CS256: Algorithm Design and Analysis **Spring 2025**

Problem Set 8 (due 05/14/2025)

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Problem 1. Suppose an extra-terrestrial being gave you a magic blackbox that can solve HAMILTONIAN-CYCLE, which is an NP-complete problem, with a worst-case running time of $O(n^8)$, where n denotes the number of vertices in the graph.

Assuming that our magic box actually works, which of the following can we conclude? Give a brief justification with your response.

(a) All problems in NP are solvable in polynomial time.
(b) All NP-hard problems are solvable in polynomial time.
(c) All NP-complete problems are solvable in $O(n^8)$ time.

Solution.

□

NP-Completeness

A Note of NP-completeness proofs. The next two questions will ask you to prove that a given problem X is NP-Complete (not just NP-hard). A complete solution would include the following:

- Problem X is in NP: you don't have to give a "tight bound" on the verification complexity just briefly reason why it can be achieved in time polynomial in the input size.
- State a *known* NP hard problem Y from class that you will use to prove X is also NP hard
- Show that $Y \leq_p X$. Remember to:
  - Prove that the reduction is correct by arguing both the "if" and "only if" directions
  - Argue that your reduction is polynomial time (again, you are don't have to give an efficient bound, just that is is polynomial time, so this piece should 1-2 sentences)

Students often ask "Which problems can I use as the *known* NP-hard problem?" You can use any of the problems whose NP-hardness was established in class, or Chapter 8 of Kleinberg Tardos or Chapter 12 of Erickson. Here is a (not necessarily complete) list:

(a) Independent Set
(b) Vertex Cover
(c) Set Cover
(d) Clique
(e) 3-SAT
(f) Hamiltonian Cycle/Path
(g) Graph 3-color
(h) Subset-Sum

Problem 2. (KT 8.5) Consider a set $A = \{a_1, \dots, a_n\}$ and a collection $B_1, B_2, \dots, B_m$ of subsets of A (i.e., $B_i \subseteq A$ for each i). We say that $H \subseteq A$ is a hitting set for the collection $B_1, \dots, B_m$ if H contains at least one element from each B_i—that is, if $H \cap B_i$ is not empty for each i (so H "hits" all the sets B_i).

We now define the *Hitting Set Problem* as follows: given $A = \{a_1, \dots, a_n\}$, a collection $B_1, B_2, \dots, B_m$ of subsets of A, and a number k, is there a hitting set $H \subseteq A$ for $B_1, B_2, \dots B_m$ such that size of H is at most k. Prove that Hitting Set is NP complete.

Solution.

□

Problem 3. Suppose you are in charge of forming a student committee at Williams. Given a set of n students and a compatibility function that maps every student to a subset of students they are compatible with, you must choose k students to be in the committee such that each student in the committee is compatible with everyone else.

(a) Show that the problem of determining whether it is possible to form such a committee is NP-complete.

(b) Suppose you were forced to pick a given student as the president of your committee. That is, you must always include this person (and each person in the committee should still be compatible with the president and everyone else). Show that this new version is still NP-complete, by modifying your reduction from part (a).

Note: You need to use a polynomial-time reduction as defined in class: you need to map "yes"-instances to "yes"-instances, and "no"-instances to "no"-instances. Any answers of the type: "choose each vertex one at a time as president and call the oracle n times" are not correct polynomial time reductions..

Solution.

□

Problem 4. (**Extra Credit: 5 points**)

Define the ODD − SUBSET − SUM − RANGE problem as follows. Given n odd numbers $s_1, \ldots s_n$, and two target integers T_1 and T_2, determine if there exists a subset of numbers that adds up to a value between T_1 and T_2.[1]

Prove that ODD − SUBSET − SUM − RANGE is NP-hard.

Solution. □

[1] "Between" is inclusive, so a subset adding up to exactly T_1 or T_2 indicates a yes instance.