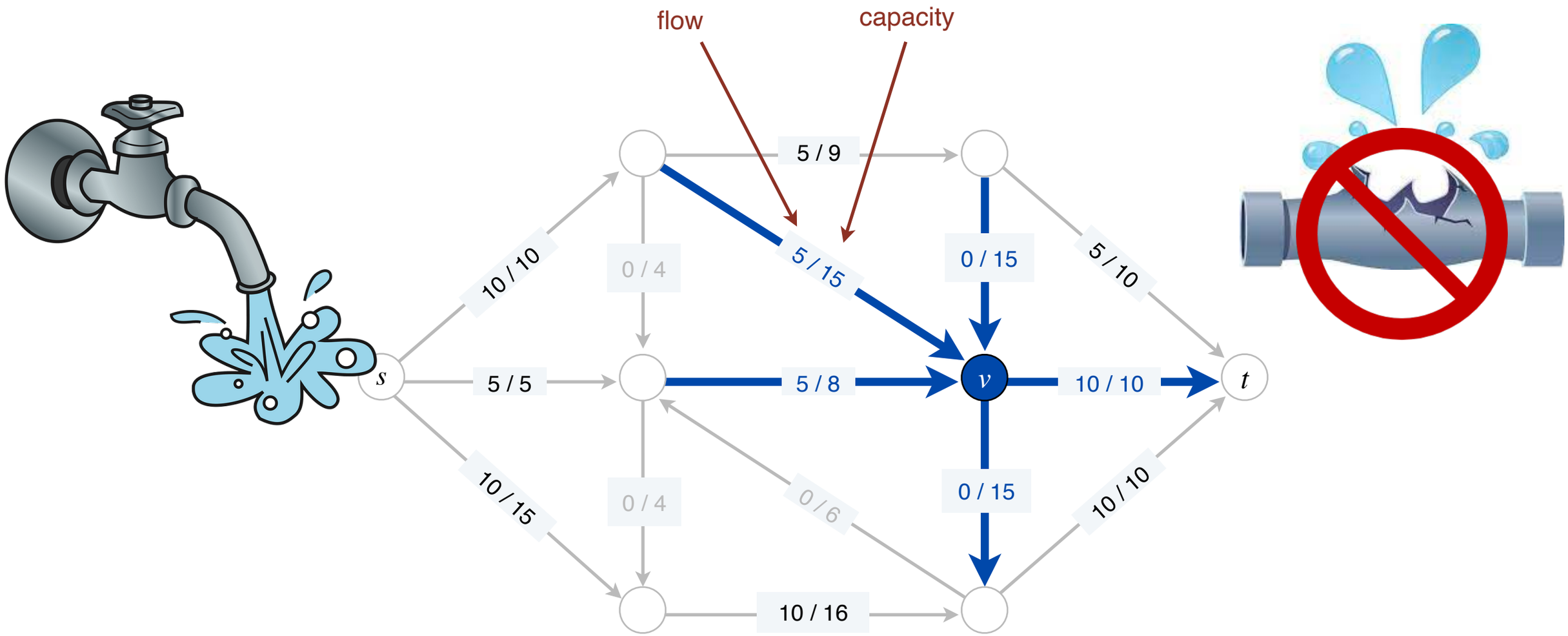


- To simplify, $f(u \to v) = 0$ if there is no edge from u to v

Feasible Flow

- And second, a feasible flow must satisfy the capacity constraints of the network, that is,

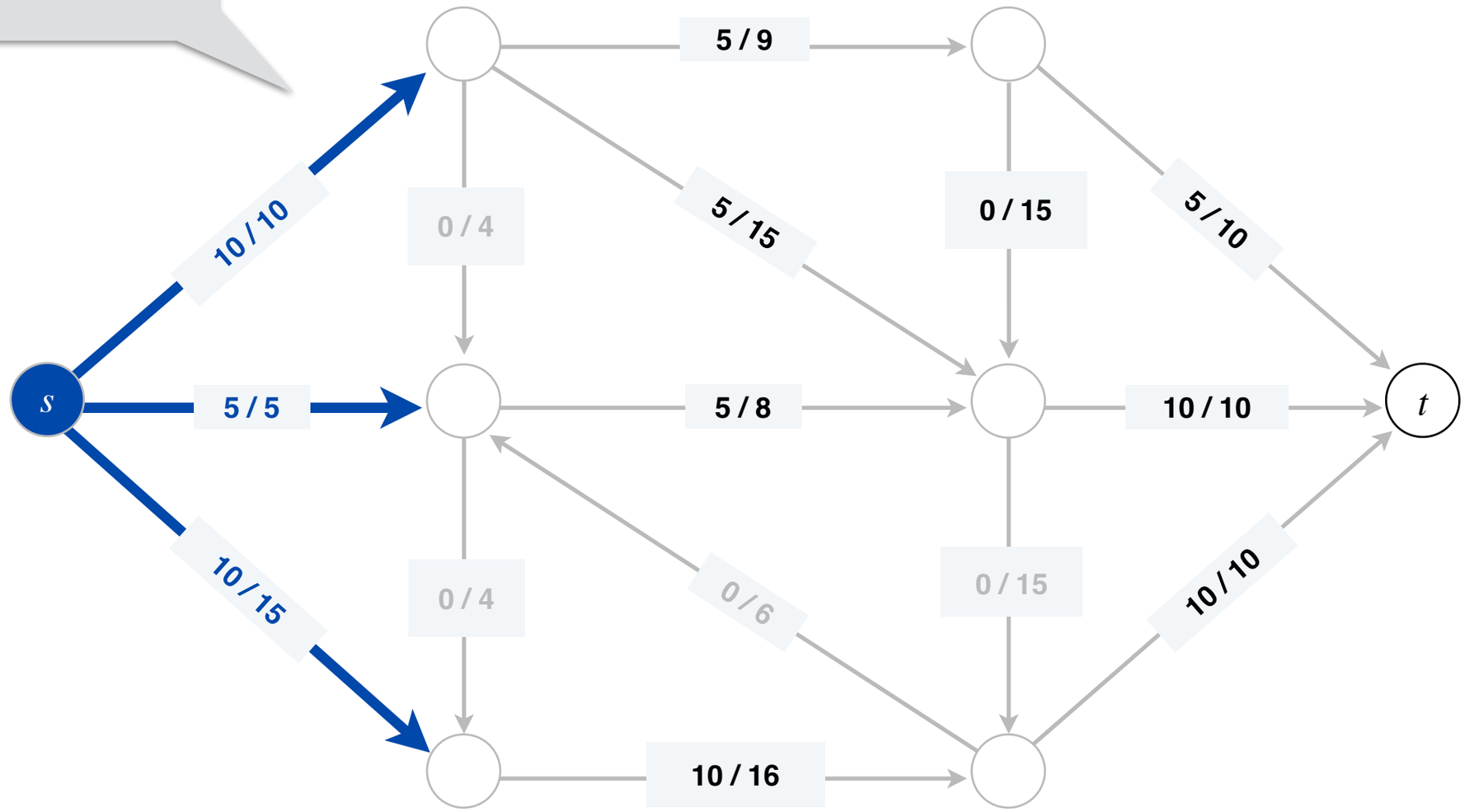
  [Capacity constraint] for each $e \in E$, $0 \leq f(e) \leq c(e)$



Value of a Flow

- **Definition.** The **value** of a flow f, written $v(f)$, is $f_{out}(s)$.

What is $v(f)$ here?

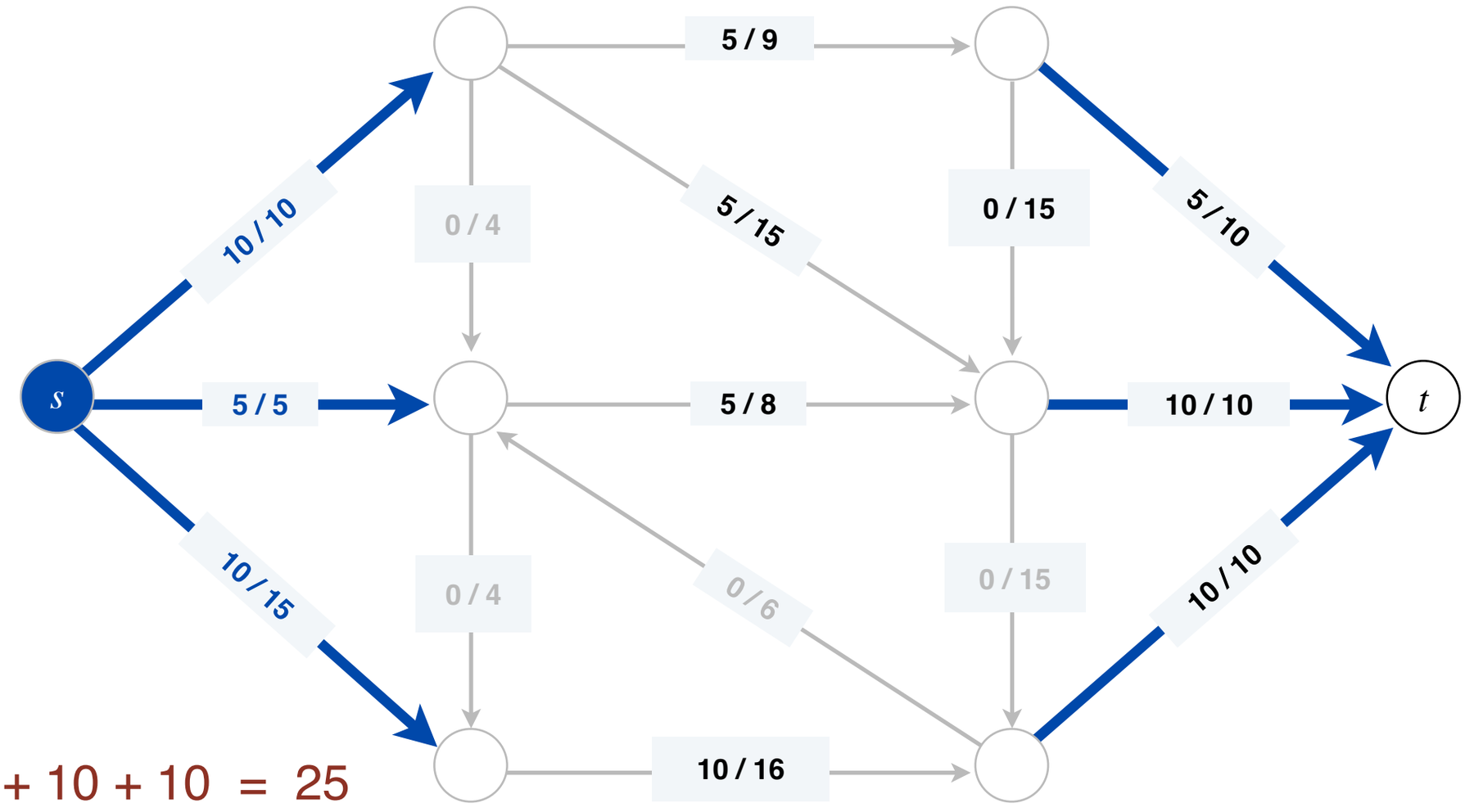


$v(f) = 5 + 10 + 10 = 25$

Value of a Flow

- **Definition.** The **value** of a flow f, written $v(f)$, is $f_{out}(s)$.
- **Lemma**. $f_{out}(s) = f_{in}(t)$

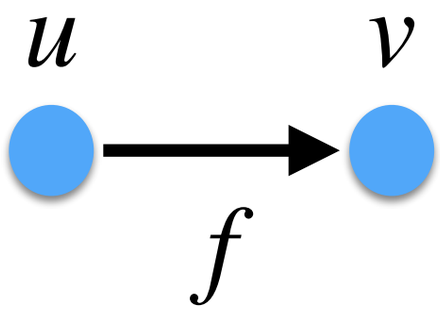
Intuitively, why do you think this is true?



value = 5 + 10 + 10 = 25

Value of a Flow

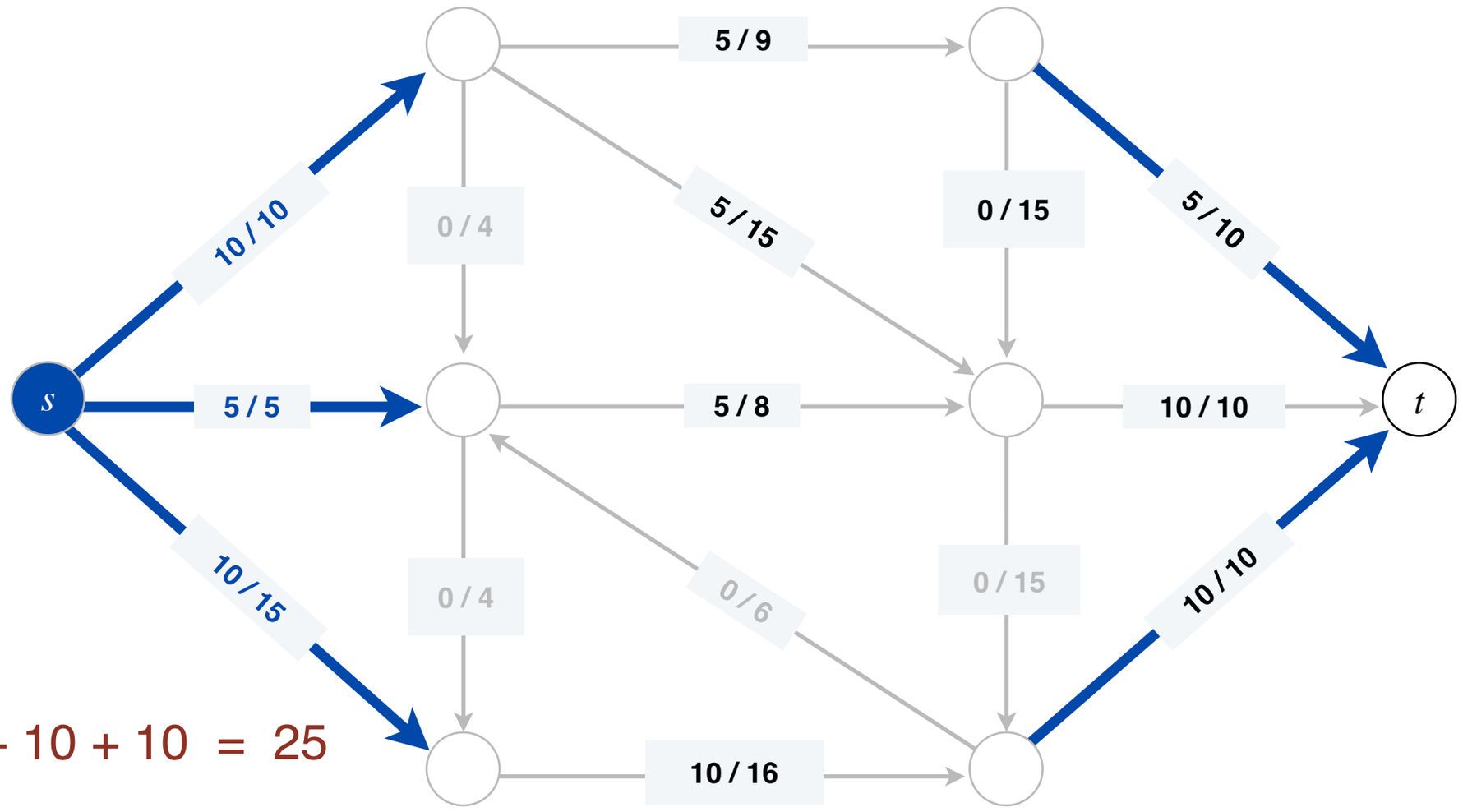
- **Lemma**. $f_{out}(s) = f_{in}(t)$
- **Proof**. Let $f(E) = \sum_{e \in E} f(e)$
  - Then, $\sum_{v \in V} f_{in}(v) = f(E) = \sum_{v \in V} f_{out}(v)$



- For every $v \neq s, t$ flow conversation implies $f_{in}(v) = f_{out}(v)$
- Thus all terms cancel out on both sides except
  $f_{in}(s) + f_{in}(t) = f_{out}(s) + f_{out}(t)$
- But $f_{in}(s) = f_{out}(t) = 0$ ∎

Value of a Flow

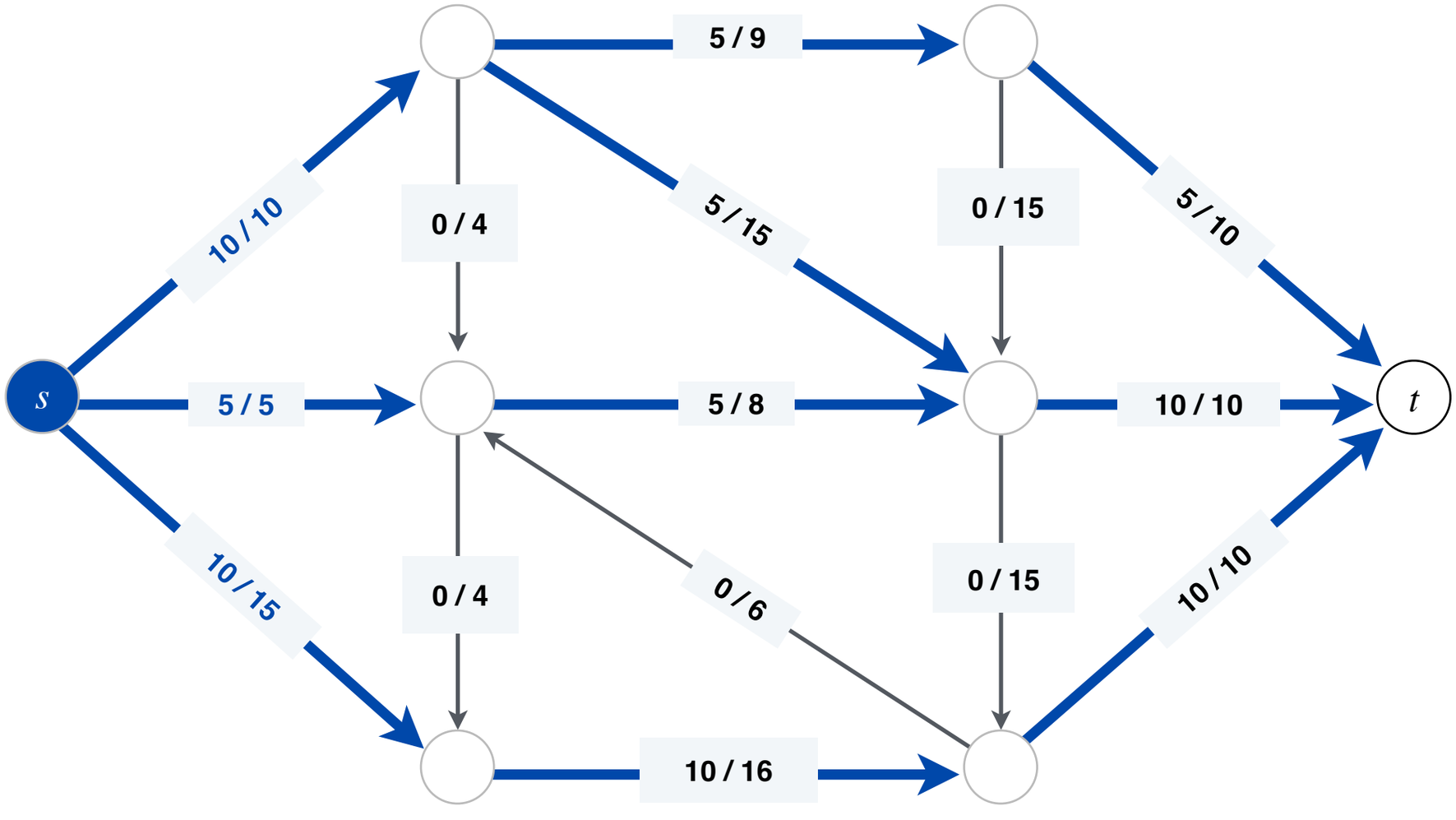
- **Lemma**. $f_{out}(s) = f_{in}(t)$
- **Corollary.** $v(f) = f_{in}(t)$.



value = 5 + 10 + 10 = 25

Max-Flow Problem

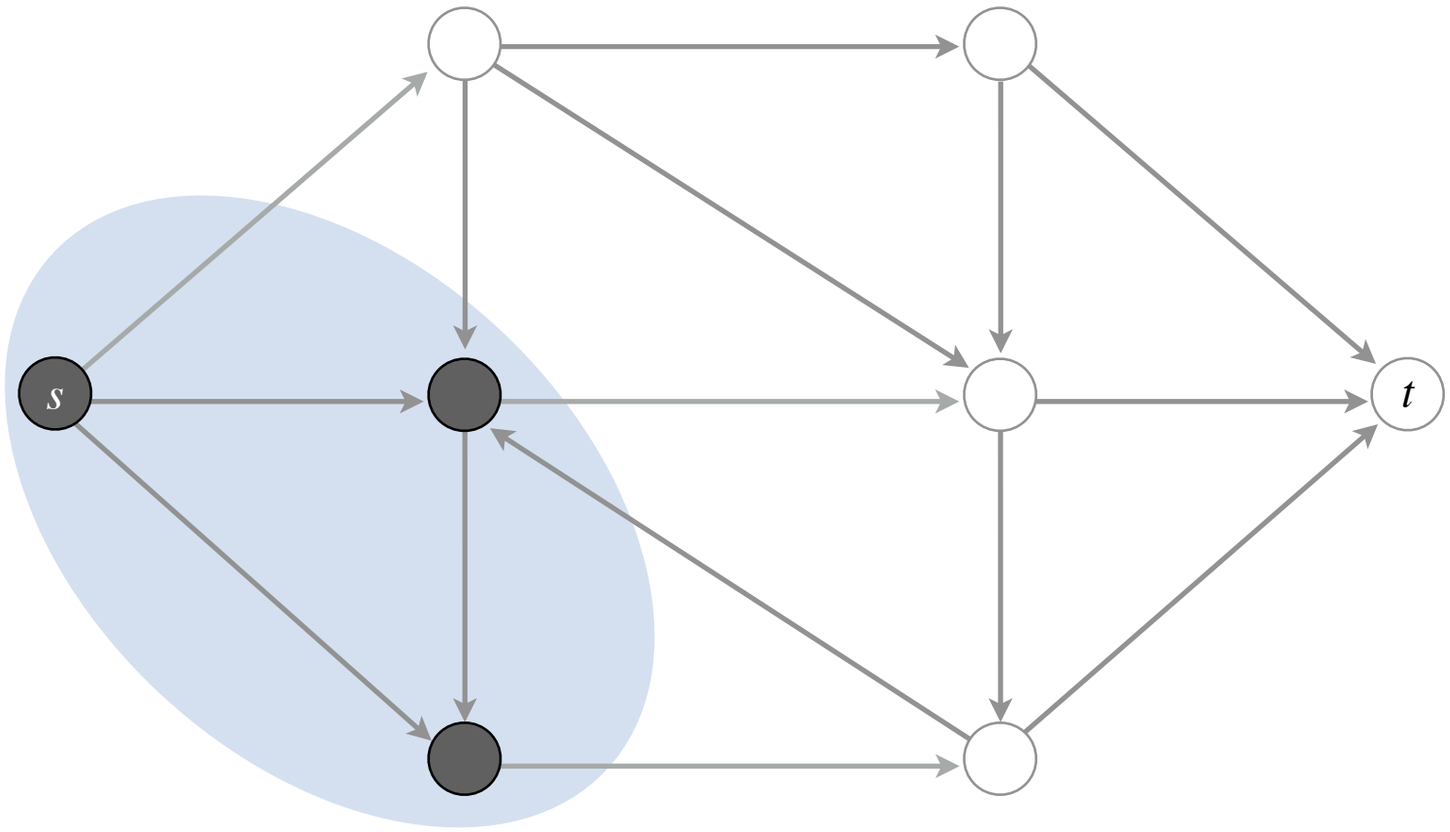
- **Problem**. Given an s-t flow network, find a feasible s-t flow of **maximum** value.



Minimum Cut Problem

Cuts are Back!

