CS 256: Algorithm Design and Analysis

Problem Set 2 (due 02/26/2024 at 9:59pm)

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Problem 1. (KT 3.5) Let $G = (V, E)$ be a connected undirected graph, and let T be a depth-first spanning tree of G rooted at some node v. Prove that if T is also a breadth-first spanning tree of G rooted at v, then $E = T$, that is, all edges of G must be present in T and vice versa.

> **Hint.** Use the properties of BFS and DFS that we proved in class.

Solution. $\square$

Problem 2. Recall that the diameter of a graph G is the "longest shortest path", that is, $diam(G) = \max\{dist(u, v) : u, v \in V\}$, where $d(u, v)$ is the length of the shortest path from u to v in G. Let $T = (V, E)$ be a tree with n vertices. In this question, we will design an $O(n)$-time algorithm to find the diameter of T by modifying recursive DFS.[1]

In particular, suppose we modify recursive DFS so that it also computes, for each vertex v: (i) the diameter of the subtree of T rooted at v, and (ii) the longest path from v to a leaf in the subtree of T rooted at v.

(a) Why do you think we need to know both (i) and (ii)?

Solution. □

(b) Show how, knowing this information for all children of some vertex u, we can determine this information for u itself.

Solution. □

(c) Argue why a modified recursive DFS (that computes the diameter as explained in part(b)) would still be $O(n)$ time. (To analyze the running time, you'd have describe how the algorithm works but only in so much detail as to justify the time bound.)

Solution. □

[1] It might seem odd to apply BFS or DFS to a tree—you just get the tree you started with! However, the traversal of a tree in a particular order can allow for efficient computation of useful quantities.

Problem 3. You're helping a group of archaeologists analyze some ancient data they've found. From this data, they've learned about a set of n ancient dynasties, which we'll denote $D_1, D_2, \ldots, D_n$.

Unfortunately, the data they've found does not contain when each dynasty began or ended.[2] Instead, they have found a sequence of facts, each of which has one of the two following two forms:

- For some i and j, dynasty D_i ended before dynasty D_j began; or
- For some i and j, dynasties D_i and D_j overlapped at least partially.

Naturally, they're not sure that all these facts are correct. So what they'd like you to determine is whether the data they've collected is at least internally consistent, in the sense that there could have existed a set of dates for which all the facts they've learned simultaneously hold.

Give an algorithm to do this: either it should produce proposed beginning and ending dates for each of the n dynasties so that all the facts hold true, or it should report (correctly) that no such dates can exist—that is, the facts collected by the ethnographers are not internally consistent. You can assume that the "dates" have the form $1, 2, \ldots$.

Justify the correctness and running time of your algorithm.

Hint. Model the collected facts as a directed graph. The direction of an edge (x, y) can represent an event x occurring before event y. What should be the nodes of this graph?

Your proof should address how you modeled the problem as a graph. In other words, you should show that the graph has a certain property if and only if there is a correct ordering of the events.

Solution. □

[2] To be clear: each dynasty has one point when it began, and one point when it permanently ended. After ending, a dynasty will not restart at a later date; that would be considered a different dynasty.

Problem 4. Suppose G is a connected undirected graph with a node v such that removing v from G makes the remaining graph disconnected. Such a v is called an *articulation point*. Let T be a DFS tree of G rooted at v. Show that v is an articulation point **if and only if** v has at least two children in T.

Solution.

$\square$