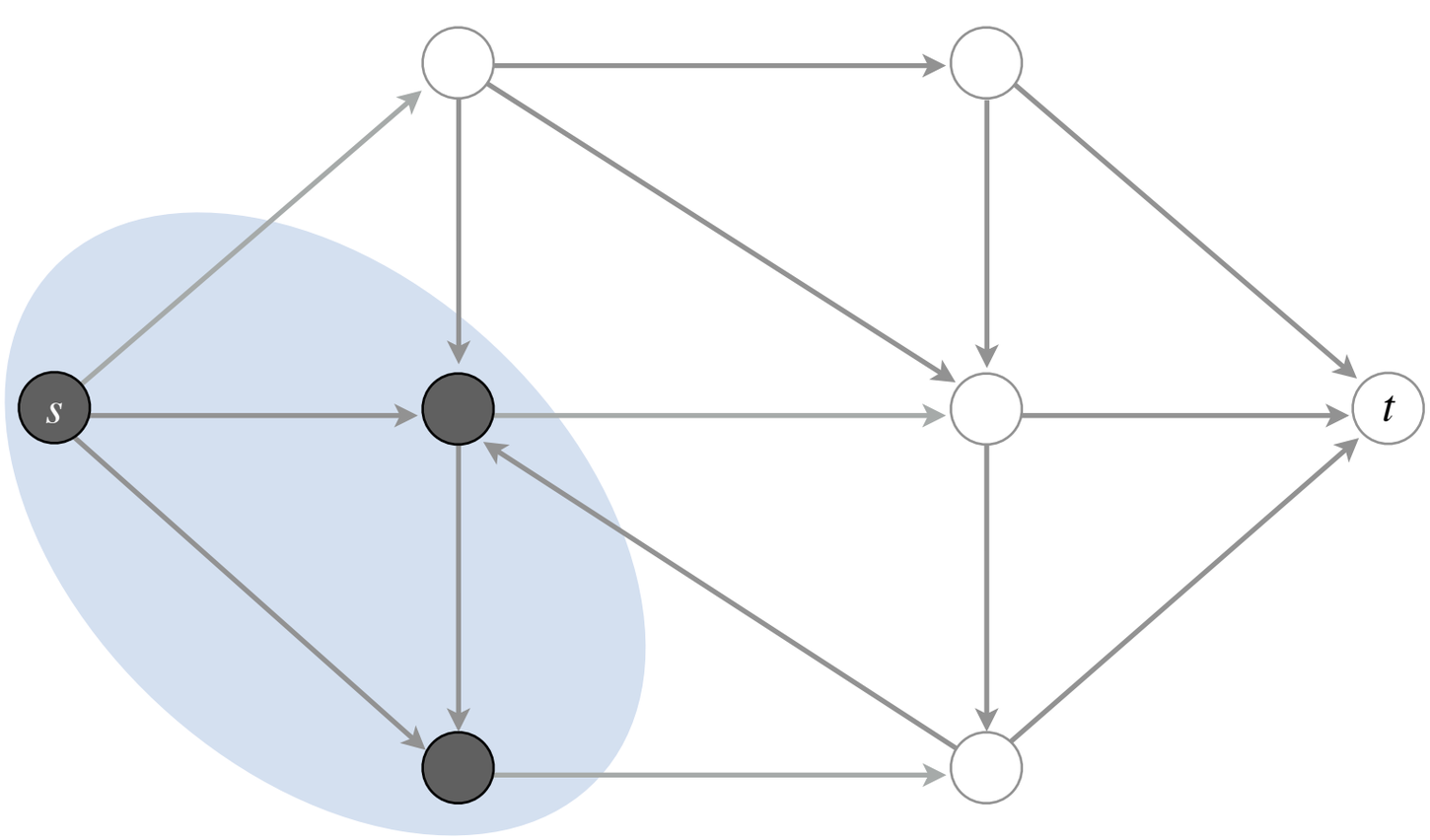
- Cuts in graphs played a lead role when we were designing algorithms for MSTs
- What is the definition of a cut?



Cuts in Flow Networks

- **Recall**. A cut (S, T) in a graph is a partition of vertices such that $S \cup T = V$, $S \cap T = \emptyset$ and S, T are non-empty.

- **Definition**. An (s, t)-*cut* is a cut (S, T) s.t. $s \in S$ and $t \in T$.



Cut Capacity

- **Recall**. A cut (S, T) in a graph is a partition of vertices such that $S \cup T = V$, $S \cap T = \emptyset$ and S, T are non-empty.
- **Definition**. An (s, t)-*cut* is a cut (S, T) s.t. $s \in S$ and $t \in T$.
- **Capacity** of a (s, t)-cut (S, T) is the sum of the capacities of edges leaving S:
  - $$c(S, T) = \sum_{v \in S, w \in T} c(v \to w)$$

Quick Quiz

Question. What is the capacity of the s-t cut given by grey and white nodes?

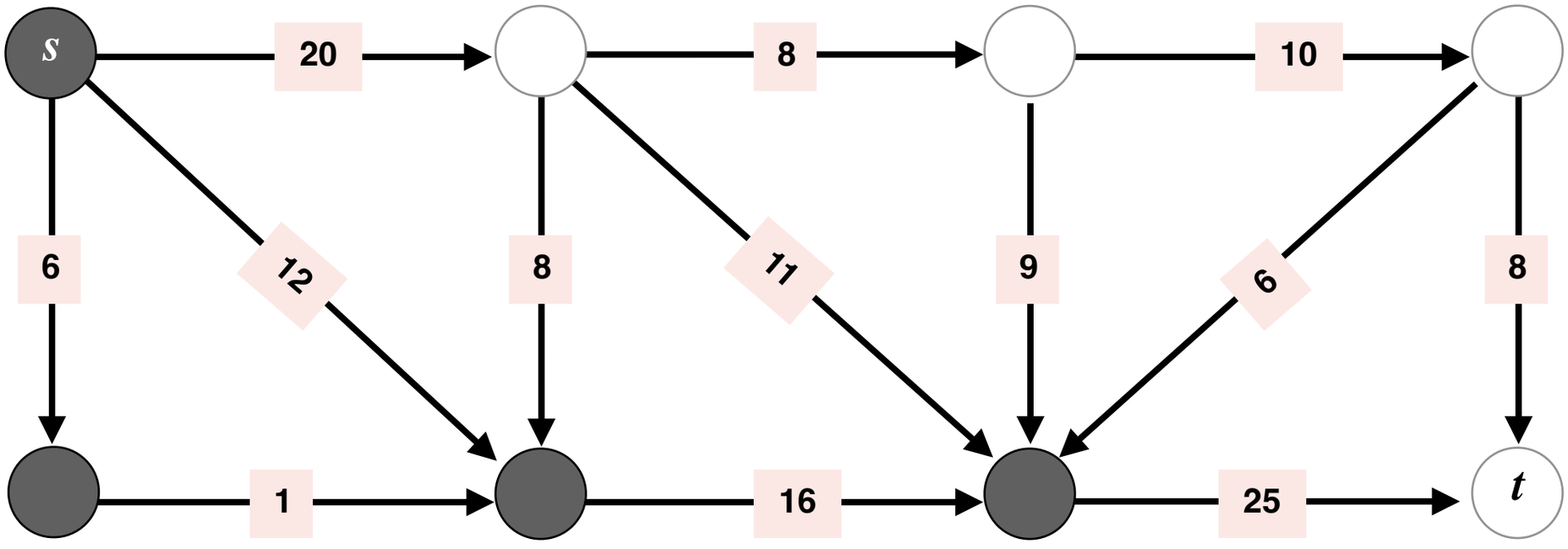
$$c(S,T) = \sum_{v \in S, w \in T} c(v \to w)$$

A. 11 (20 + 25 – 8 – 11 – 9 – 6)

B. 34 (8 + 11 + 9 + 6)

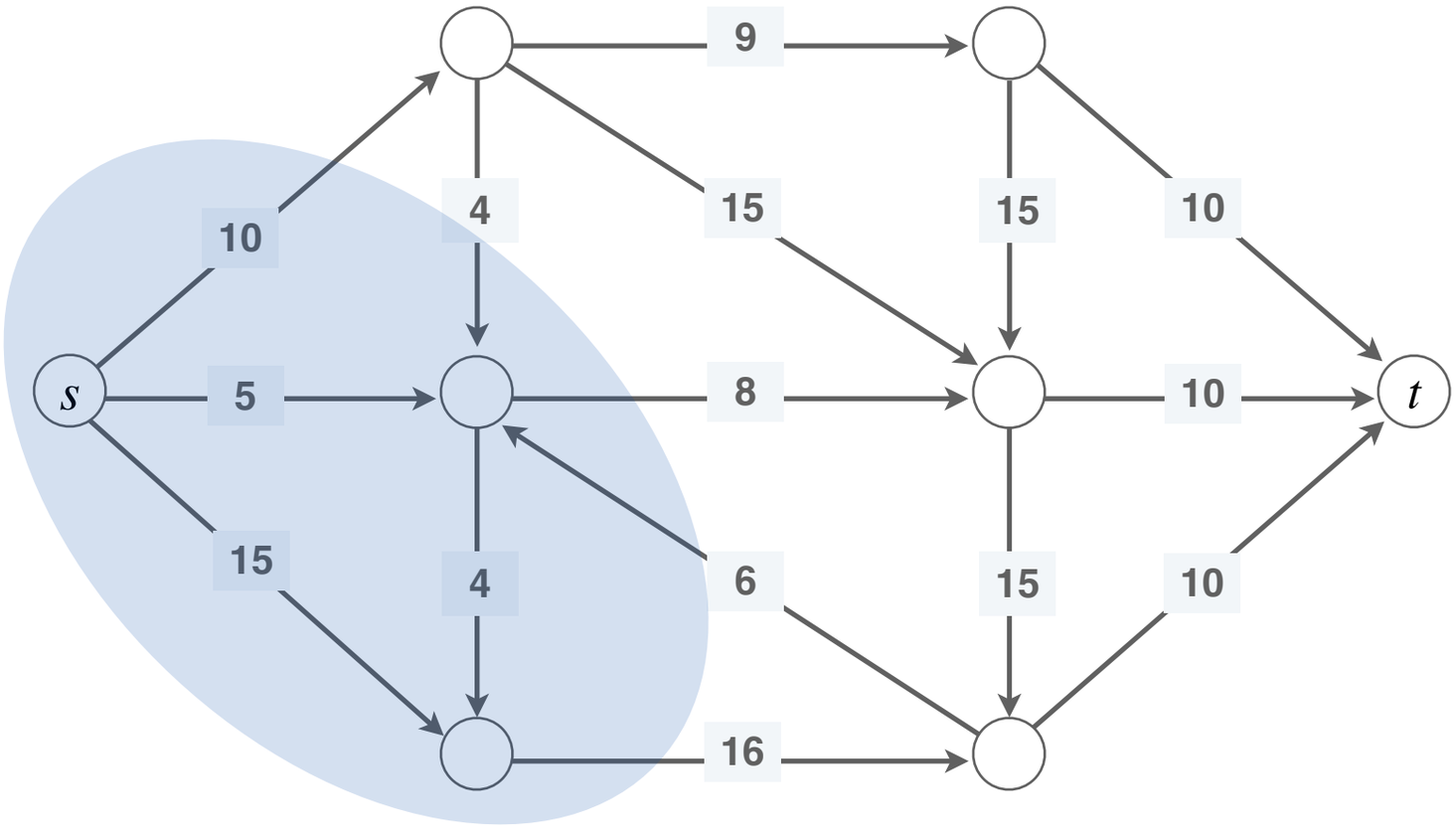
C. 45 (20 + 25)

D. 79 (20 + 25 + 8 + 11 + 9 + 6)



Min Cut Problem

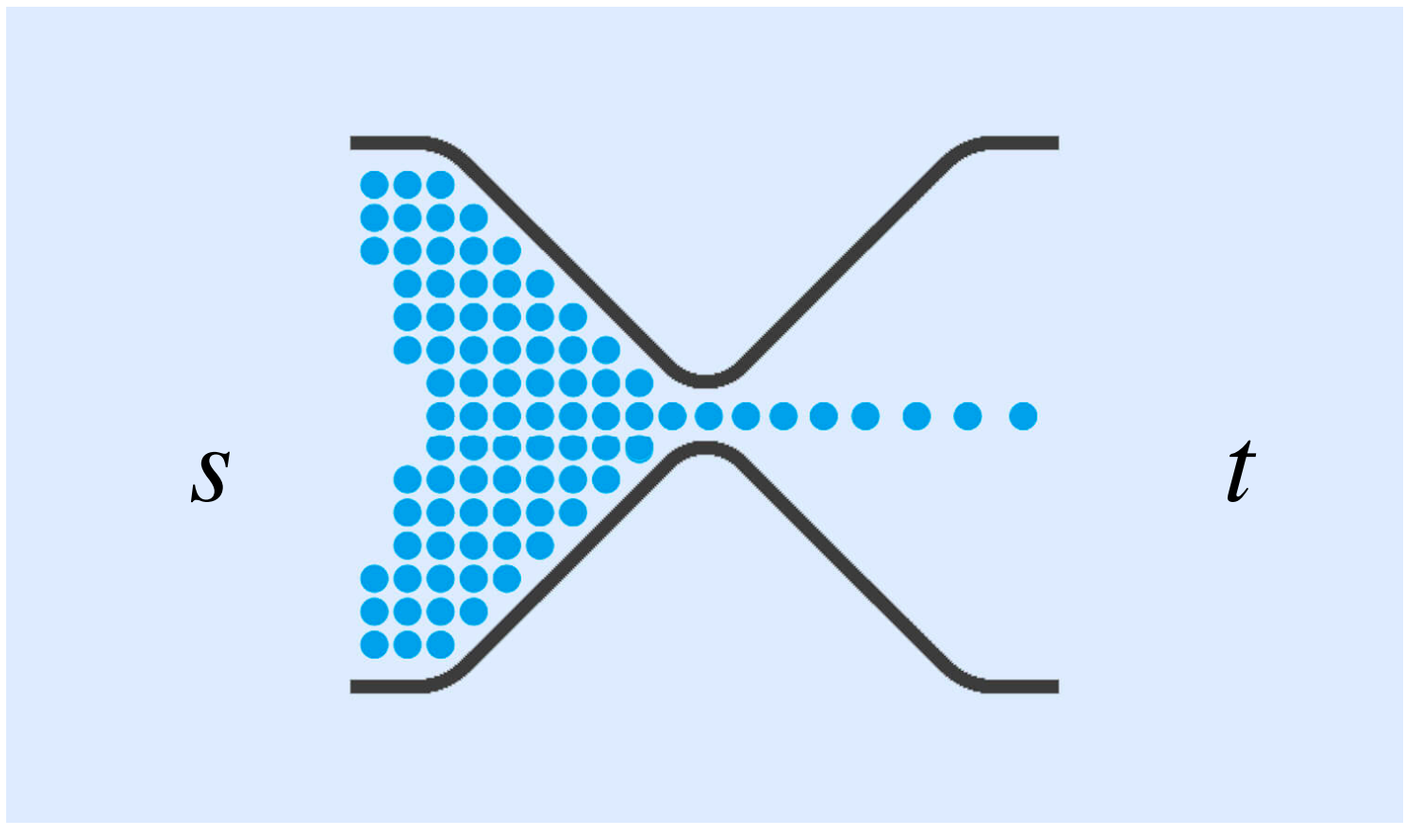
- **Problem**. Given an s-t flow network, find an s-t cut of **minimum** capacity.



Relationship between Flows and Cuts

Flows and Cuts

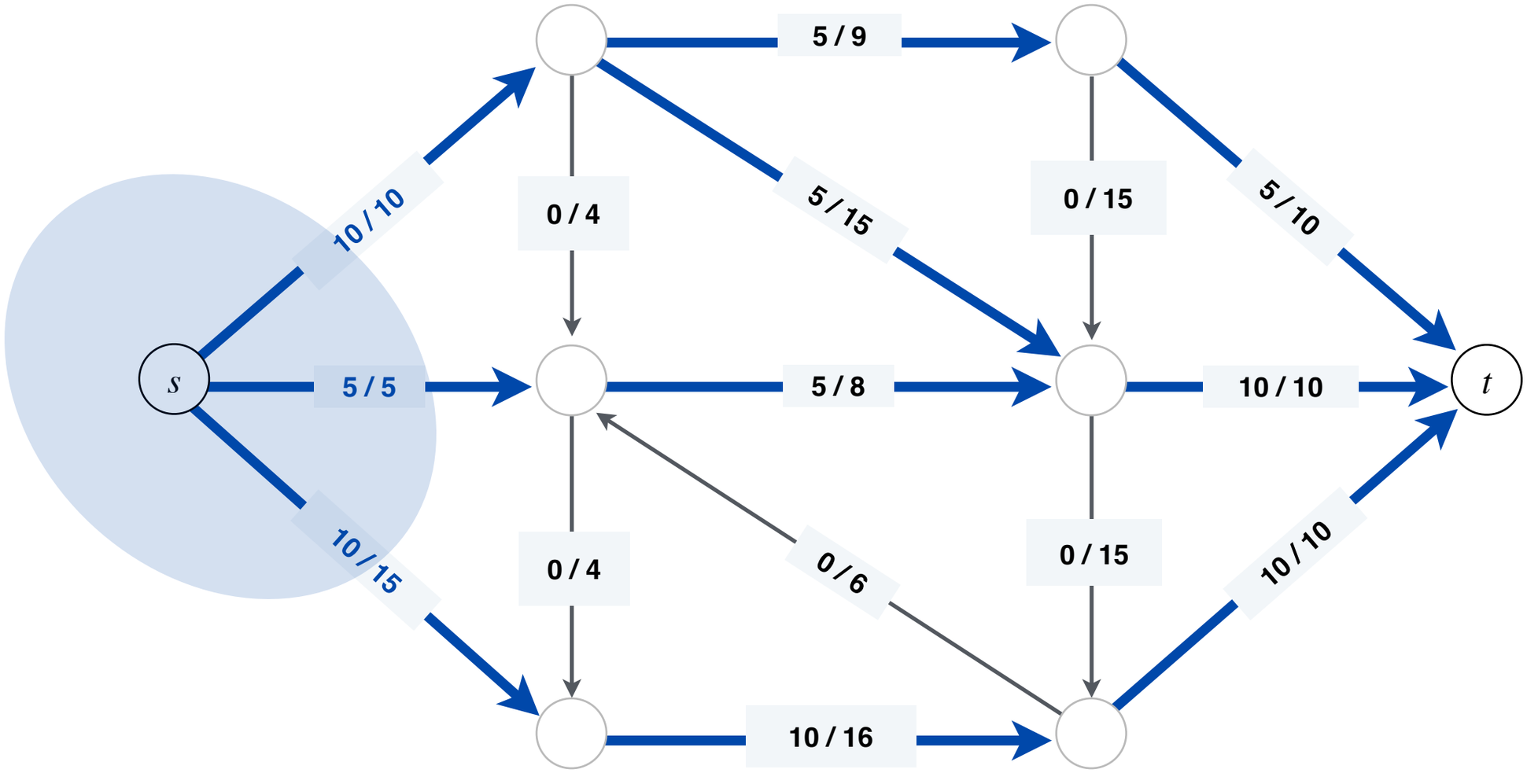
- Cuts represent "**bottlenecks**" in a flow network
- For any cut, our flow needs to “get out” of that cut on its route from s to t
- Let us formalize this intuition



Flows and Cuts

- **Claim**. Let f be **any** s-t flow and (S, T) be **any** s-t cut then $v(f) \leq c(S, T)$

- There are two s-t cuts for which this is easy to see, which ones?



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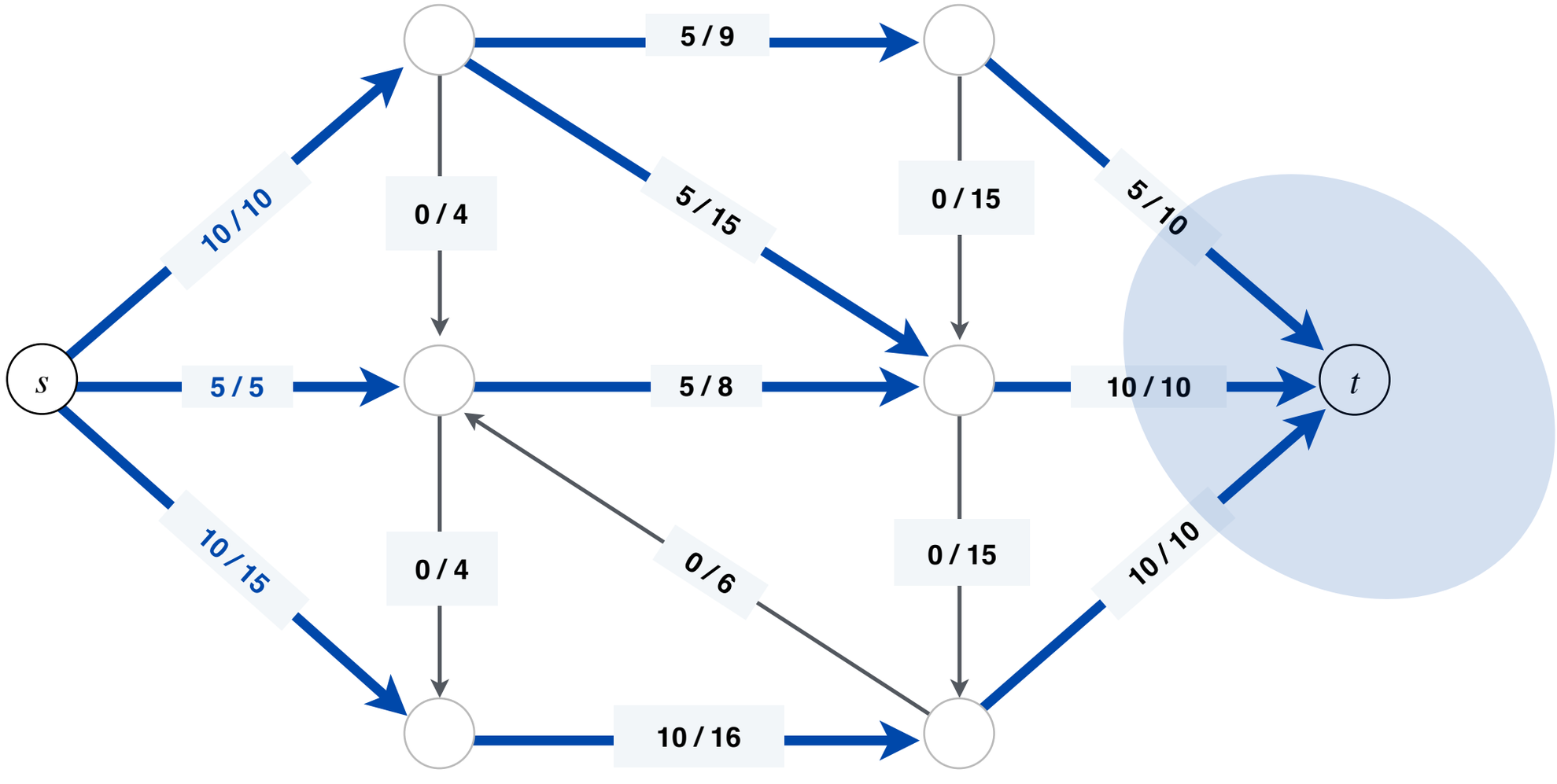


Flows and Cuts

- To prove this for any cut, we first relate the flow value in a network to the net flow leaving a cut
- **Lemma**. For any feasible (s, t)-flow f on $G = (V, E)$ and any (s, t)-cut , $v(f) = f_{out}(S) - f_{in}(S)$, where
- $f_{out}(S) = \sum_{v \in S, w \in T} f(v \to w)$ (sum of flow 'leaving' S)
- $f_{in}(S) = \sum_{v \in S, w \in T} f(w \to v)$ (sum of flow 'entering' S)
- Note: $f_{out}(S) = f_{in}(T)$ and $f_{in}(S) = f_{out}(T)$

Flows and Cuts

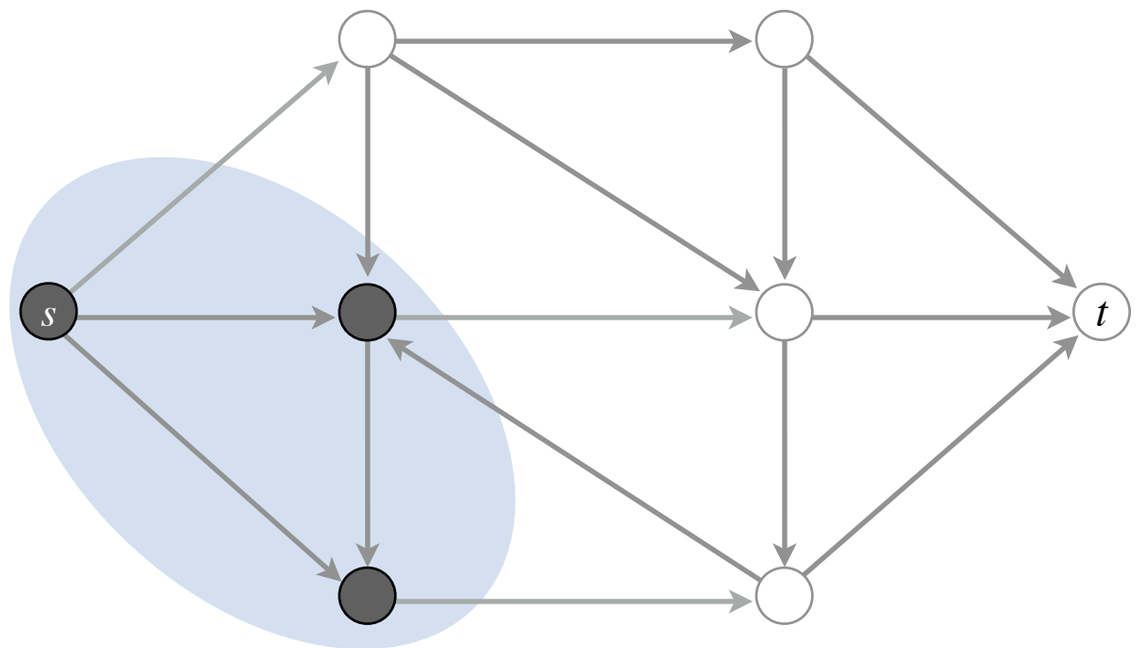
Proof. $f_{out}(S) - f_{in}(S)$

$$= \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v) \quad \text{[by definition]}$$

Adding zero terms

$$= \left[\sum_{v,w \in S} f(v \to w) - \sum_{v,u \in S} f(u \to v) \right] + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v)$$

These are the same sum: they sum the flow of all edges with both vertices in S



Flows and Cuts

Proof. $f_{out}(S) - f_{in}(S)$

$$= \left[\sum_{v,w \in S} f(v \to w) - \sum_{v,u \in S} f(u \to v) \right] + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v \in S, u \in T} f(u \to v)$$

Rearranging terms

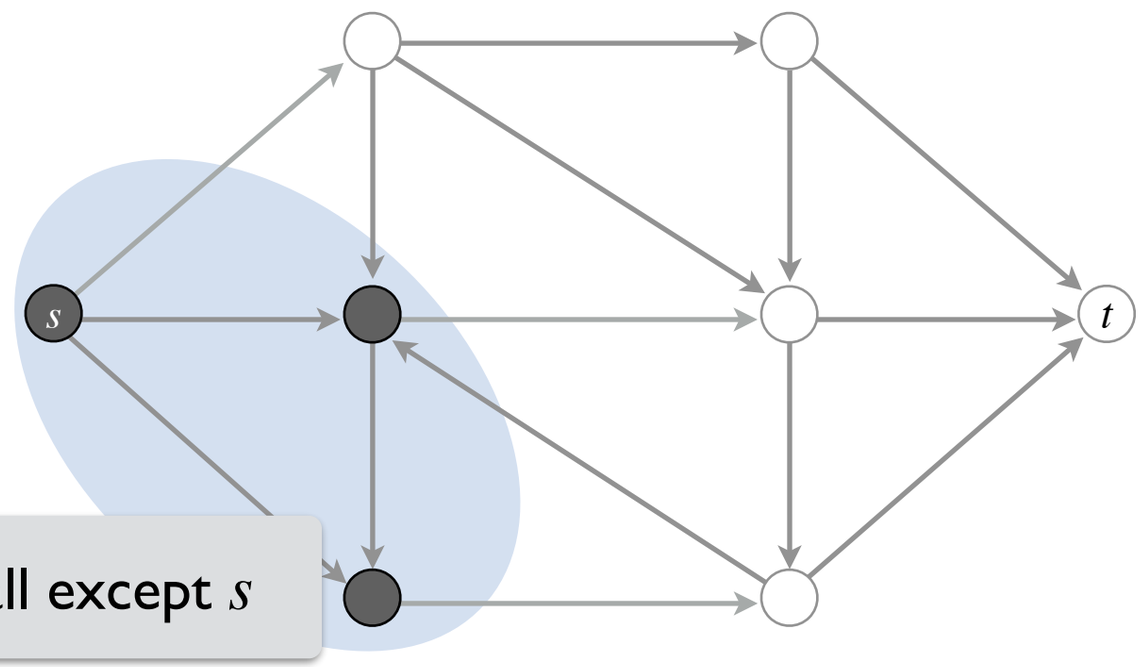
$$= \sum_{v,w \in S} f(v \to w) + \sum_{v \in S, w \in T} f(v \to w) - \sum_{v,u \in S} f(u \to v) - \sum_{v \in S, u \in T} f(u \to v)$$

$$= \sum_{v \in S} \left(\sum_{w} f(v \to w) - \sum_{u} f(u \to v) \right)$$

$$= \sum_{v \in S} f_{out}(v) - f_{in}(v)$$

$$= f_{out}(s) = v(f) \quad \blacksquare$$

Cancels out for all except s



Flows and Cuts

- We use this result to prove that the value of a flow cannot exceed the capacity of any cut in the network
- **Claim**. Let f be any s-t flow and (S, T) be any s-t cut then $v(f) \leq c(S, T)$

- **Proof.** $v(f) = f_{out}(S) - f_{in}(S)$

$$\leq f_{out}(S) = \sum_{v \in S, w \in T} f(v \to w)$$

$$\leq \sum_{v \in S, w \in T} c(v, w) = c(S, T)$$

When is $v(f) = c(S, T)$?

$f_{in}(S) = 0,\ f_{out}(S) = c(S, T)$

Max-Flow & Min-Cut

- Suppose the $c_{\min}$ is the capacity of the minimum cut in a network
- What can we say about the feasible flow we can send through it
    - cannot be more than $c_{\min}$
- In fact, whenever we find any s-t flow f and any s-t cut (S, T) such that, $v(f) = c(S, T)$ we can conclude that:
    - f is the maximum flow, and,
    - (S, T) is the minimum cut
- The question now is, given any flow network with min cut $c_{\min}$, is it always possible to route a feasible s-t flow f with $v(f) = c_{\min}$

Max-Flow Min-Cut Theorem

- A beautiful, powerful relationship between these two problems in given by the following theorem
- **Theorem**. Given any flow network G, there exists a feasible (s, t)-flow f and a (s, t)-cut (S, T) such that,

$$v(f) = c(S, T)$$

- Informally, in a flow network, the max-flow = min-cut
- This will guide our algorithm design for finding max flow
- (Will prove this theorem by construction in a bit—our algorithm will prove the theorem! (like with Gale-Shapley))

Network Flow History

- In 1950s, US military researchers Harris and Ross wrote a classified report about the rail network linking Soviet Union and Eastern Europe
    - Vertices were the geographic regions
    - Edges were railway links between the regions
        - Edge weights were the rate at which material could be shipped from one region to next
- Ross and Harris determined:
    - Maximum amount of stuff that could be moved from Russia to Europe **(max flow)**
    - Cheapest way to disrupt the network by removing rail links **(min cut)**

Network Flow History

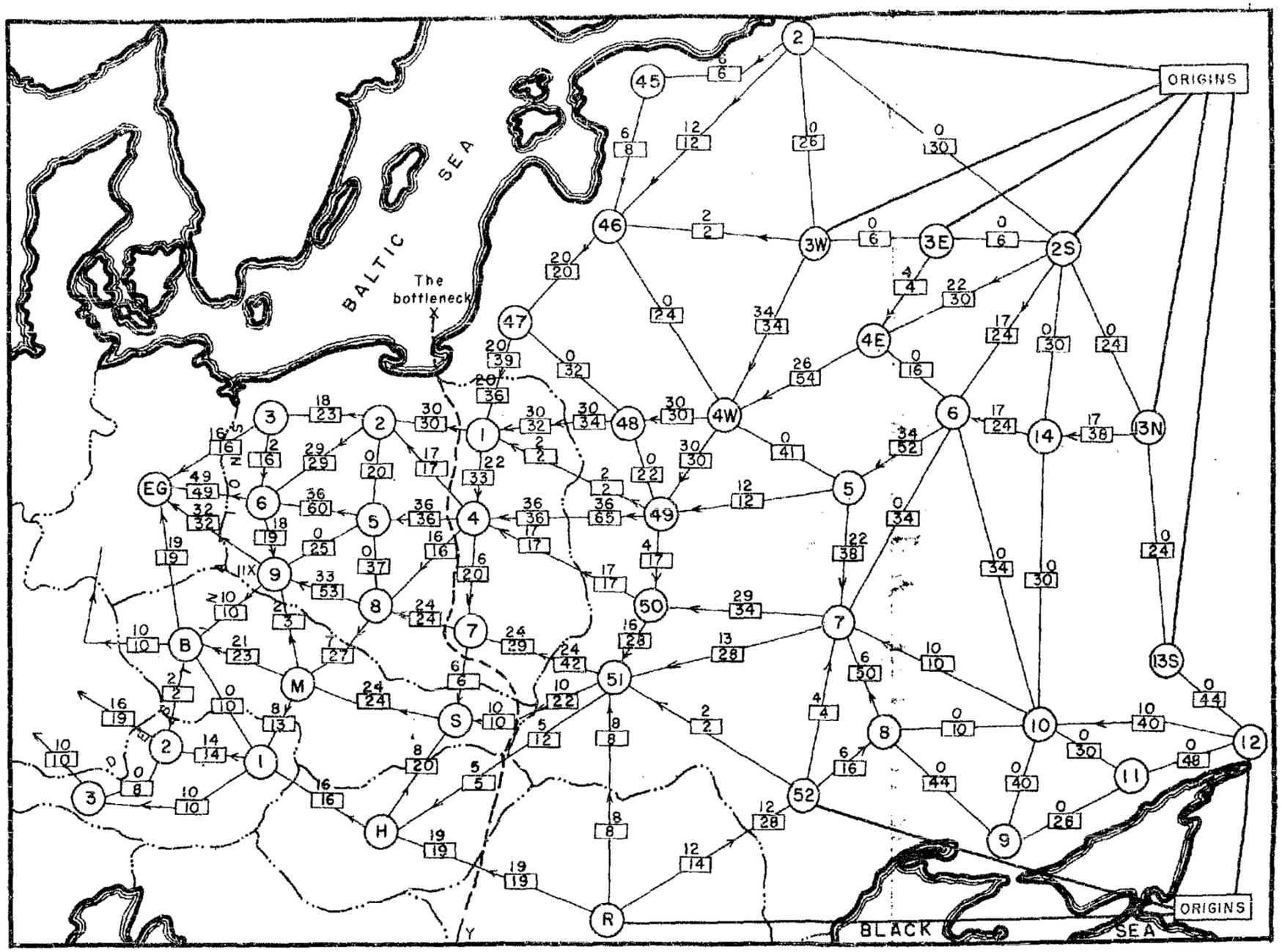


Image Credits: — Jeff Erickson's book and T[homas] E. Harris and F[rank] S. Ross. Fundamentals of a method for evaluating rail net capacities. The RAND Corporation, Research Memorandum RM-1517, October 24, 1955. United States Government work in the public domain. http://www.dtic.mil/dtic/tr/fulltext/u2/093458.pdf

Towards a Max-Flow Algorithm

- Today: we will prove the max-flow min-cut theorem *constructively*
- We will design a max-flow algorithm and show that there is a s-t cut s.t. value of flow computed by algorithm = capacity of cut
- Let's start with a greedy approach
    - Push as much flow as possible down a s-t path
    - This won't actually work
    - But gives us a sense of what we need to keep track off to improve upon it

Towards a Max-Flow Algorithm

- Greedy strategy:
    - Start with $f(e) = 0$ for each edge
    - Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$
    - "Augment" flow (as much as possible) along path P
        - Repeat until you get stuck
- Let's take an example

Towards a Max-Flow Algorithm

- Start with $f(e) = 0$ for each edge
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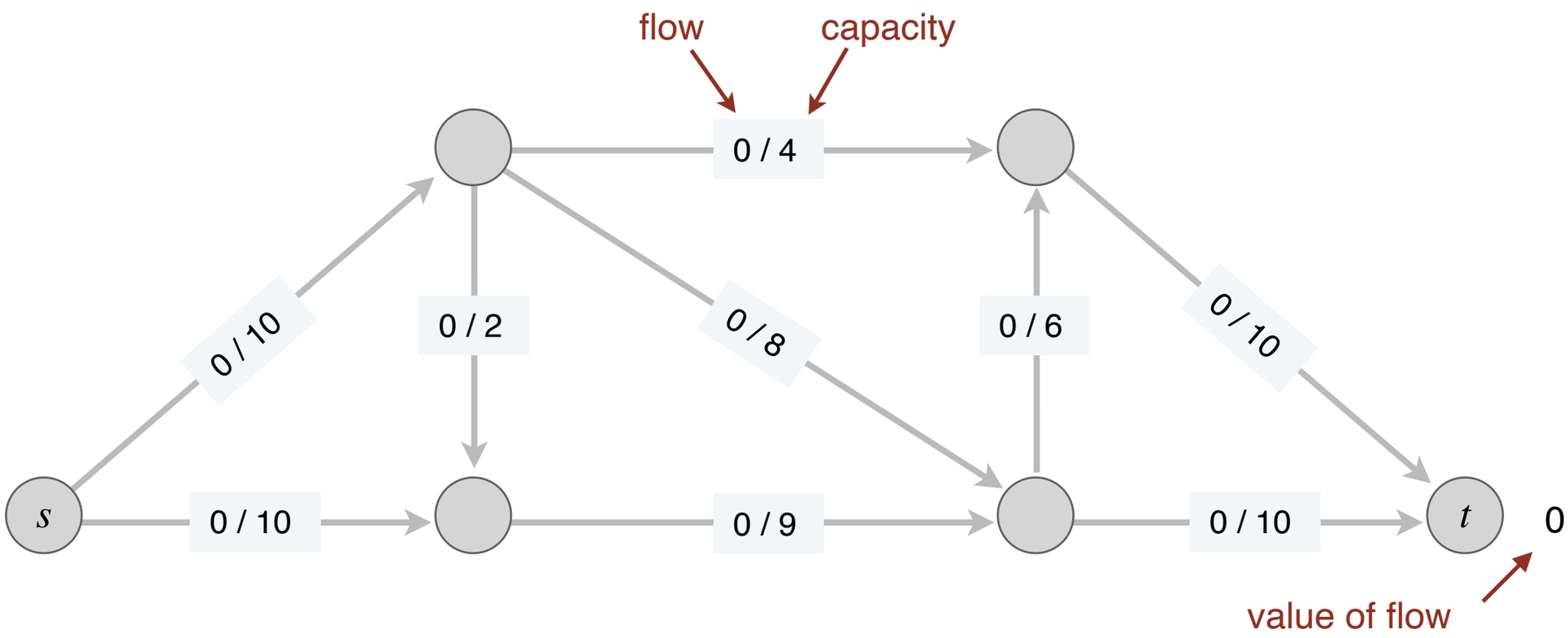

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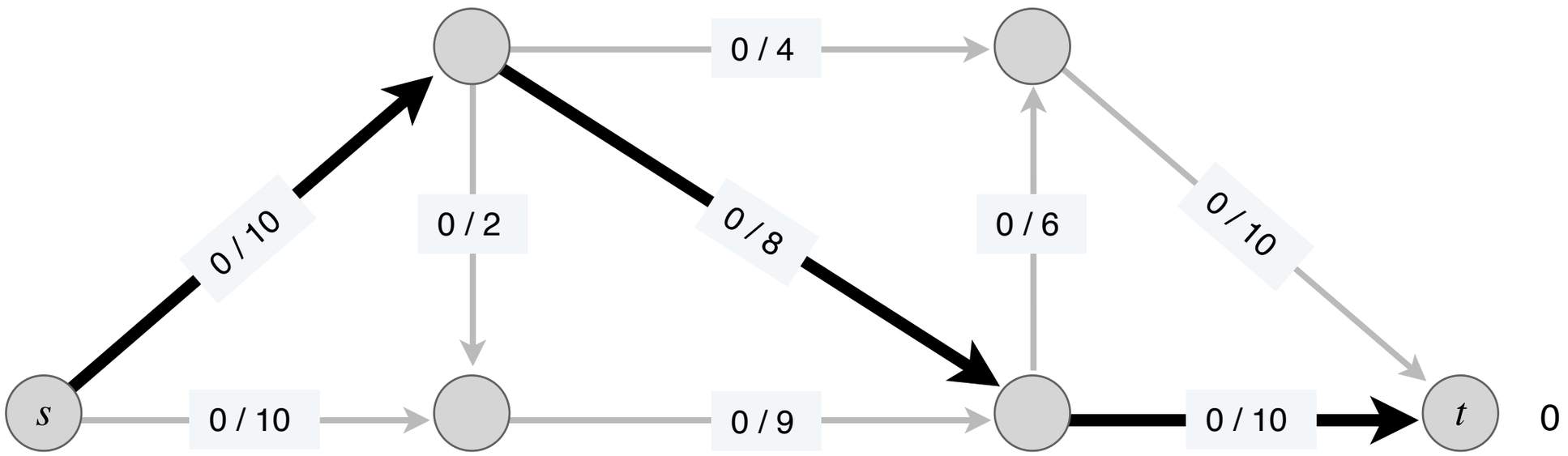

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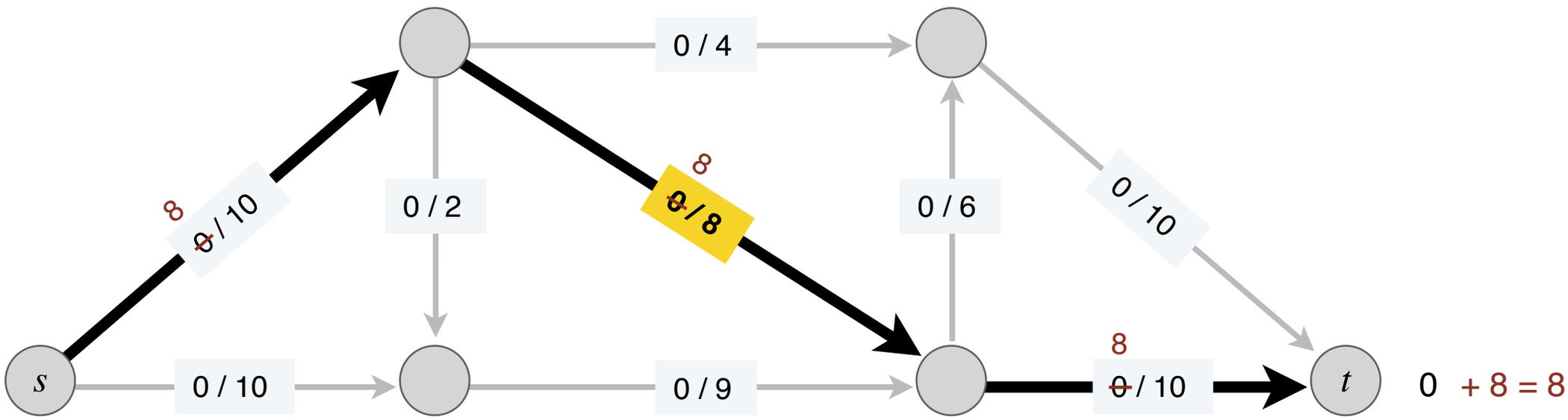

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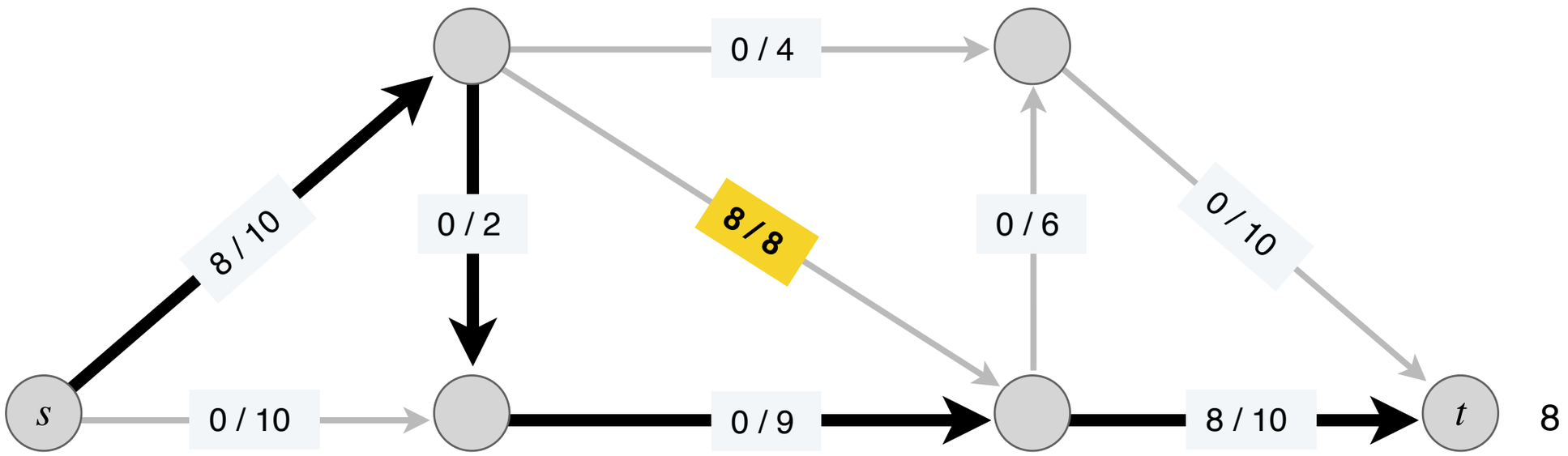

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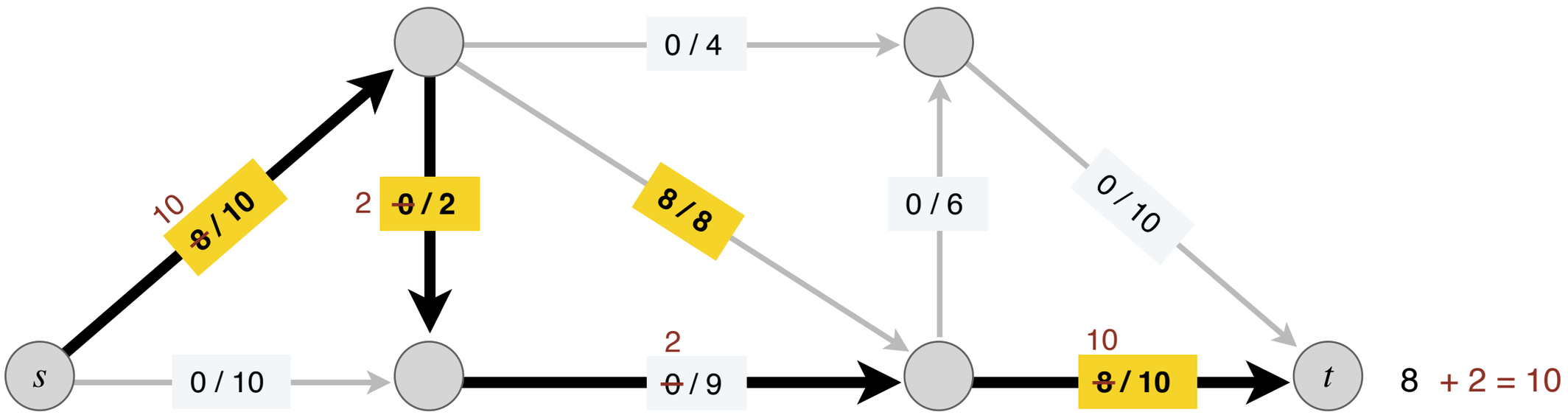

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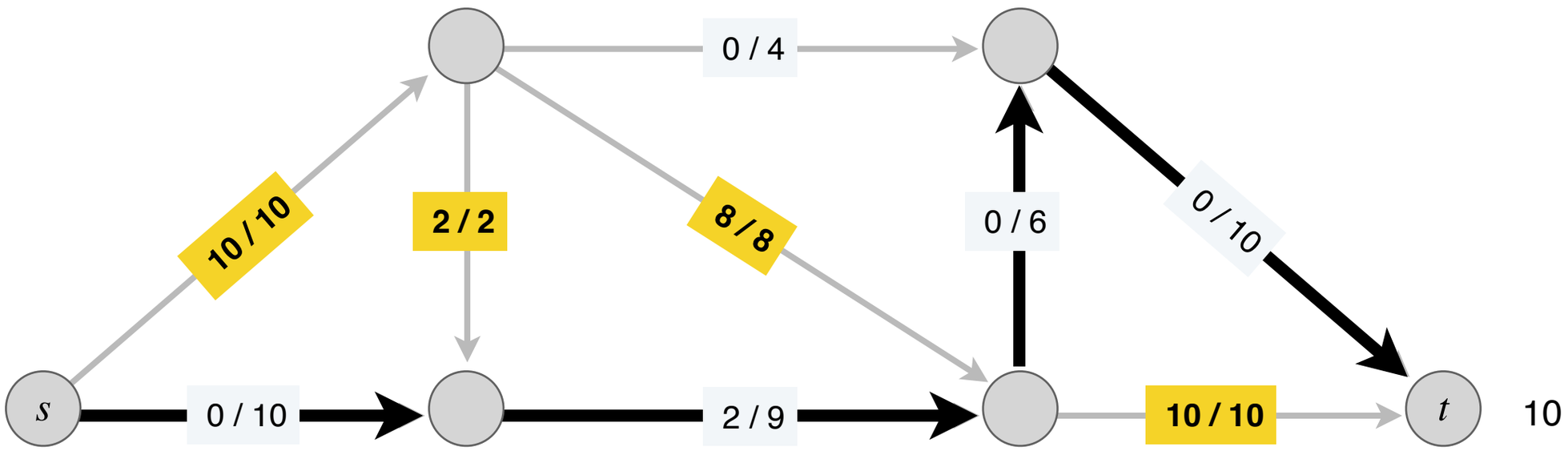

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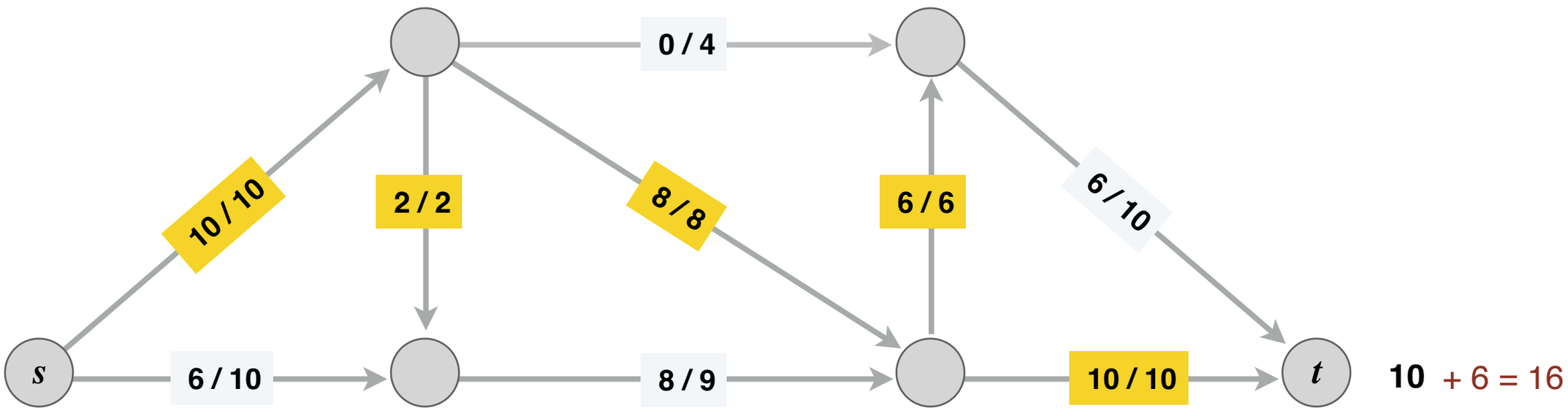
Is this the best we can do?

ending flow value = 16



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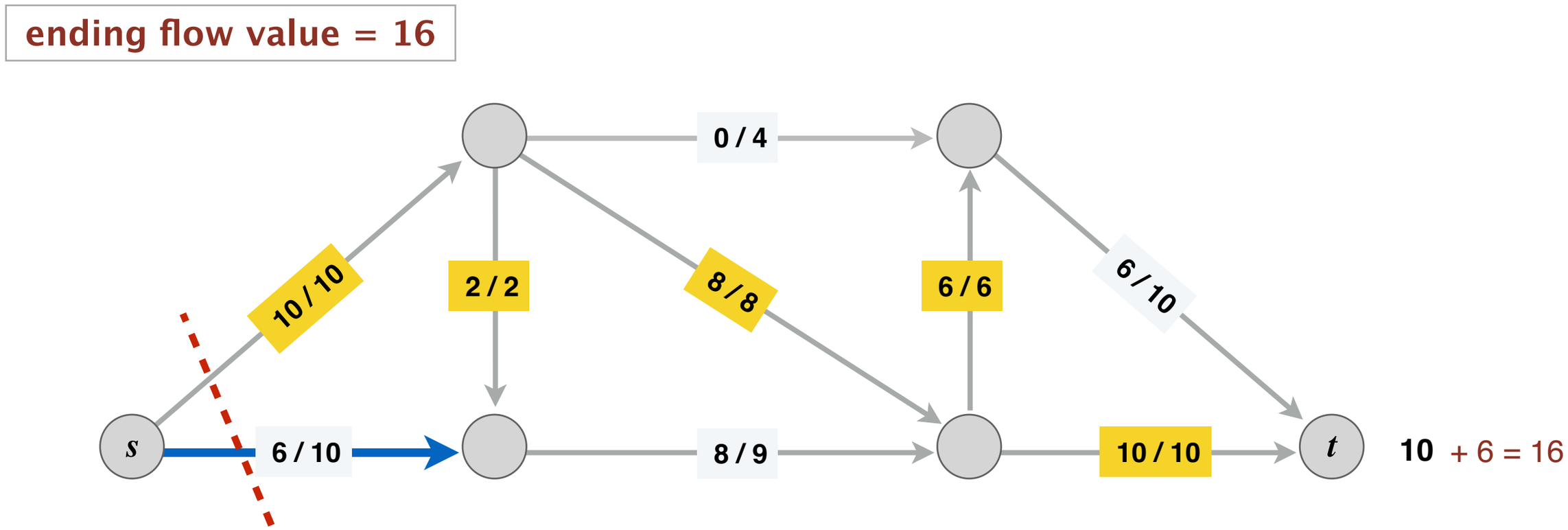

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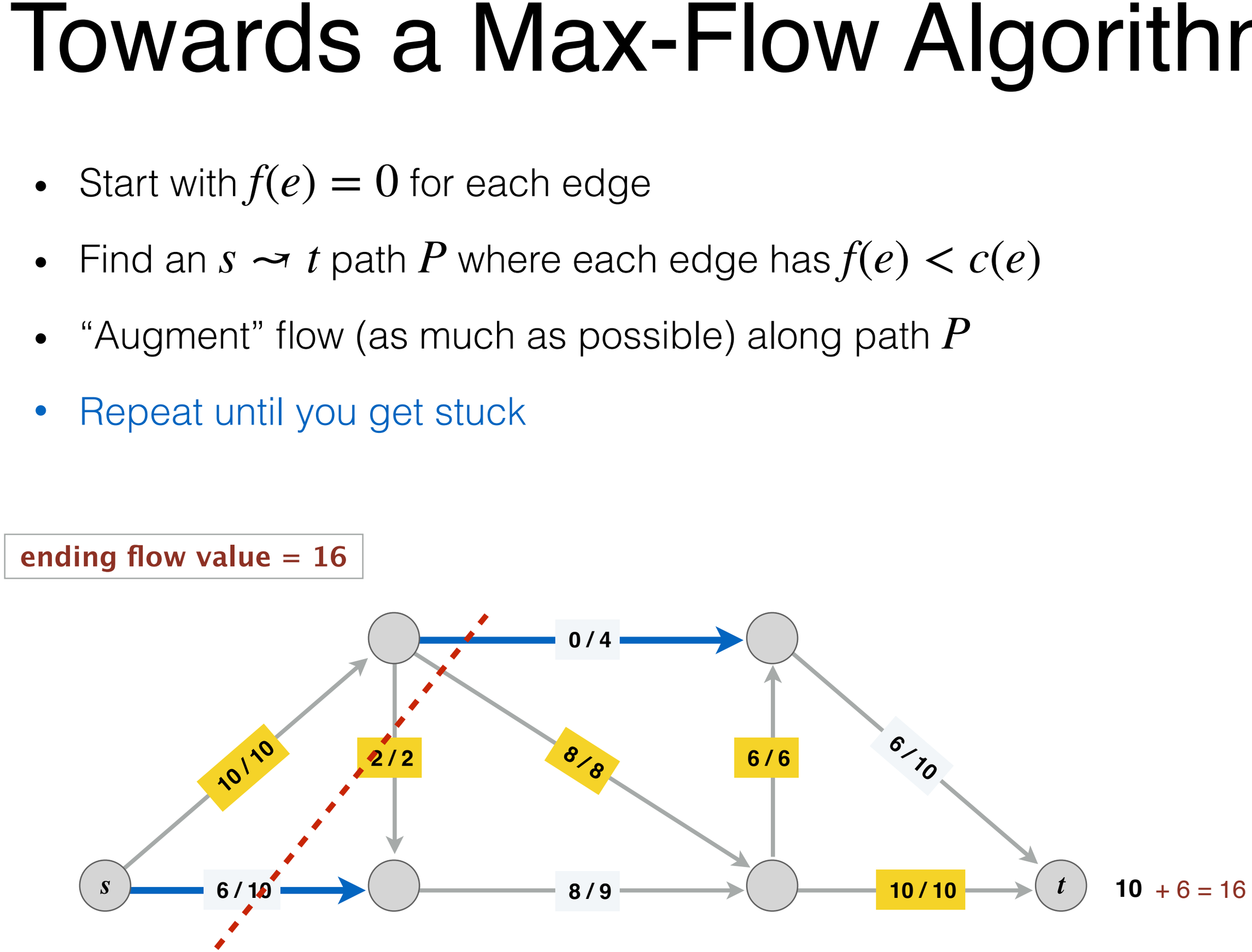

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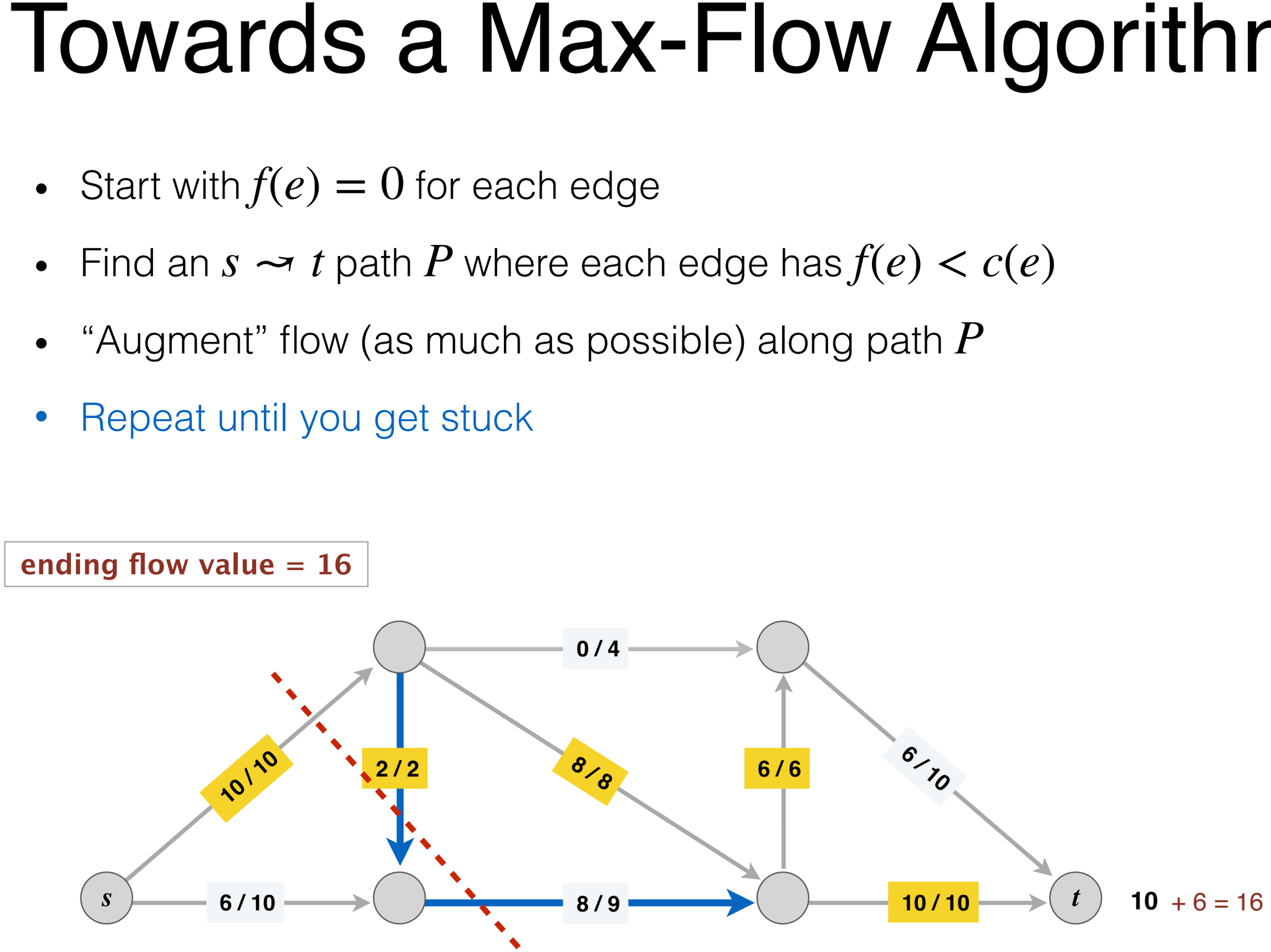
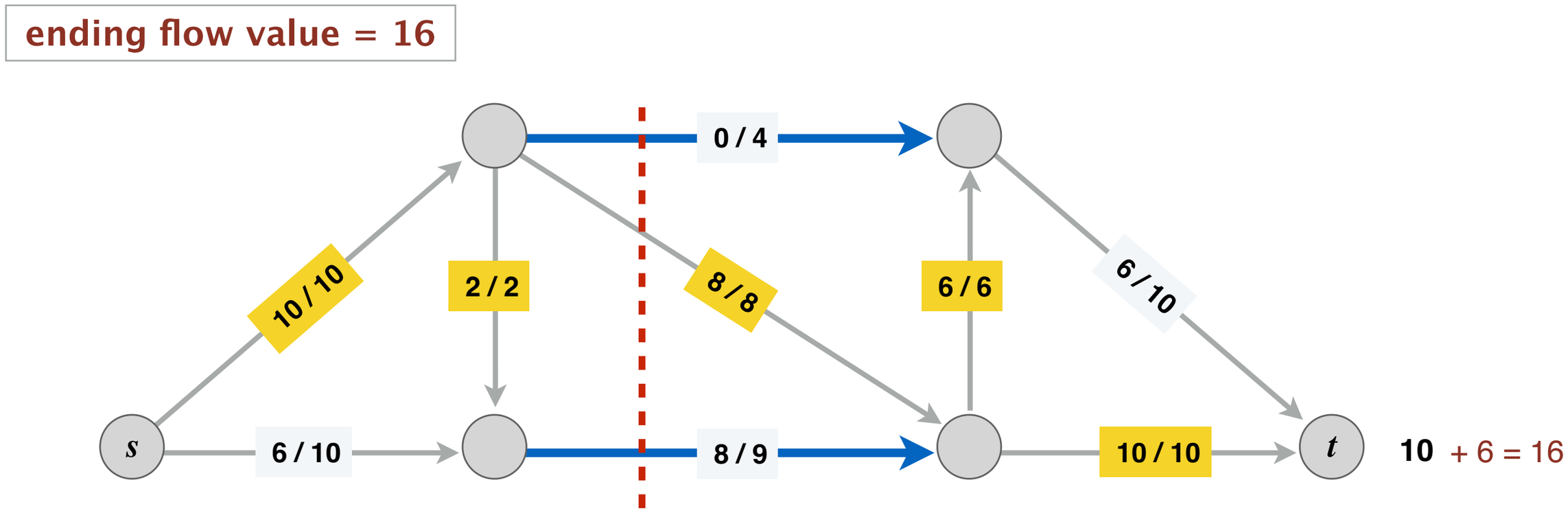


Towards a Max-Flow Algorithm

- Start with $f(e) = 0$ for each edge
- Find an $s \rightsquigarrow t$ path P where each edge has $f(e) < c(e)$
- "Augment" flow (as much as possible) along path P
- Repeat until you get stuck



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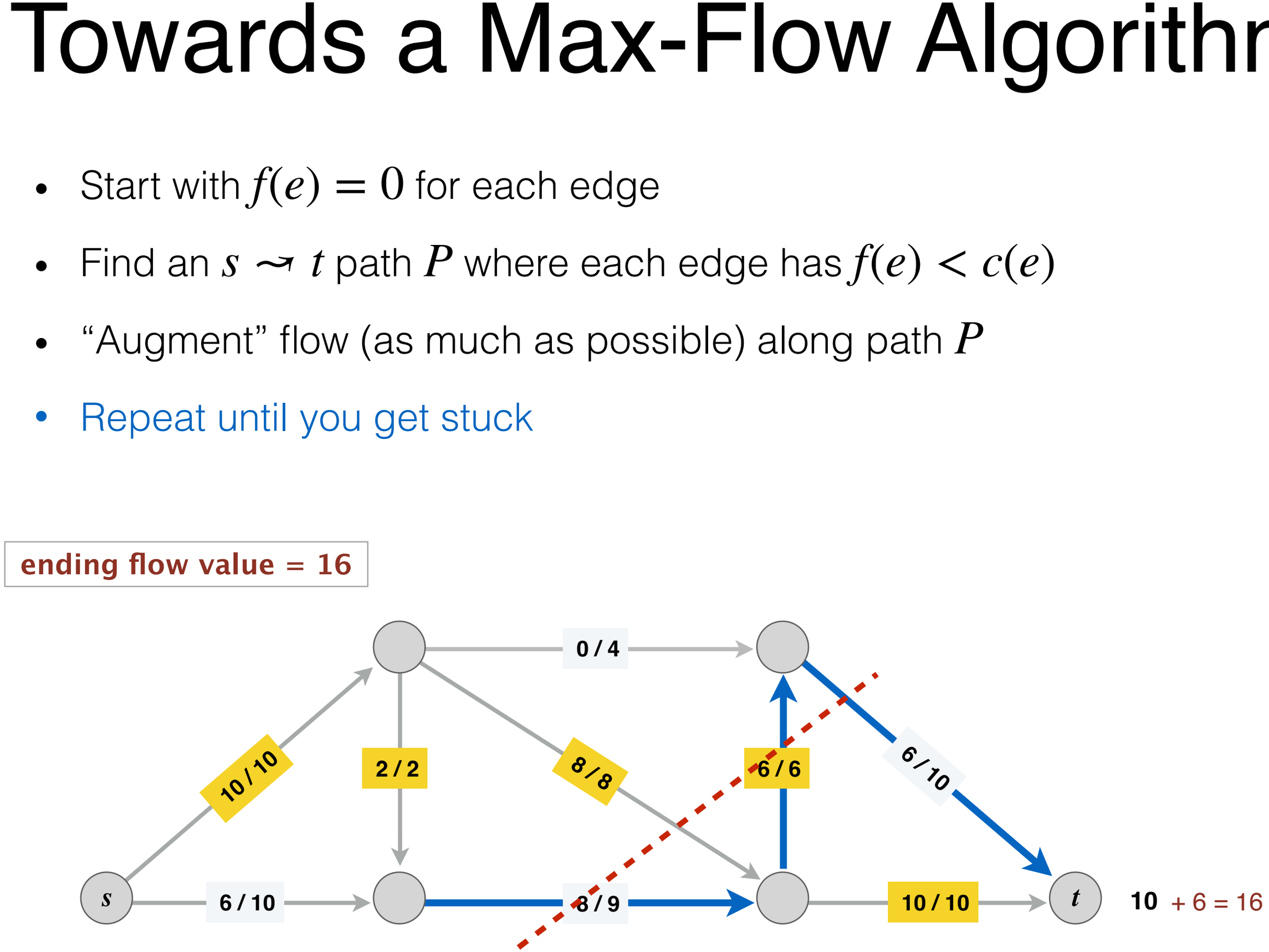

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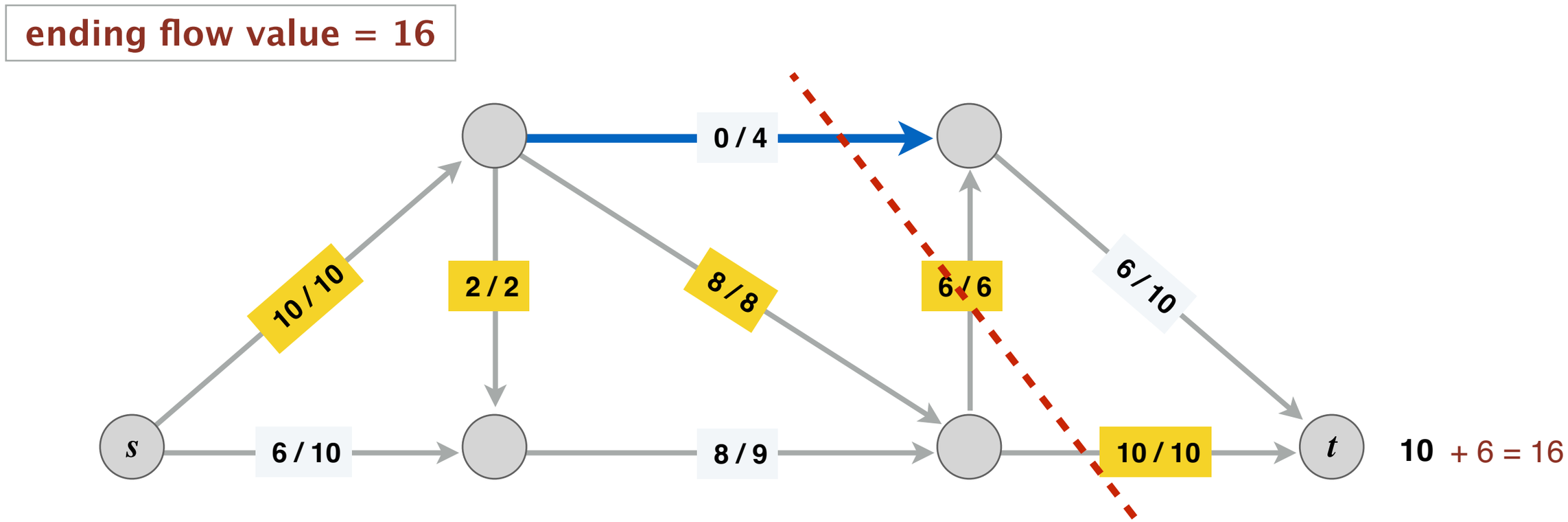

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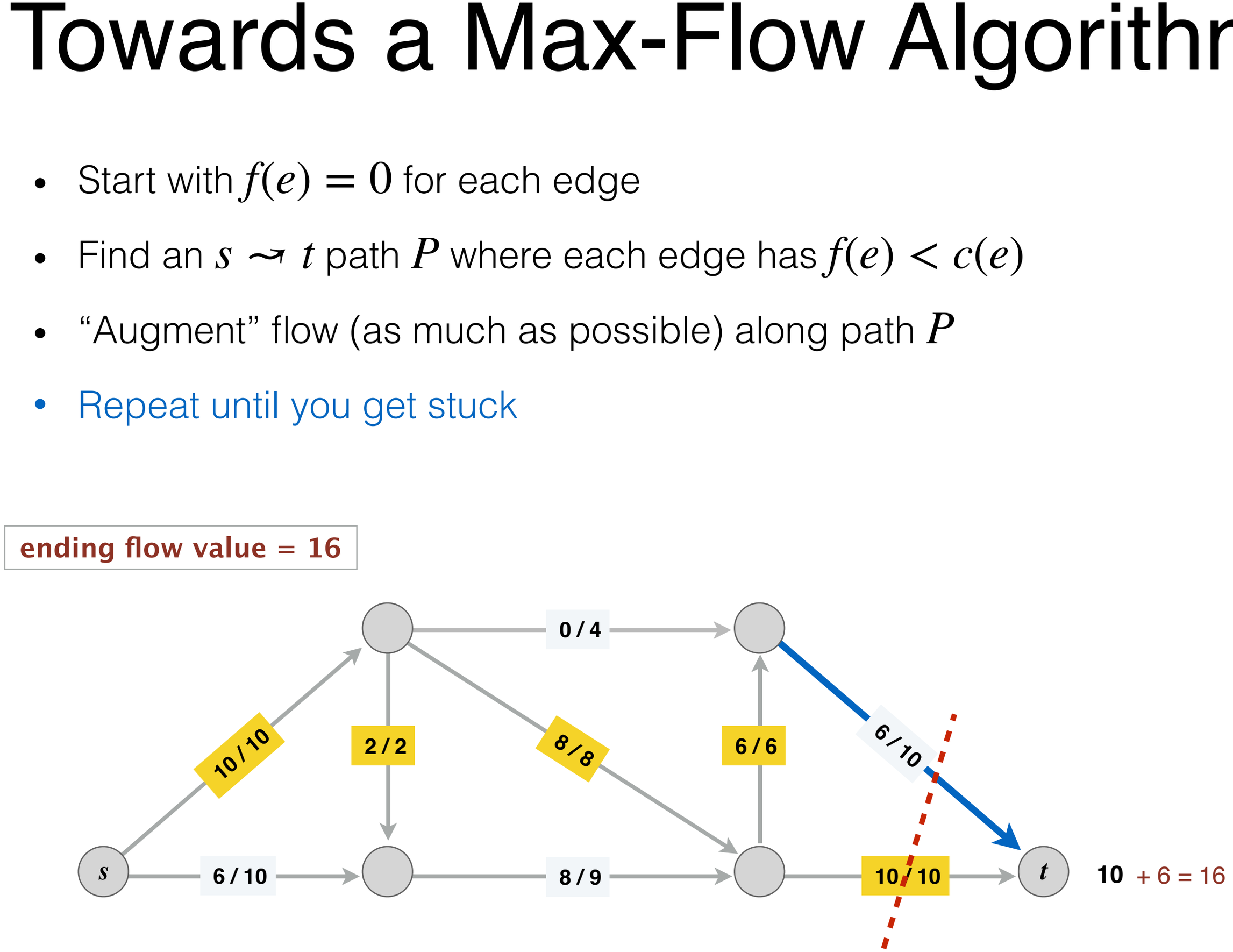

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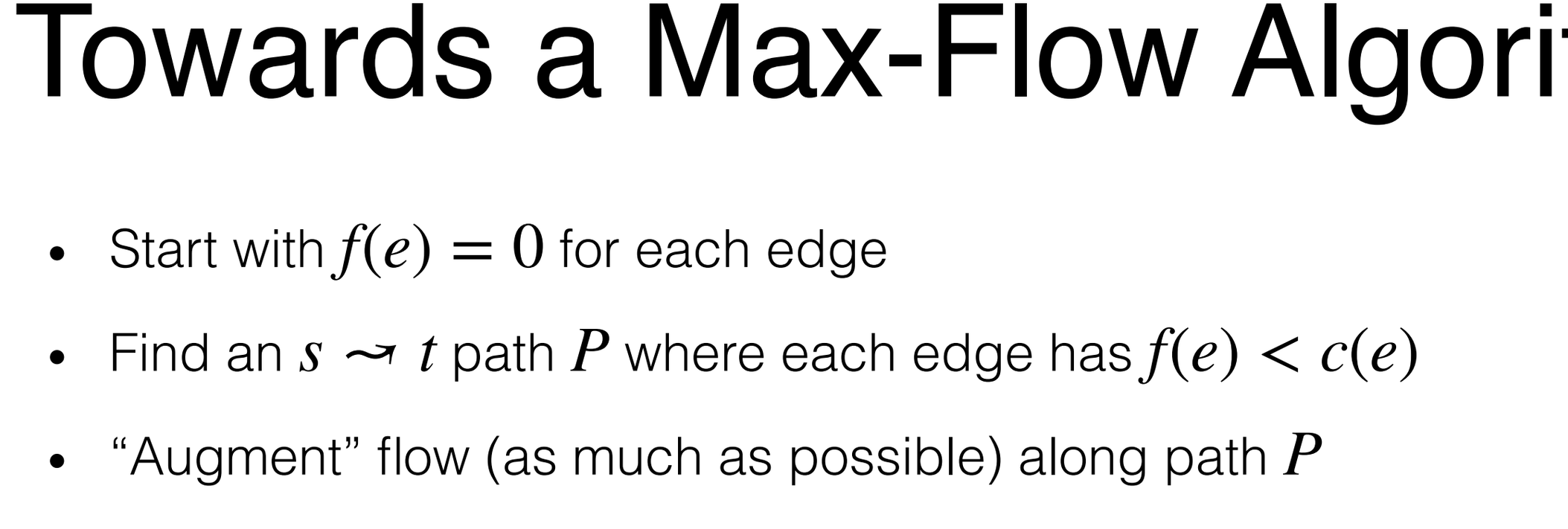

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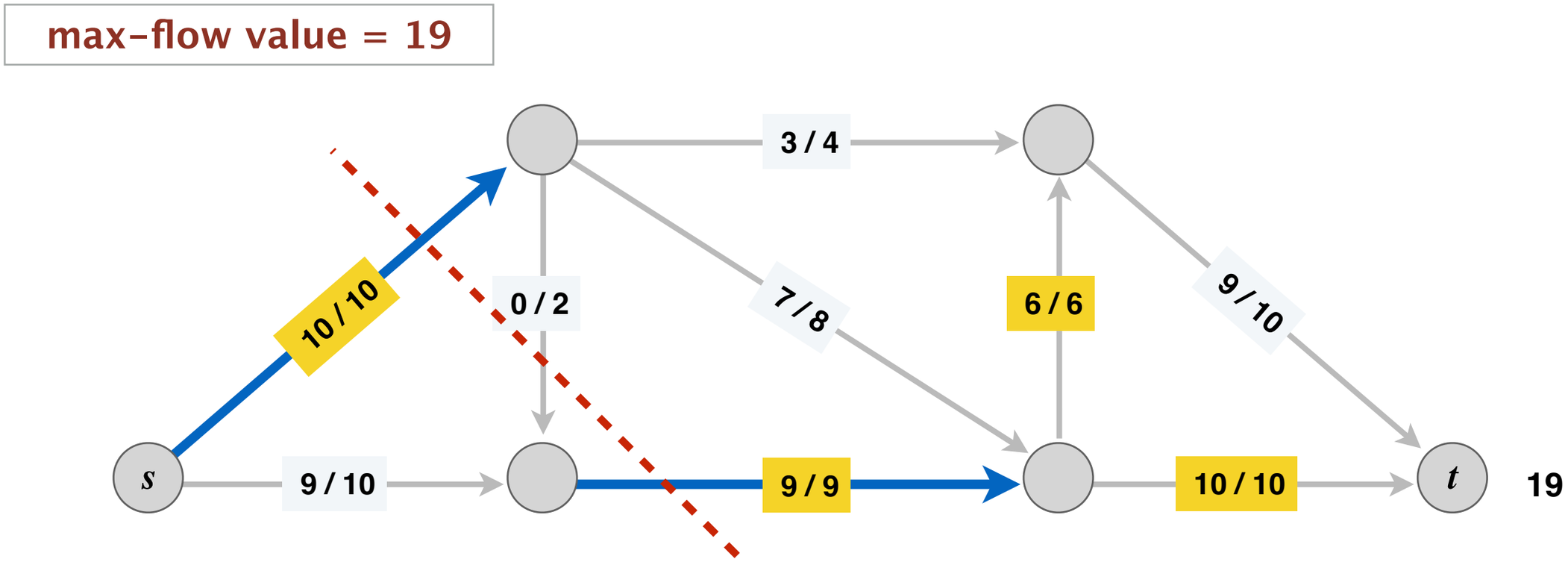
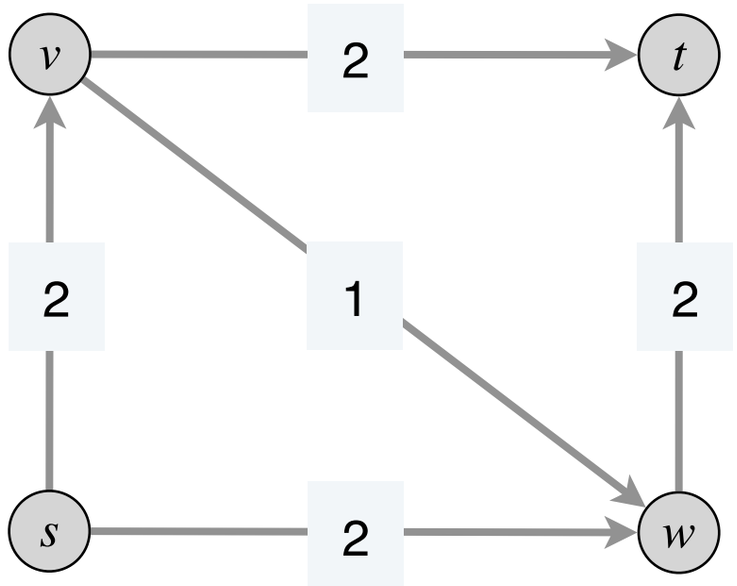


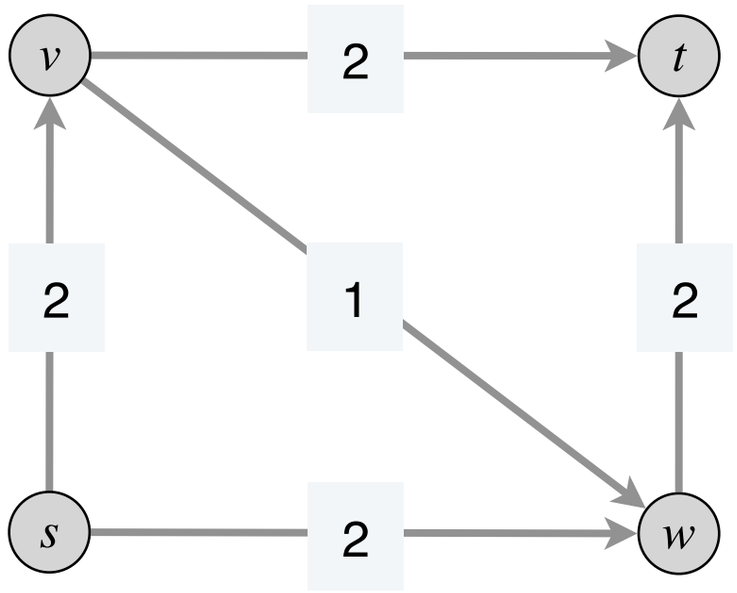
Why Greedy Fails

- **Problem**: greedy can never “undo” a bad flow decision
- Consider the following flow network



Why Greedy Fails

- **Problem**: greedy can never "undo" a bad flow decision
- Consider the following flow network
  - Unique max flow has $f(v \to w) = 0$
  - Greedy could choose $s \to v \to w \to t$ as first P



- **Takeaway**: Need a mechanism to "undo" bad flow decisions

Ford-Fulkerson Algorithm

Ford Fulkerson: Idea

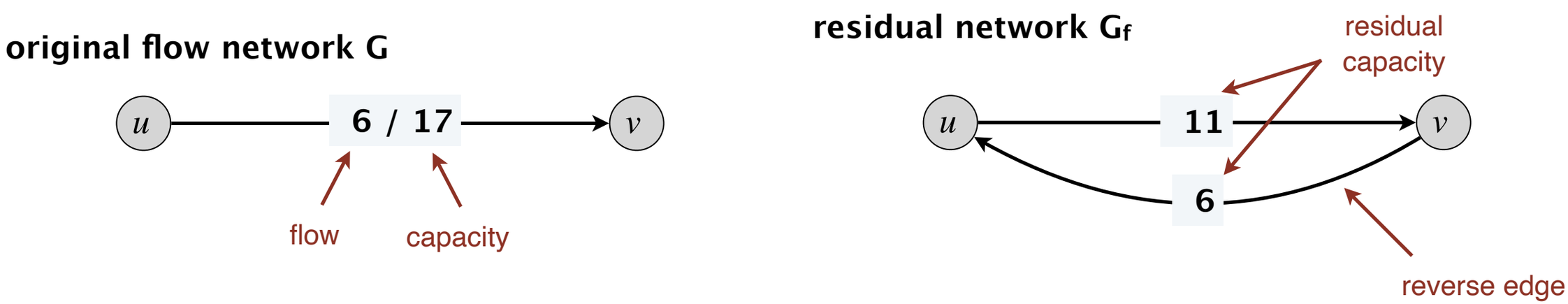
- Want to make “forward progress” while letting ourselves undo previous decisions if they’re getting in our way

- **Idea**: keep track of where we can push flow
  - Can push more flow along an edge with remaining capacity
  - Can also push flow “back” along an edge that already has flow down it

- Need a way to systematically track these decisions

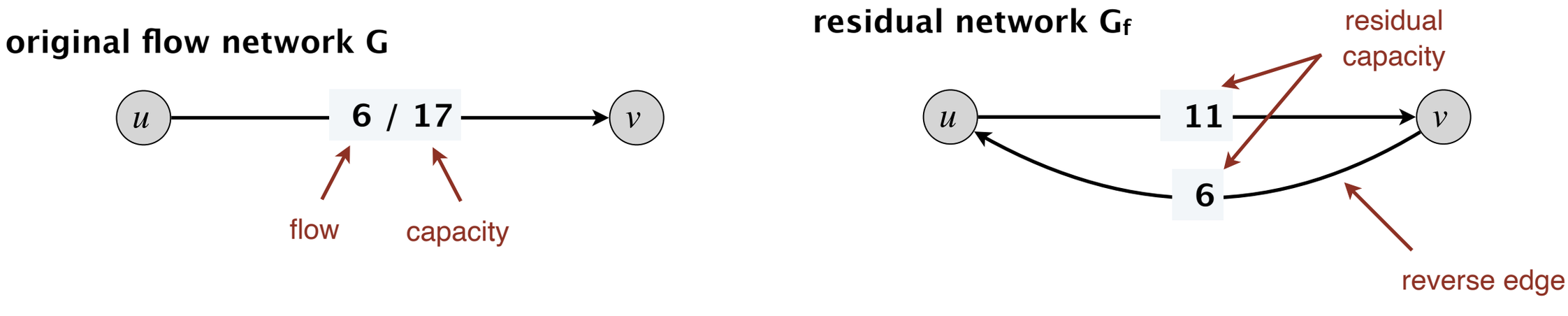
Residual Graph

- Given flow network $G = (V, E, c)$ and a feasible flow f on G, **the residual graph** $G_f = (V, E_f, c_f)$ is defined as:
  - Vertices in G_f same as G
  - **(Forward edge)** For $e \in E$ with residual capacity $c(e) - f(e) > 0$, create $e \in E_f$ with capacity $c(e) - f(e)$
  - **(Backward edge)** For $e \in E$ with $f(e) > 0$, create $e_{\text{reverse}} \in E_f$ with capacity $f(e)$



Residual Graph

- Idea: we are capturing how the flow can change in each direction
  - Forward edge: how much *more* flow can be pushed?
  - Backward edge: how much can we *decrease* the flow?



Flow Algorithm Idea

- Now we have a residual graph that lets us make forward progress or push back existing flow
- We will look for $s \rightsquigarrow t$ paths in G_f rather than G
- Once we have a path, we will "augment" flow along it similar to greedy
  - find bottleneck capacity edge on the path and push that much flow through it in G_f
- When we translate this back to G, this means:
  - We increment existing flow on a forward edge
  - Or we decrement flow on a backward edge

Augmenting Path & Flow

- An **augmenting path** P is a **simple** $s \rightsquigarrow t$ path in the residual graph G_f
- The **bottleneck capacity** b of an augmenting path P is the minimum capacity of any edge in P.

The path P is in G_f

AUGMENT(f, P)

$b \leftarrow$ bottleneck capacity of augmenting path P.

FOREACH edge $e \in P$:

  IF ($e \in E$, *that is*, e *is forward edge*)

    Increase $f(e)$ in G by b

  ELSE

    Decrease $f(e)$ in G by b

RETURN f.

Updating flow in G

Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for each edge $e \in E$
- Find a simple $s \rightsquigarrow t$ path P in the residual network G_f
- Augment flow along path P by bottleneck capacity b
- Repeat until you get stuck

FORD–FULKERSON(G)

FOREACH edge $e \in E$: $f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f.

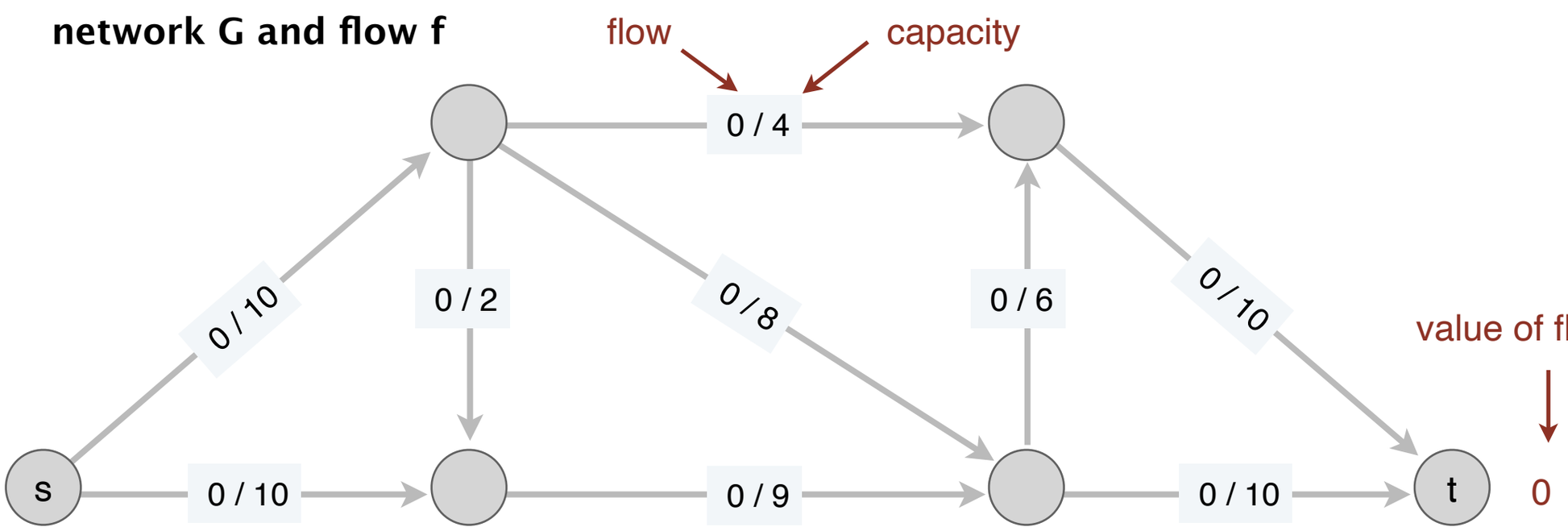
WHILE (there exists an s$\rightsquigarrow$t path P in G_f)

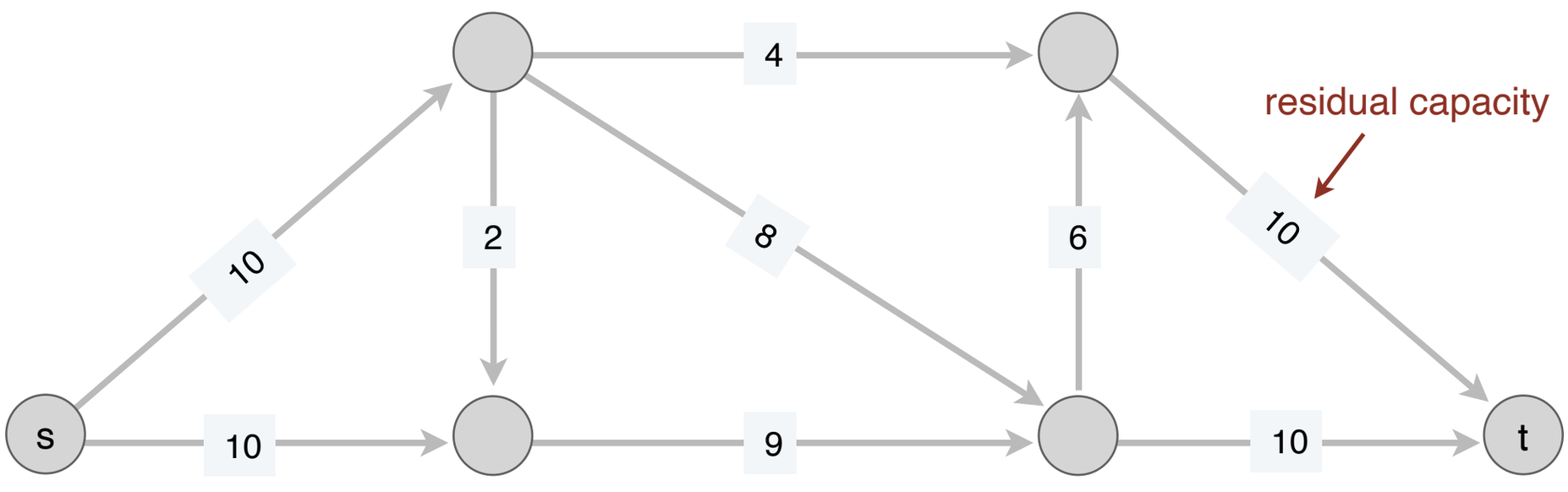
  $f \leftarrow$ AUGMENT(f, P).

  Update G_f.

RETURN f.

Ford-Fulkerson Example





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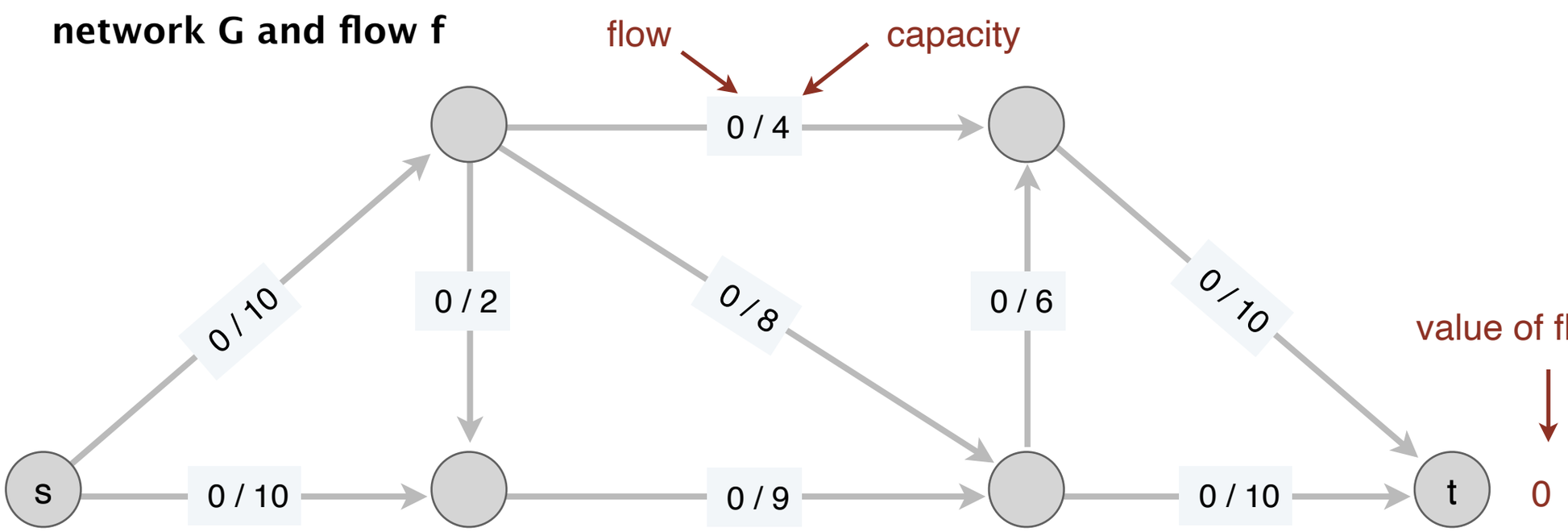

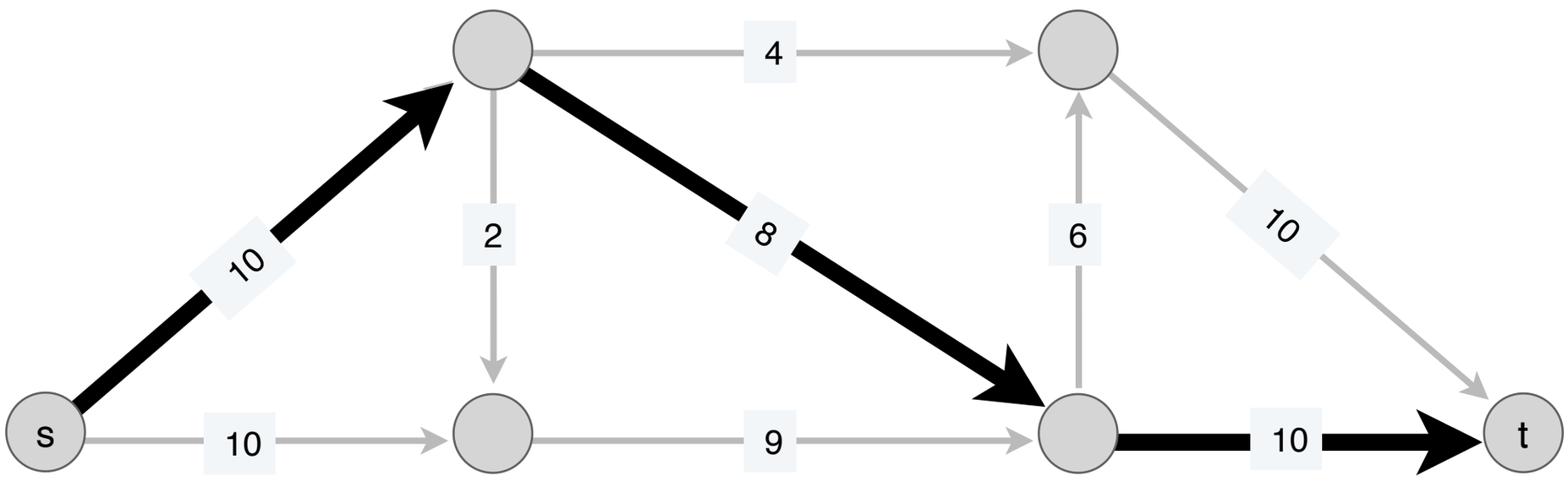

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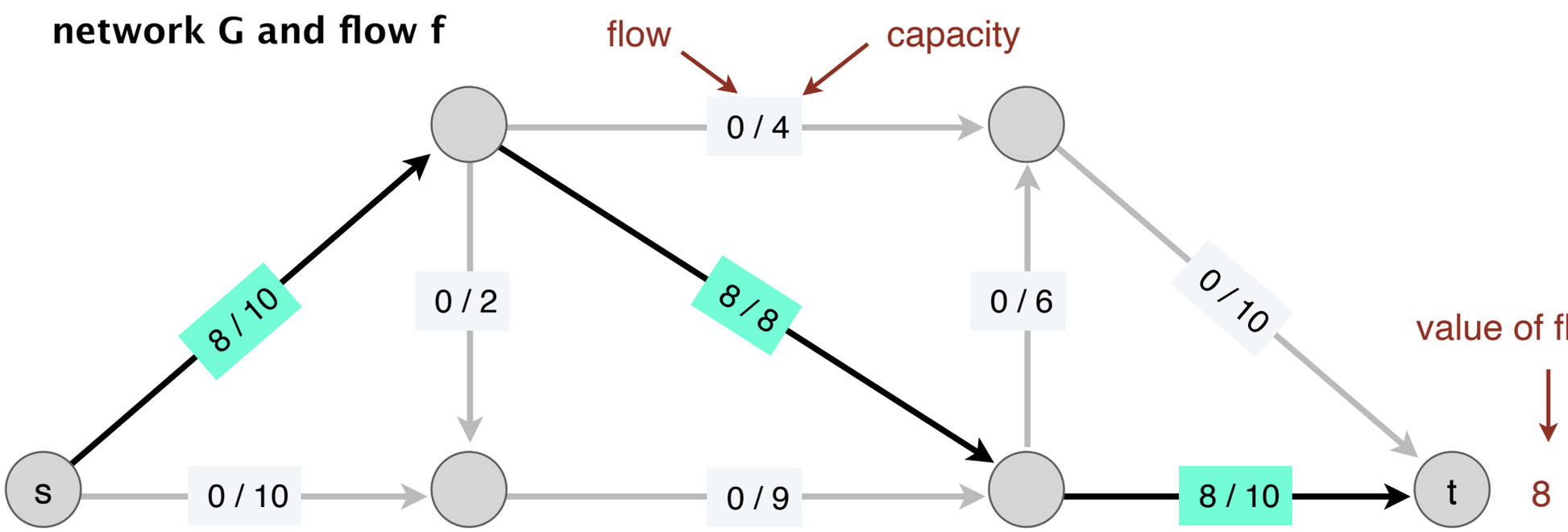

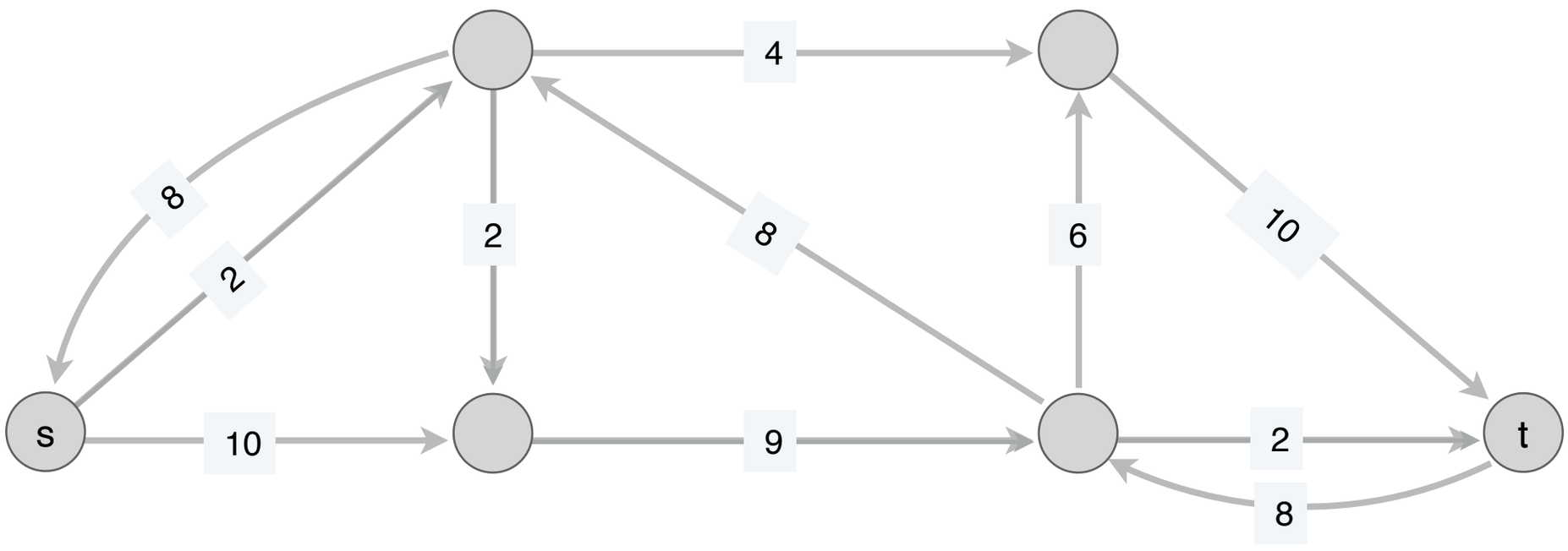

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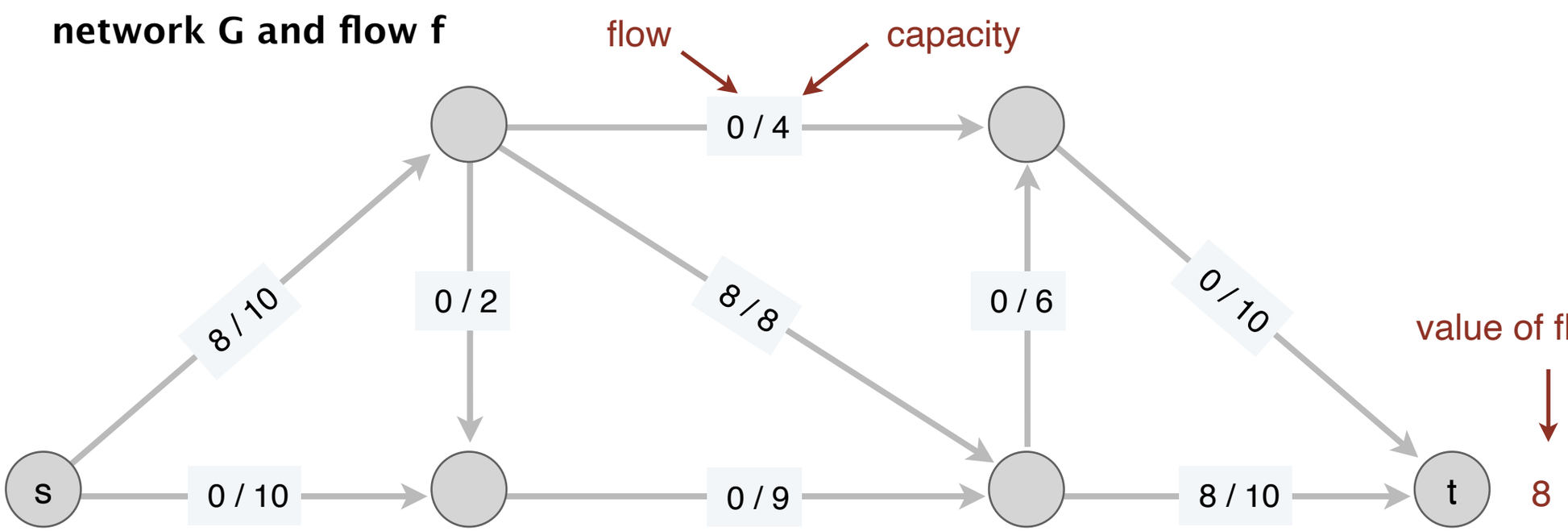

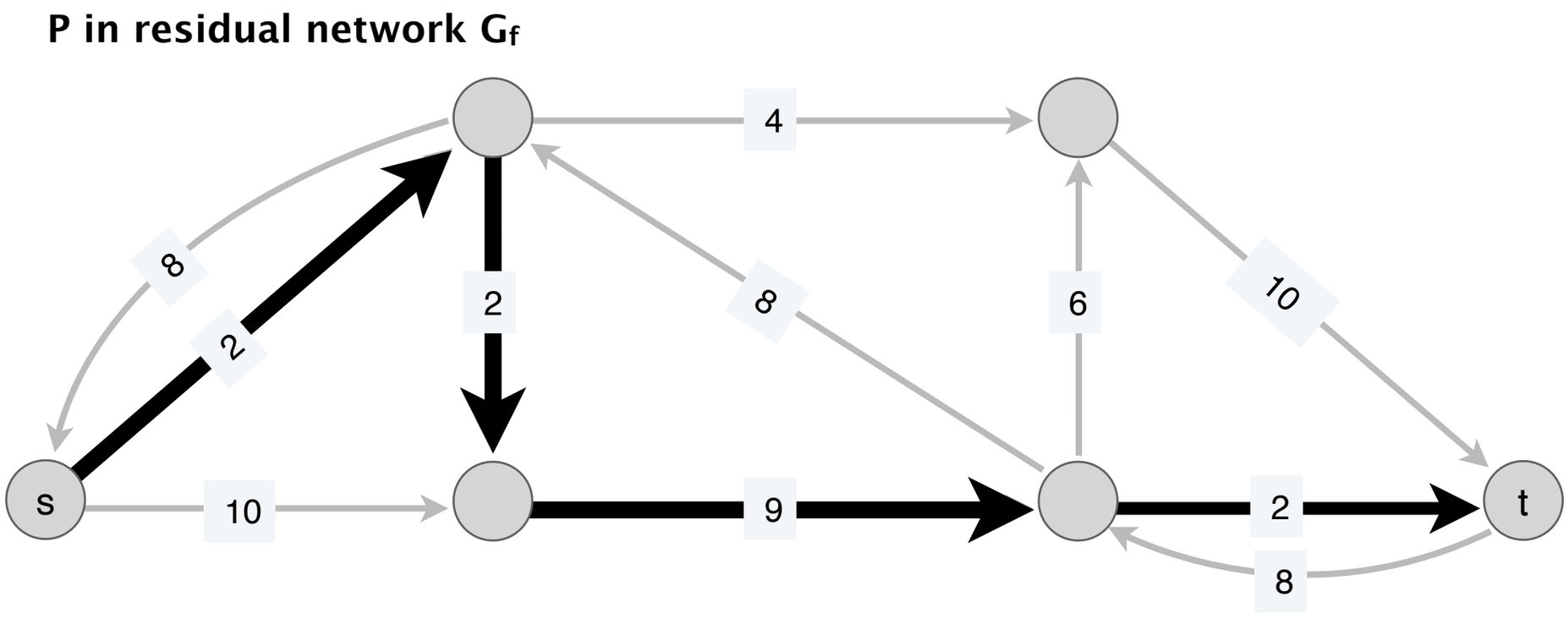

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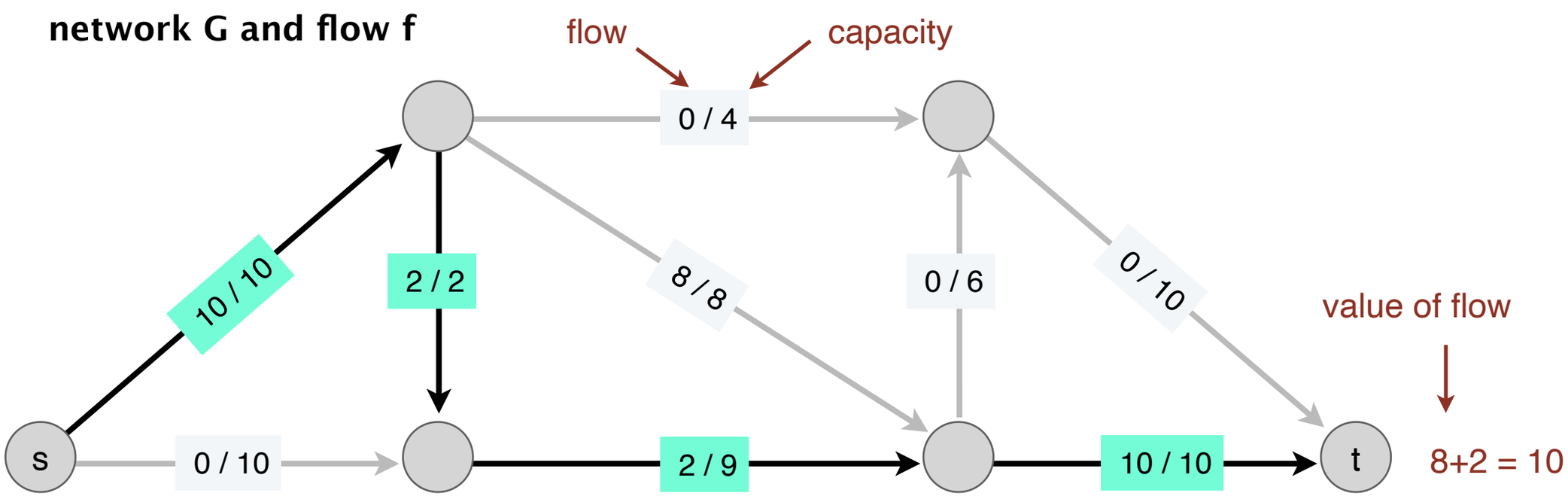

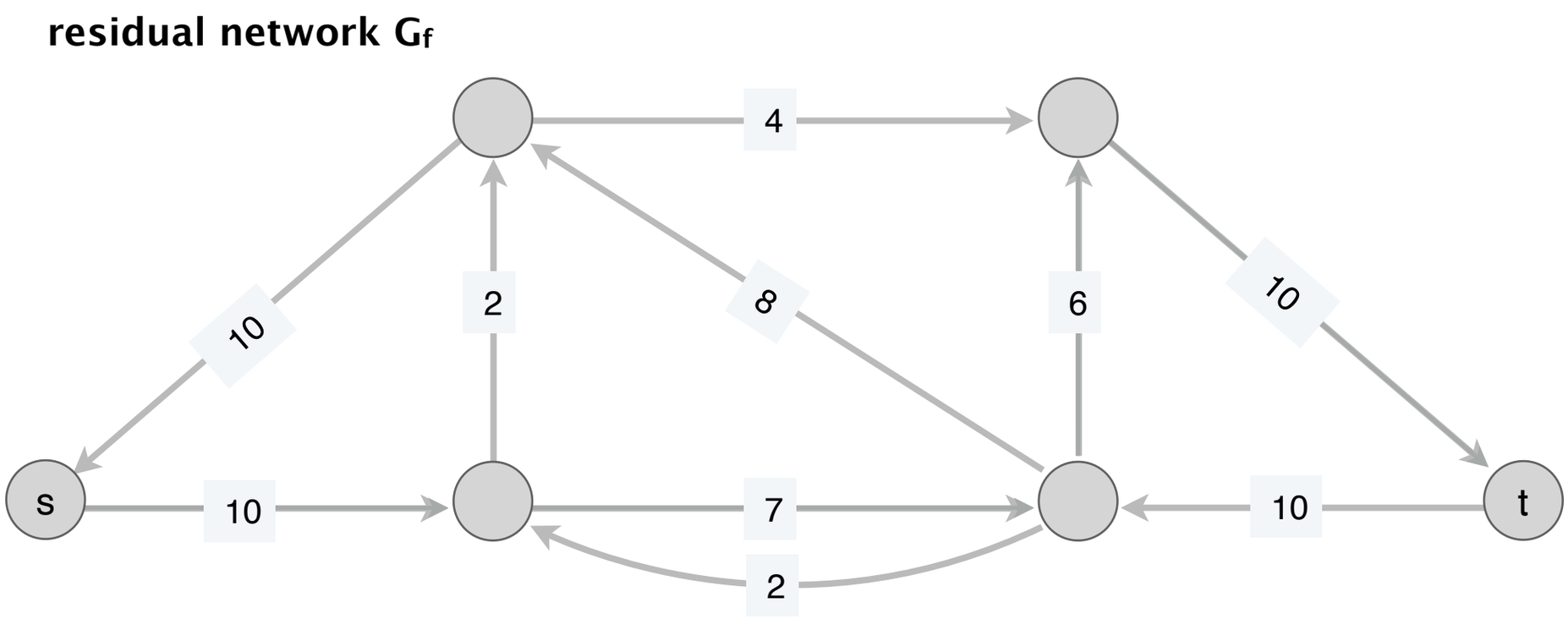

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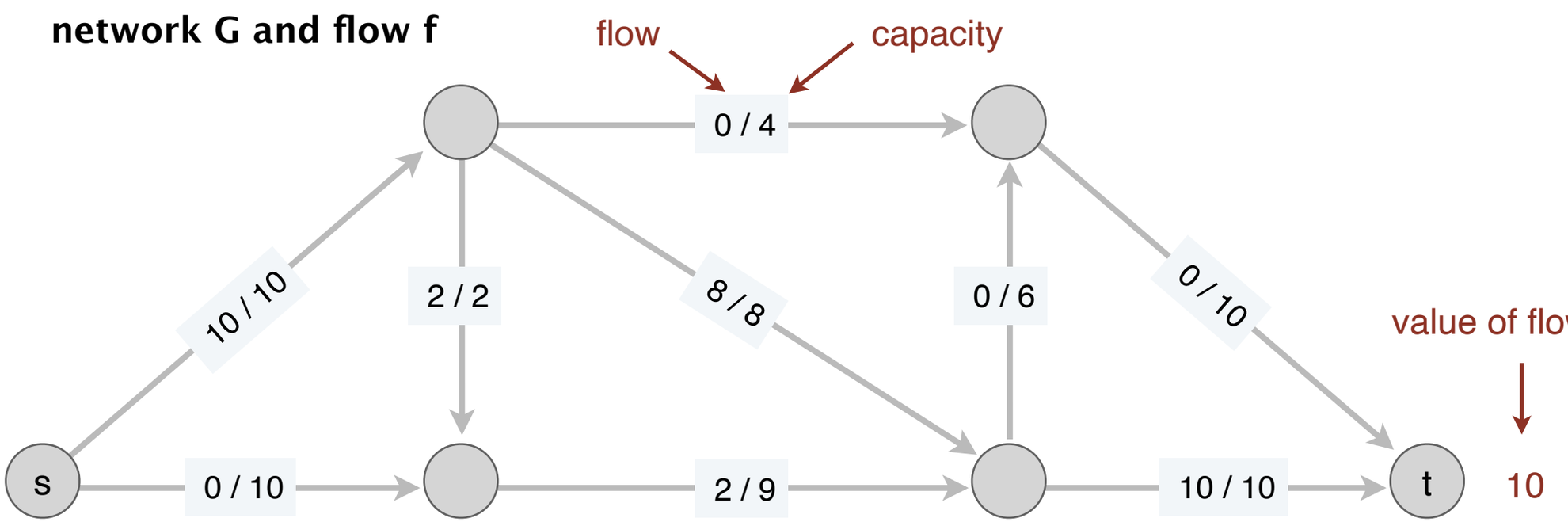

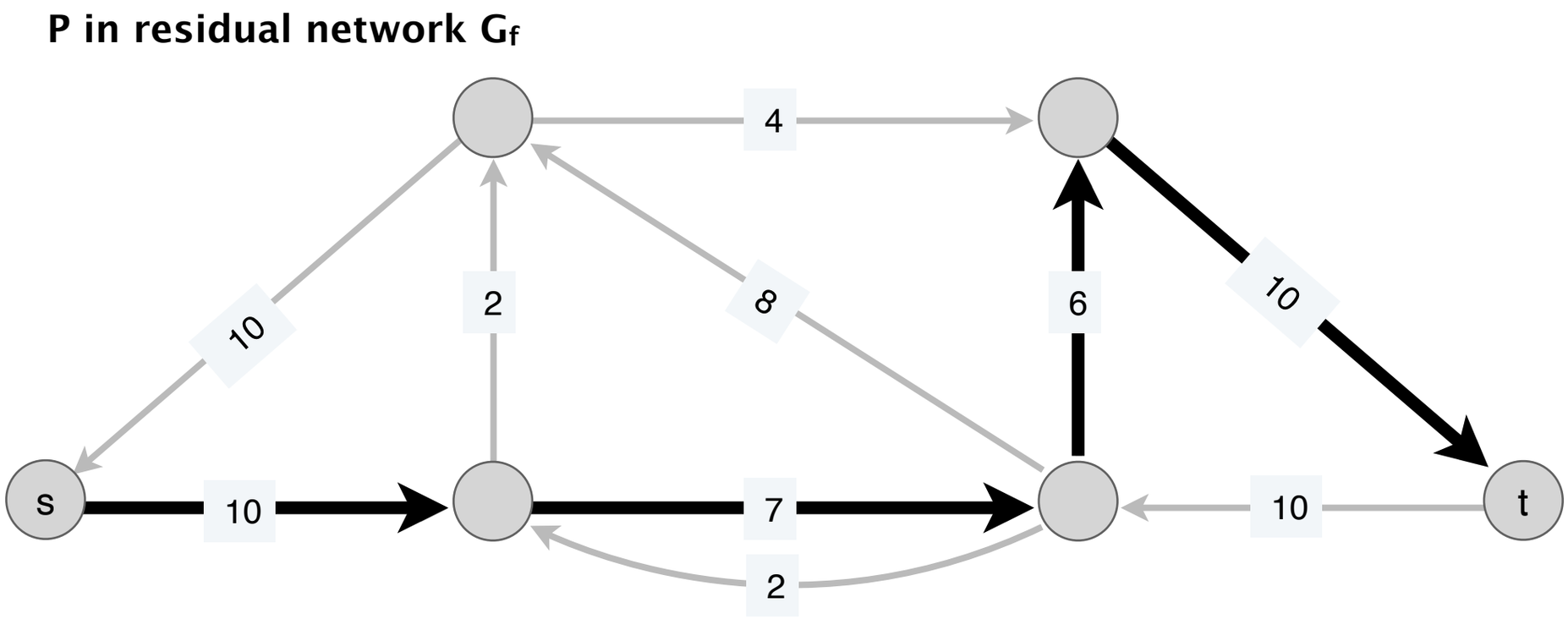

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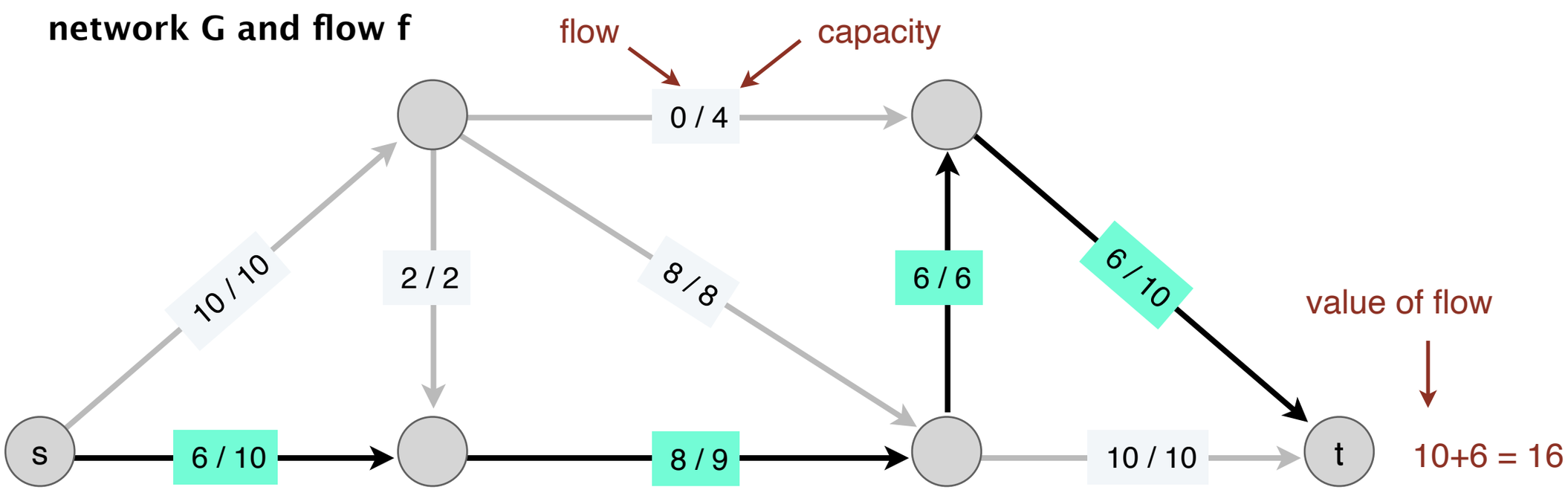

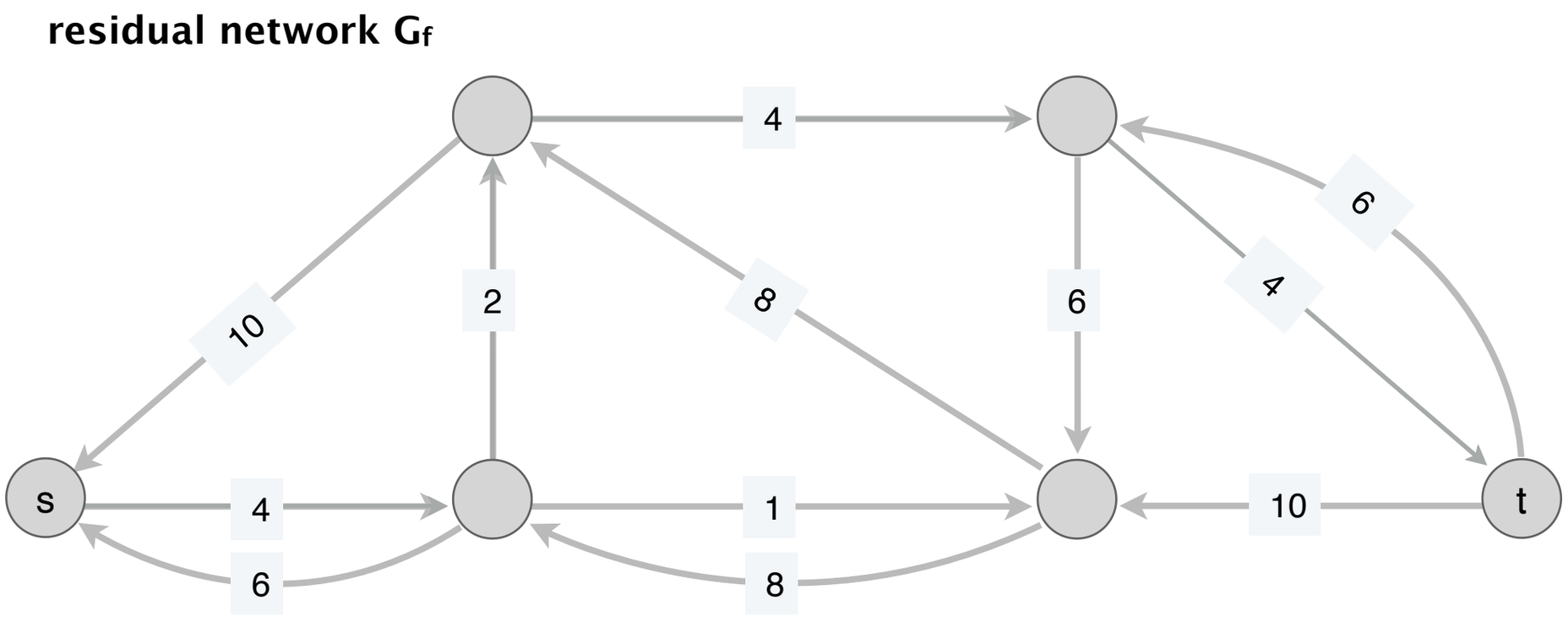

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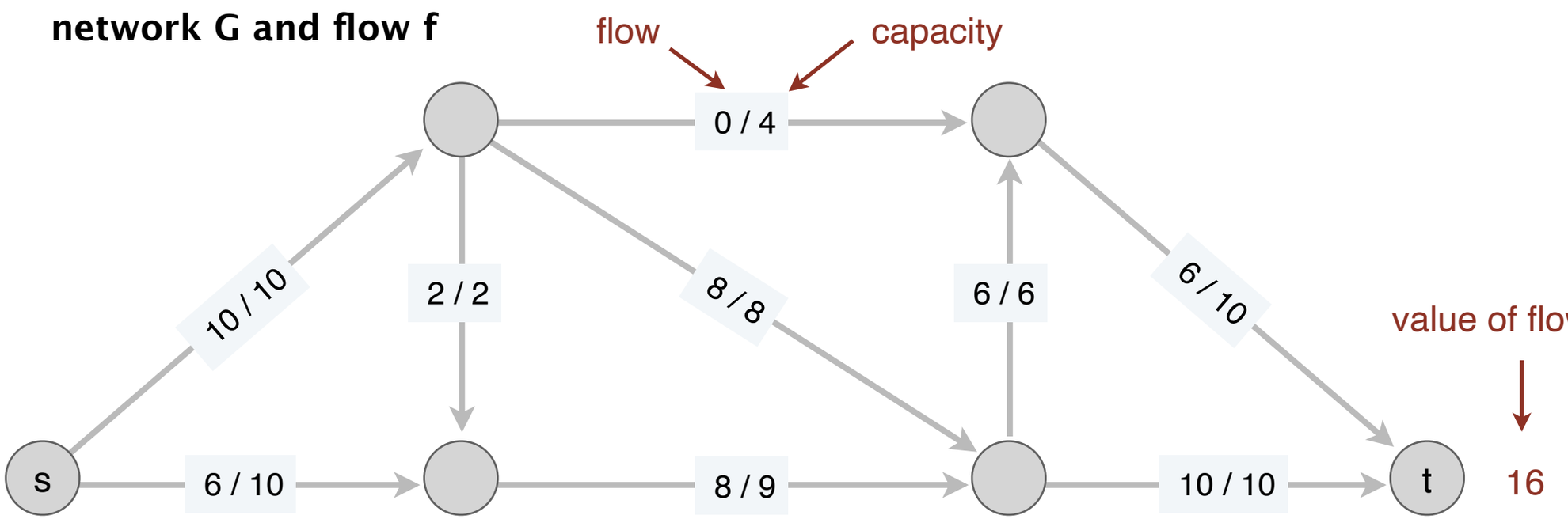

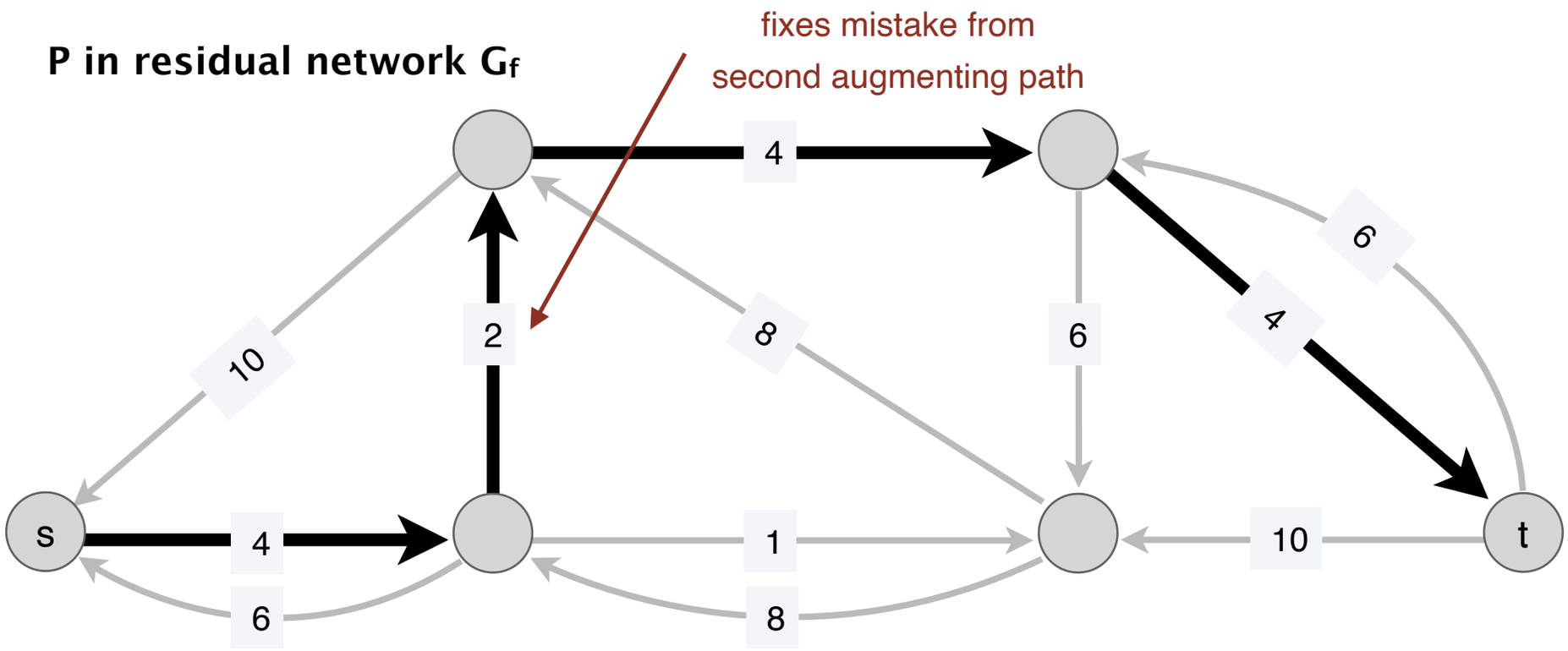

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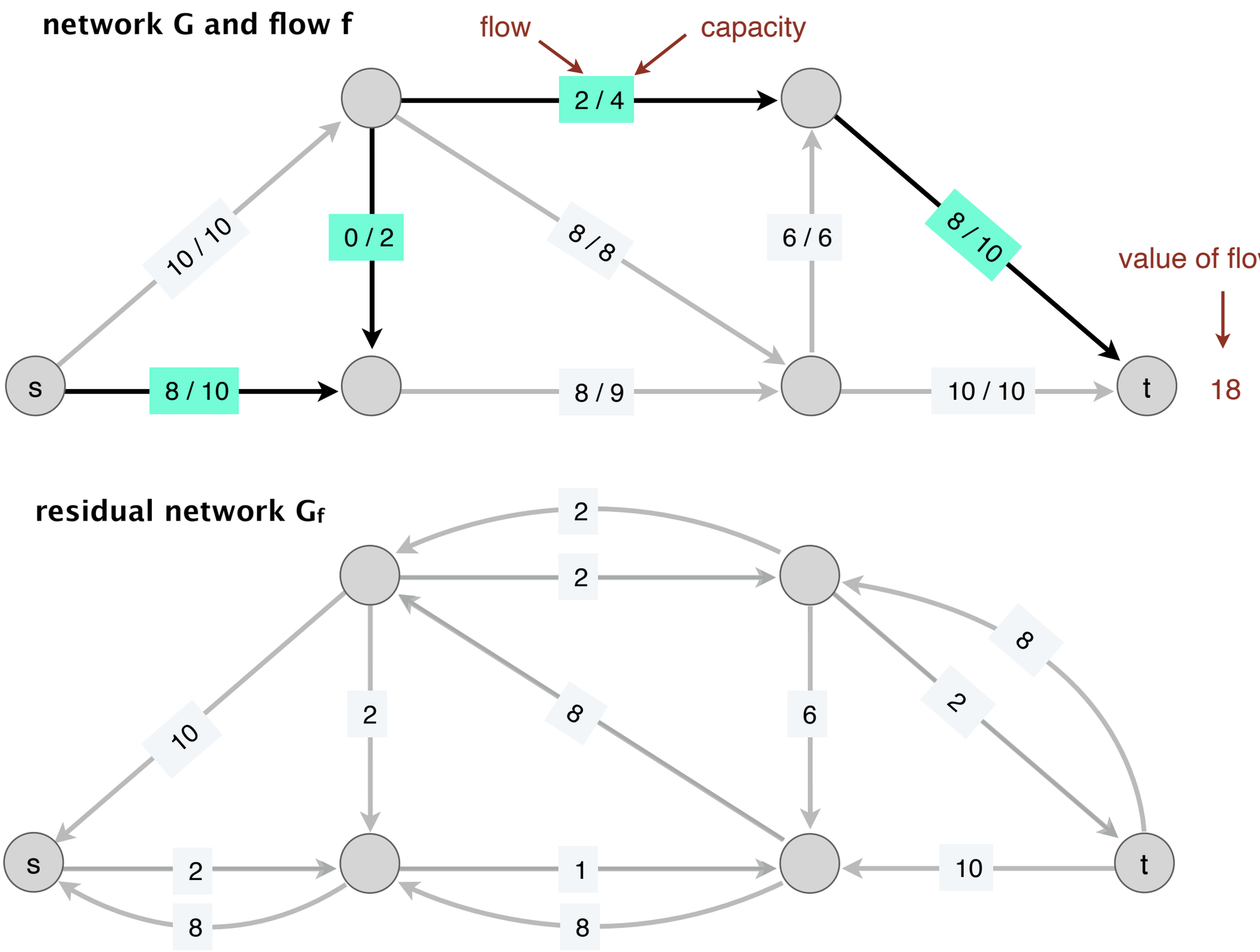

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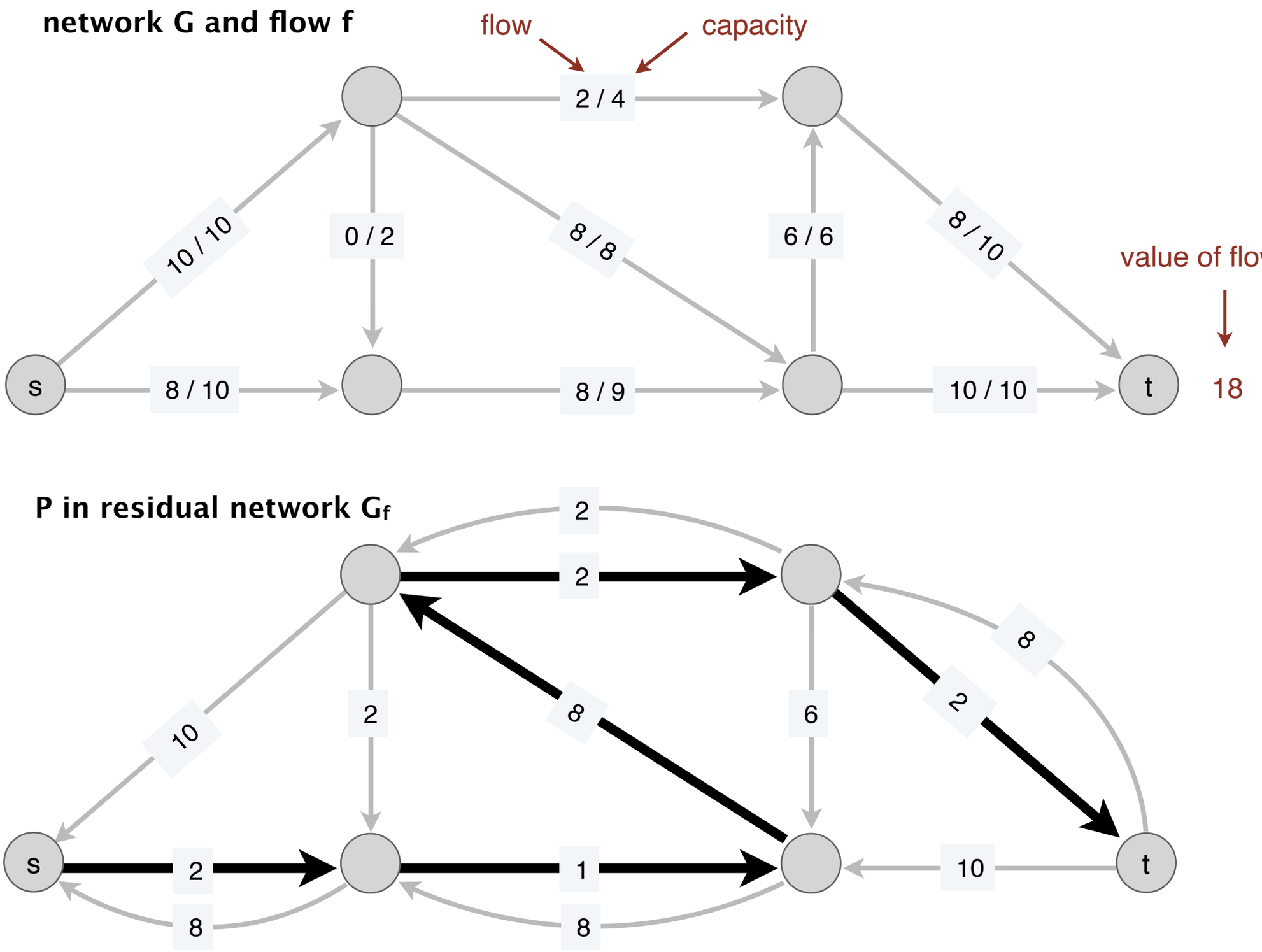

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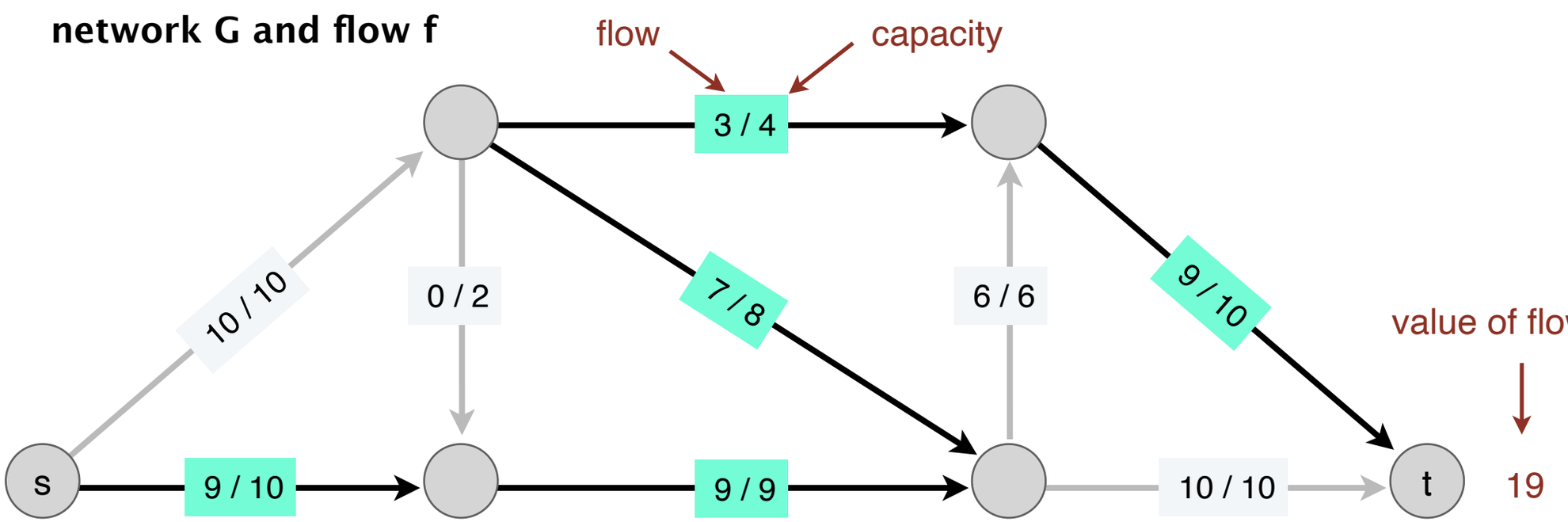

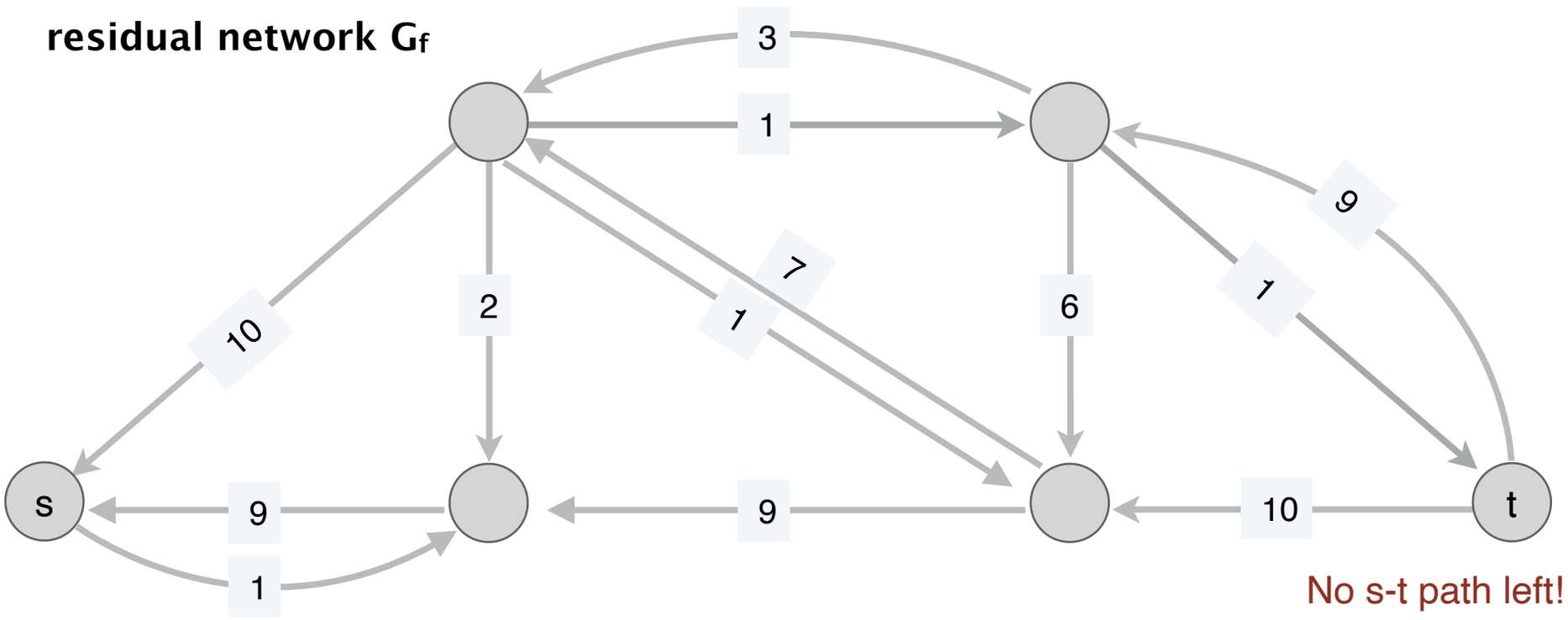

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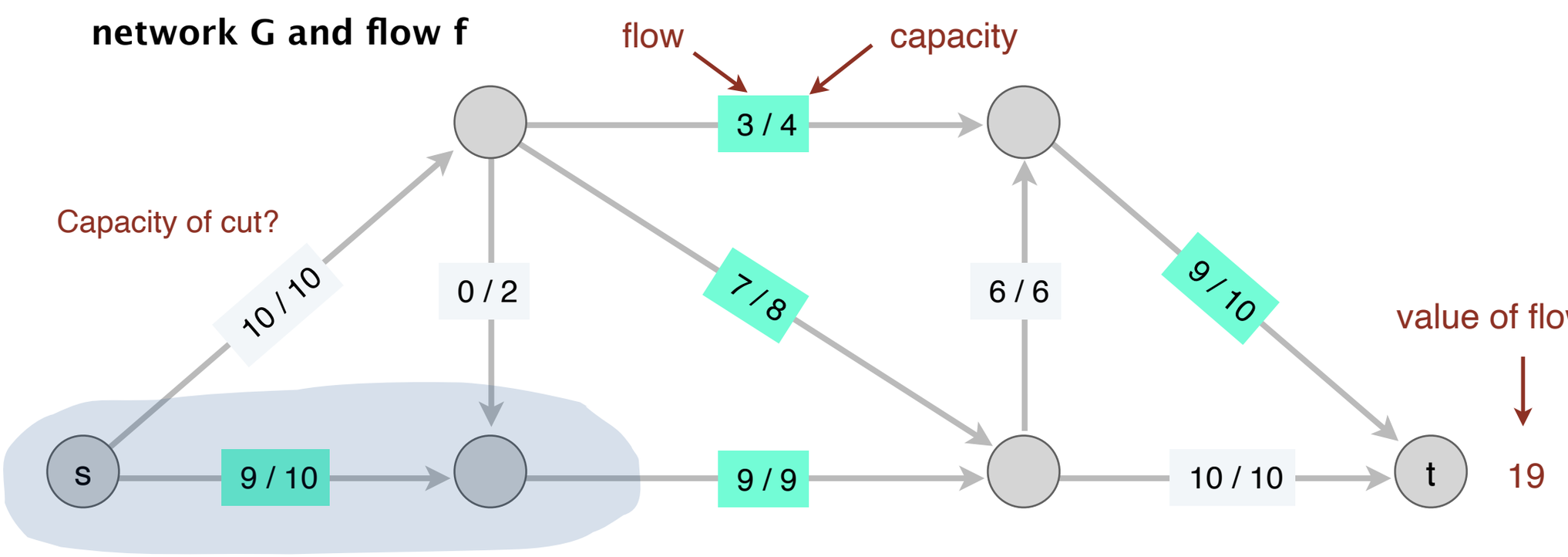
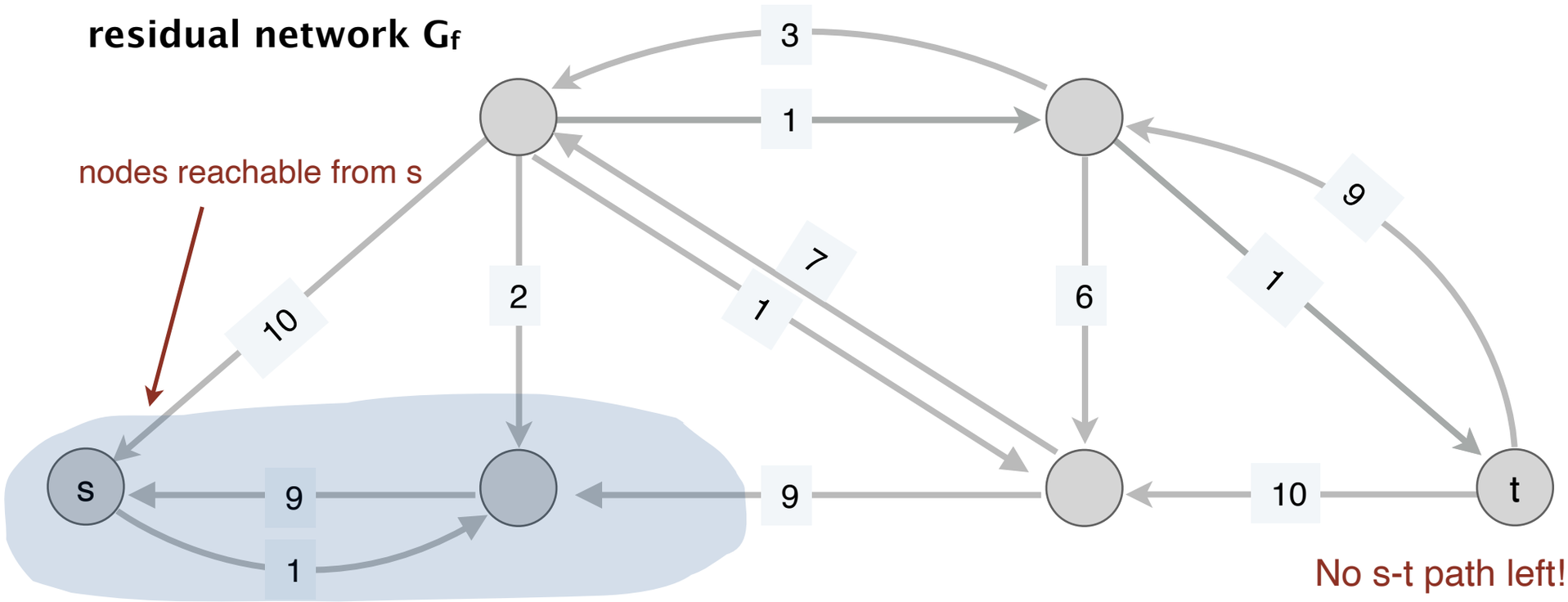




Analysis: Ford-Fulkerson

Analysis Outline

- Feasibility and value of flow:
    - Show that each time we update the flow, we are routing a feasible s-t flow through the network
    - And that value of this flow increases each time by that amount
- Optimality:
    - Final value of flow is the maximum possible
- Running time:
    - How long does it take for the algorithm to terminate?
- Space:
    - How much total space are we using

Feasibility of Flow

- **Claim**. Let f be a feasible flow in G and let P be an augmenting path in G_f with bottleneck capacity b. Let $f' \leftarrow \text{AUGMENT}(f, P)$, then f' is **a feasible flow**.

- **Proof**. Only need to verify constraints on the edges of P (since $f' = f$ for other edges). Let $e = (u, v) \in P$
  - If e is a forward edge: $f'(e) = f(e) + b$

    $$\leq f(e) \ + \ (c(e) - f(e)) \ = \ c(e)$$

  - If e is a backward edge: $f'(e) = f(e) - b$

    $$\geq f(e) \ - \ f(e) \ = \ 0$$

- Conservation constraint hold on any node in $u \in P$:
  - $f_{in}(u) = f_{out}(u)$, therefore $f'_{in}(u) = f'_{out}(u)$ for both cases

Value of Flow: Making Progress

- **Claim**. Let f be a feasible flow in G and let P be an augmenting path in G_f with bottleneck capacity b. Let $f' \leftarrow \text{AUGMENT}(f, P)$, then $v(f') = v(f) + b$.
- **Proof**.
  - First edge $e \in P$ must be out of s in G_f
  - (P is simple so never visits s again)
  - e must be a forward edge (P is a path from s to t)
  - Thus $f(e)$ increases by b, increasing $v(f)$ by b ∎
- Note. Means the algorithm makes forward progress each time!

Optimality

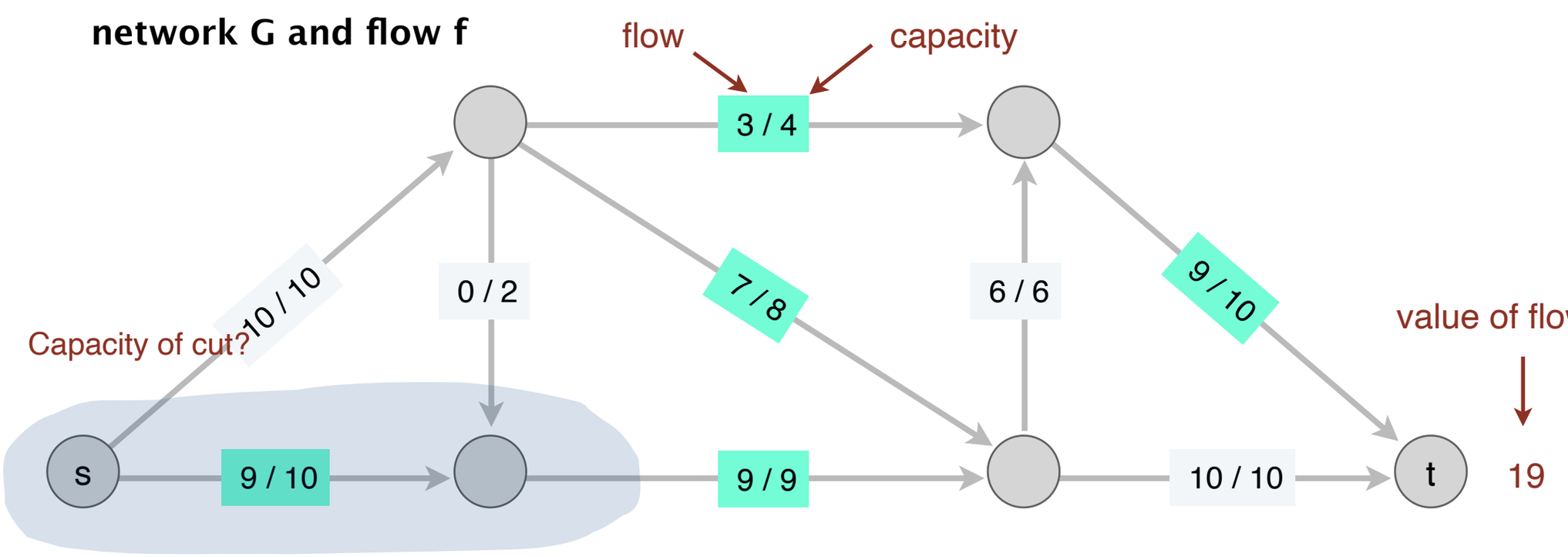
Ford-Fulkerson Optimality

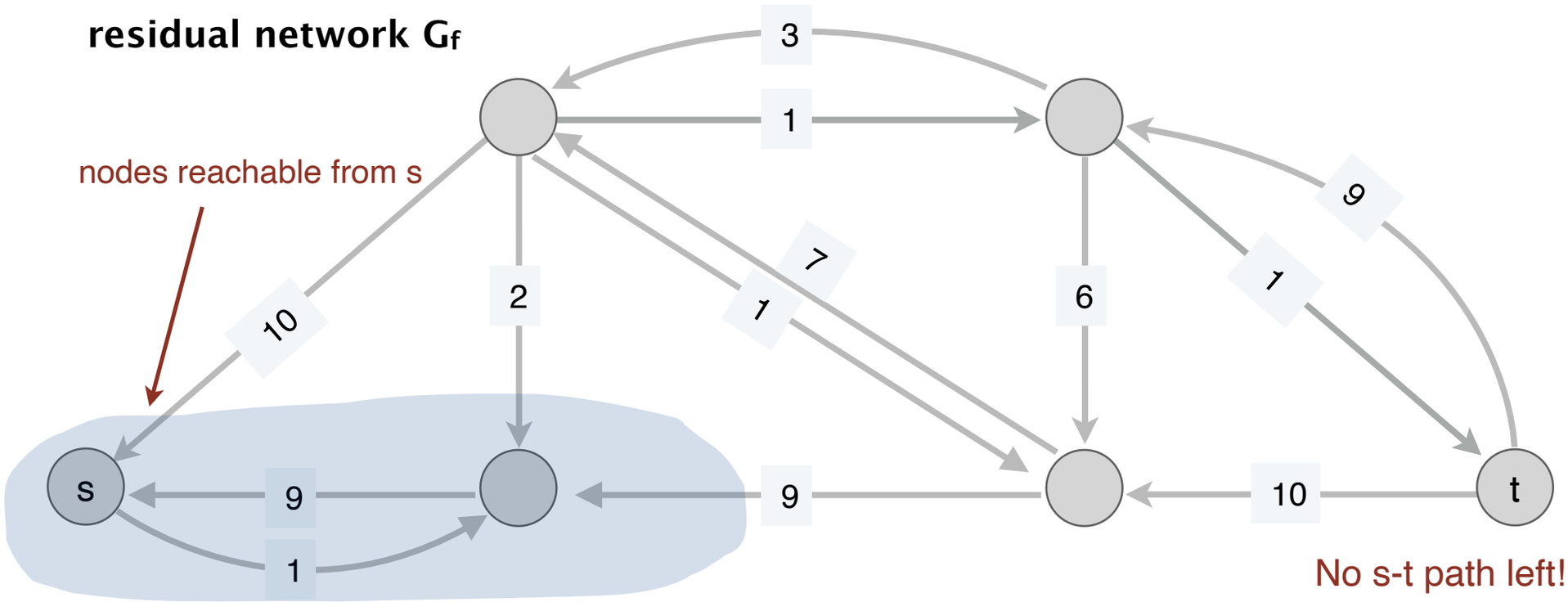
- **Recall**: If f is any feasible s-t flow and (S, T) is any s-t cut then $v(f) \leq c(S, T)$.
- We will show that the Ford-Fulkerson algorithm terminates in a flow that achieves equality, that is,
- Ford-Fulkerson finds a flow f^* and there exists a cut (S^*, T^*) such that, $v(f^*) = c(S^*, T^*)$
- Proving this shows that it finds the maximum flow (and the min cut)
- This also **proves the max-flow min-cut theorem**

Ford-Fulkerson Optimality

- **Lemma**. Let f be a s-t flow in G such that there is no augmenting path in the residual graph G_f, then there exists a cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
- **Proof**.
  - Let $S^* = \{v \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T^* = V - S^*$
  - Is this an s-t cut?
    - $s \in S, t \in T$, $S \cup T = V$ and $S \cap T = \emptyset$
  - Consider an edge $e = u \rightarrow v$ with $u \in S^*, v \in T^*$, then what can we say about $f(e)$?

Recall: Ford-Fulkerson Example





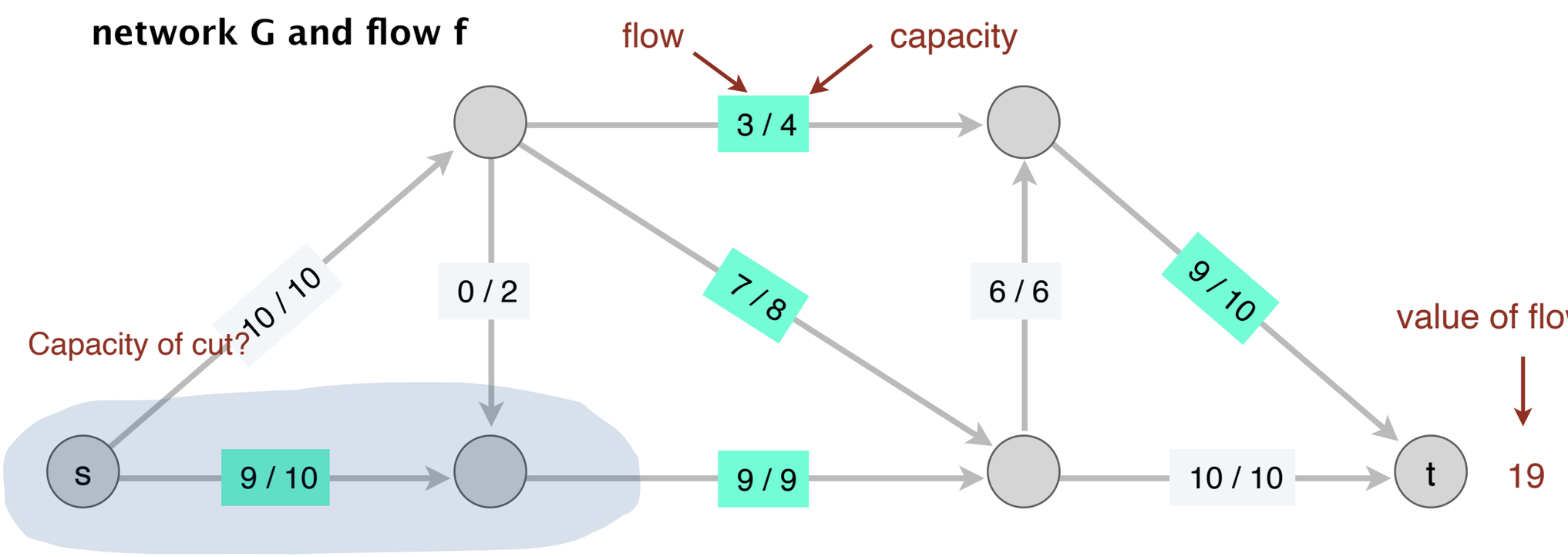
Ford-Fulkerson Optimality

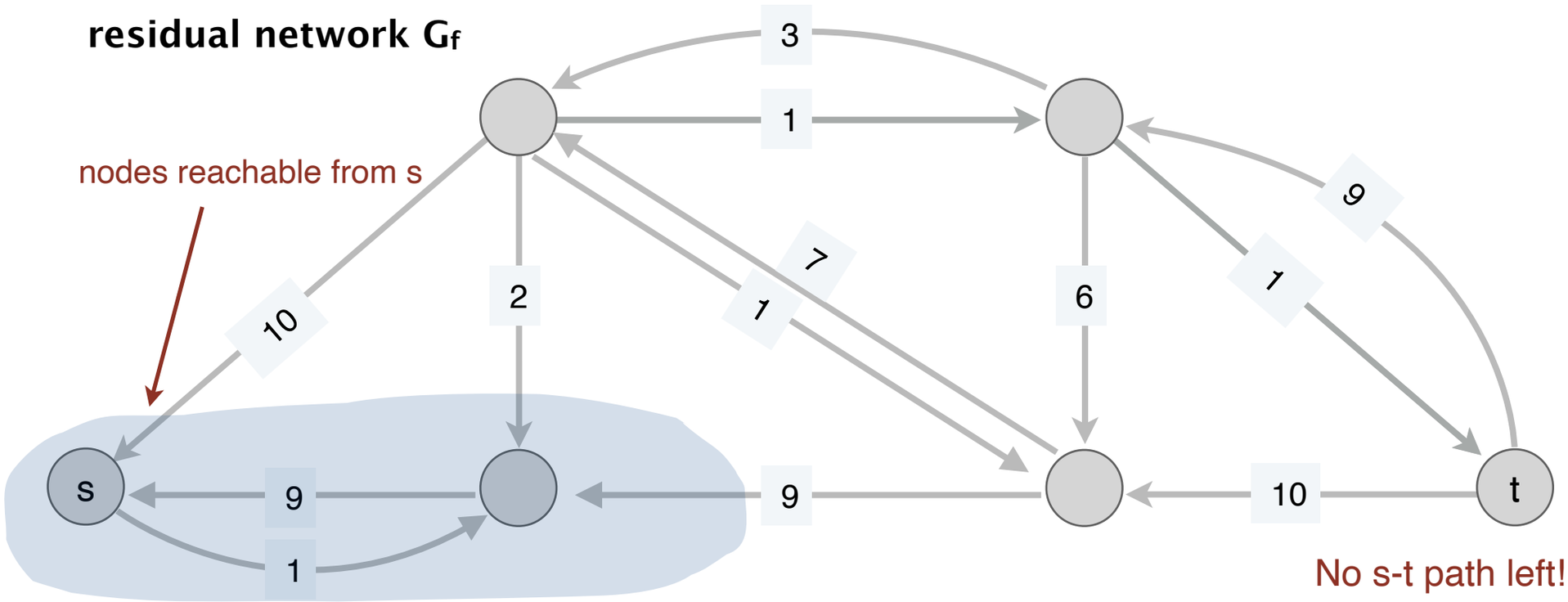
- **Lemma**. Let f be a s-t flow in G such that there is no augmenting path in the residual graph G_f, then there exists a cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
- **Proof**.
  - Let $S^* = \{v \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T^* = V - S^*$
  - Is this an s-t cut?
    - $s \in S, t \in T$, $S \cup T = V$ and $S \cap T = \emptyset$
  - Consider an edge $e = u \rightarrow v$ with $u \in S^*, v \in T^*$, then what can we say about $f(e)$?
    - $f(e) = c(e)$

Ford-Fulkerson Optimality

- **Lemma**. Let f be a s-t flow in G such that there is no augmenting path in the residual graph G_f, then there exists a cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
- **Proof**. **(Cont.)**
  - Let $S^* = \{v \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T^* = V - S^*$
  - Is this an s-t cut?
    - $s \in S, t \in T$, $S \cup T = V$ and $S \cap T = \emptyset$
  - Consider an edge $e = w \rightarrow v$ with $v \in S^*, w \in T^*$, then what can we say about $f(e)$?

Recall: Ford-Fulkerson Example





Ford-Fulkerson Optimality

- **Lemma**. Let f be a s-t flow in G such that there is no augmenting path in the residual graph G_f, then there exists a cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
- **Proof**. **(Cont.)**
  - Let $S^* = \{v \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T^* = V - S^*$
  - Is this an s-t cut?
    - $s \in S, t \in T$, $S \cup T = V$ and $S \cap T = \emptyset$
  - Consider an edge $e = w \rightarrow v$ with $v \in S^*, w \in T^*$, then what can we say about $f(e)$?
    - $f(e) = 0$

Ford-Fulkerson Optimality

- **Lemma**. Let f be a s-t flow in G such that there is no augmenting path in the residual graph G_f, then there exists a cut (S^*, T^*) such that $v(f) = c(S^*, T^*)$.
- **Proof**. **(Cont.)**
- Let $S^* = \{v \mid v \text{ is reachable from } s \text{ in } G_f\}$, $T^* = V - S^*$
- Thus, all edges leaving S^* are completely saturated and all edges entering S^* have zero flow
- $v(f) = f_{out}(S^*) - f_{in}(S^*) = f_{out}(S^*) = c(S^*, T^*)$ ∎
- **Corollary**. Ford-Fulkerson returns the maximum flow.

Ford-Fulkerson Algorithm

Running Time

Ford-Fulkerson Performance

FORD–FULKERSON(G)

FOREACH edge $e \in E$: $f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f.

WHILE (there exists an s↝t path P in G_f)

  $f \leftarrow$ AUGMENT(f, P).

  Update G_f.

RETURN f.

- Does the algorithm terminate?
- Can we bound the number of iterations it does?
- Running time?

Ford-Fulkerson Running Time

- Recall we proved that with each call to AUGMENT, we increase **value of flow** by $b = \text{bottleneck}(G_f, P)$
- **Assumption**. Suppose all capacities $c(e)$ are integers.
- **Integrality invariant.** Throughout Ford–Fulkerson, every edge flow $f(e)$ and corresponding residual capacity is an integer. Thus $b \geq 1$.
- Let $C = \max_u c(s \to u)$ be the maximum capacity among edges leaving the source s.
- It must be that $v(f) \leq (n-1)C$
- Since, $v(f)$ increases by $b \geq 1$ in each iteration, it follows that FF algorithm terminates in at most $v(f) = O(nC)$ iterations.

Ford-Fulkerson Performance

FORD–FULKERSON(G)

FOREACH edge $e \in E$: $f(e) \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f.

WHILE (there exists an s↝t path P in G_f)

  $f \leftarrow$ AUGMENT(f, P).

  Update G_f.

RETURN f.

- Operations in each iteration?
  - Find an augmenting path in G_f
  - Augment flow on path
  - Update G_f

Ford-Fulkerson Running Time

- **Claim.** Ford-Fulkerson can be implemented to run in time $O(nmC)$, where $m = |E| \geq n - 1$ and $C = \max_u c(s \to u)$.

- **Proof**. Time taken by each iteration:

- Finding an augmenting path in G_f
  - G_f has at most $2m$ edges, using BFS/DFS takes $O(m + n) = O(m)$ time

- Augmenting flow in P takes $O(n)$ time

- Given new flow, we can build new residual graph in $O(m)$ time

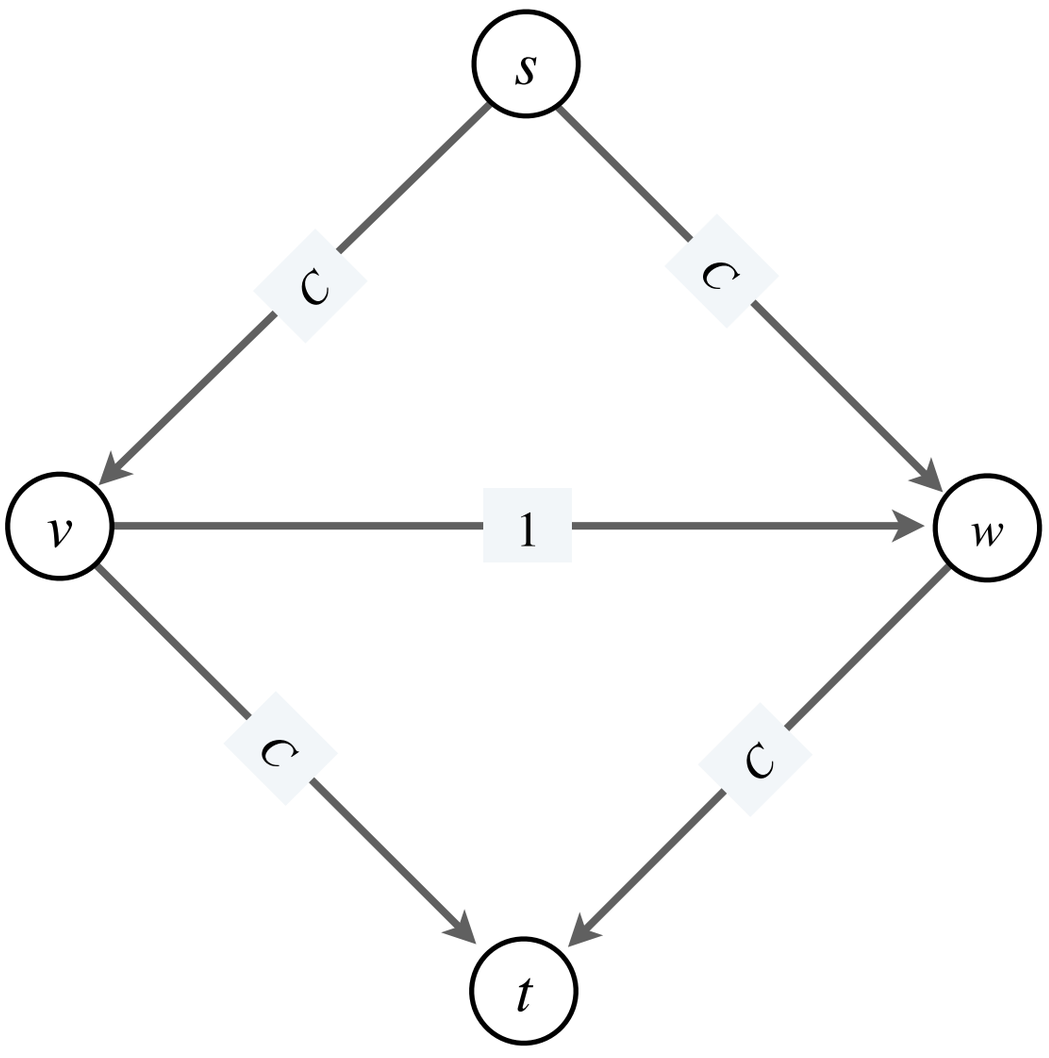
- Overall, $O(m)$ time per iteration ∎

[Digging Deeper] Polynomial time?

- Does the Ford-Fulkerson algorithm run in time polynomial in the input size?
- Running time is $O(nmC)$, where $C = \max_u c(s \to u)$
- What is the input size?
  - n vertices, m edges, m capacities
  - C represents the magnitude of the maximum capacity leaving the source node
  - How many bits to represent C?
- Let us take an example

[Digging Deeper] Polynomial time?

- **Question**. Does the Ford-Fulkerson algorithm run in polynomial-time in the size of the input? ← $\sim m$, n, and $\log C$
- **Answer**. No. if max capacity is C, the algorithm can take $\geq C$ iterations. Consider the following example.



- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$
- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$
- …
- $s \rightarrow v \rightarrow w \rightarrow t$
- $s \rightarrow w \rightarrow v \rightarrow t$

← each augmenting path sends only 1 unit of flow (# augmenting paths = $2C$)

[Digger Deeper] Pseudo-Polynomial

- Input graph has n nodes and $m = O(n^2)$ edges, each with capacity c_e
- $C = \max_{e \in E} c(e)$, then $c(e)$ takes $O(\log C)$ bits to represent
- Input size: $\Omega(n \log n + m \log n + m \log C)$ bits
- Running time: $O(nmC) = O(nm2^{\log C})$
    - Exponential in the *size* of C
- Such algorithms are called **pseudo-polynomial**
    - If the running time is polynomial in the **magnitude** but **not size** of an input parameter.
    - We saw this for knapsack as well!

Non-Integral Capacities?

- If the capacities are rational, can just multiply to obtain a large integer (massively increases running time)

- If capacities are irrational, Ford-Fulkerson can run infinitely!
    - Improvement at each step can be arbitrarily small
    - Can create bad instances where it doesn't terminate in finite steps

Network Flow:

Beyond Ford Fulkerson

Edmond and Karp's Algorithms

- Ford and Fulkerson's algorithm does not specify which path in the residual graph to augment
- Poor worst-case behavior of the algorithm can be blamed on bad choices on augmenting path
- **Better choice of augmenting paths.** In 1970s, Jack Edmonds and Richard Karp published two natural rules for choosing augmenting paths
  - Widest path first: paths with largest bottleneck capacity
  - Shortest (in terms of edges) augmenting paths first (Dinitz independently discovered & analyzed this rule)

Widest Augmenting Paths First

- Ford Fulkerson can be improved with a greedy algorithm way of choosing augmenting paths:
    - Choose the augmenting path with largest bottleneck capacity
- Largest bottleneck path can be computed in $O(m \log n)$ time in a directed graph
    - Similar to Dijkstra's analysis
- How many iterations if we use this rule?
    - Won't prove this: but takes $O(m \log C)$ iterations
- Overall running time is $O(m^2 \log n \log C)$ (polynomial time!)
    - Still depends on C though

Shortest Augmenting Paths First

- Choose the augmenting path with the smallest # of edges
- Can be found using BFS on G_f in $O(m+n) = O(m)$ time
- Surprisingly, this resulting a polynomial-time algorithm independent of the actual edge capacities !
    - Analysis looks at “level” of vertices in the BFS tree of G_f rooted at s —levels only grow over time
    - Analyzes # of times an edge $u \to v$ disappears from G_f
- Takes $O(mn)$ iterations overall
- Thus overall running time is $O(m^2 n)$

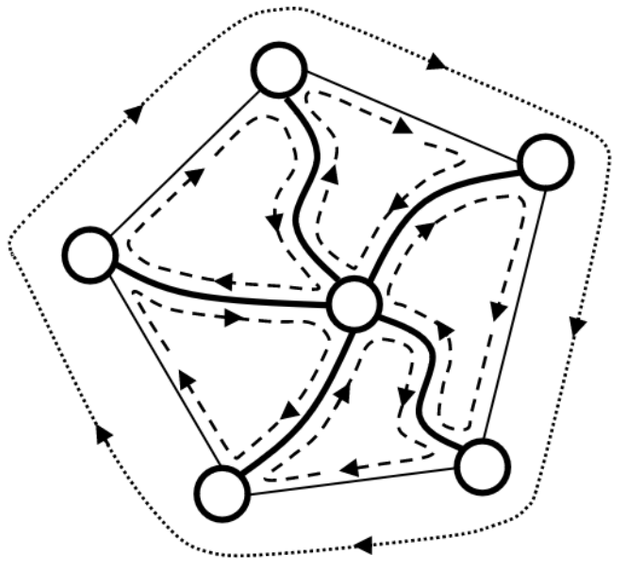
Progress on Network Flows

1951	$O(m\, n^2\, C)$	Dantzig
1955	$O(m\, n\, C)$	Ford–Fulkerson
1970	$O(m\, n^2)$	Edmonds–Karp, Dinitz
1974	$O(n^3)$	Karzanov
1983	$O(m\, n \log n)$	Sleator–Tarjan
1985	$O(m\, n \log C)$	Gabow
1988	$O(m\, n \log (n^2 / m))$	Goldberg–Tarjan
1998	$O(m^{3/2} \log (n^2 / m) \log C)$	Goldberg–Rao
2013	$O(m\, n)$	Orlin

Best among “combinatorial” approaches that push flow through the graph

Progress on Network Flows

- More recently: [Chen et al. 2022] achieve running time better than $O(m^{1+\epsilon})$ for *any* constant ϵ
- Specifically: $O\left(m^{1+1/\log^{1/168} m}\right)$
  - (don't worry about this running time)
  - Not combinatorial: uses "interior point methods"
    - "Jumps" between solutions with drastically different, non-integral flow values
    - (Very intense math)



Progress on Network Flows

- Let's say that the best known: $O(nm)$
- **For the purpose of this class**, network flows can be solved in $O(nm)$ time
- Some of these algorithms do REALLY well in "practice" basically $O(n + m)$