CS256: Algorithm Design and Analysis

Assignment 1 (due 2/19/25)

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LaTeX typesetting is worth 5 points on this assignment. Remember to turn your assignment in on Gradescope. Please use the "partner submission" feature so that you and your partner only turn in one assignment. Partner assignments can be found on Glow.

Problem 1 (10 points)**.** For any n, consider the following input: For all $1 \leq i \leq n$, the preferences for h_i are $s_1, s_2, \ldots, s_n$. For all $1 \leq i \leq n$, the preferences for s_i are $h_1, h_2, \ldots, h_n$.

Show that for any n, under the above input, the `while` loop of Gale-Shapeley iterates $\Omega(n^2)$ times (in other words, the hospitals make $\Omega(n^2)$ offers). You are not required to give a formal inductive proof—a clear English explanation suffices.

> **Technical Clarification.** For this question, we will assume that the list of free hospitals is implemented using a queue, exactly as we saw in class, and assume that the queue begins with all hospitals in order $h_1, \ldots, h_n$ (so h_1 will be the first hospital removed from the queue).

Solution. □

Problem 2 (10 points)**.** Decide whether you think the following statement (Statement 1) is True or False. If the statement is true give a proof; if false give a counterexample.

> **Statement 1.** Consider an instance of the Stable Matching problem in which there is a hospital h and a student s such that h is ranked first on the preference list of s, and s is ranked first on the preference list of h. Then h and s must be matched to each other in every stable matching for the instance.

Solution. □

> **Note.** This question relies on material from the BFS portion of the course, which we'll start on Thursday but likely will not finish covering until Monday 2/17.

Problem 3 (10 points)**.** (KT 3.9) There is a natural intuition that two nodes that are far apart in a communication network, i.e. separated by many hops, have a more tenuous connection than two nodes that are close together. Here is one way of making this intuition precise.

Suppose that an n-node undirected graph $G = (V, E)$ contains two nodes s and t such that the distance between s and t is strictly greater than $n/2$.

We will prove the following: there must exist some node v, not equal to either s or t, such that deleting v from G destroys all s-t paths. (In other words, the graph obtained from G by deleting v contains no path from s to t.)

(a) Let's begin by looking at an interesting idea for a proof that doesn't quite work.

> **Proof Attempt.** Let P be the shortest path from s to t in the graph. Let P' be another path from s to t. Since P' has length $> n/2$ and P has length $> n/2$, and there are only n vertices, there must be at least one vertex v (where v is not s or t) that is in both P and P'. Then deleting v disconnects s and t.

Give a counterexample to show that this proof attempt is incorrect. The counterexample should be a graph where the shortest path P from s to t in the graph has length at least $n/2$, but after we delete v as defined in the proof (based on some P'), there is still a path from s to t.

Solution. □

(b) Fill in a correct proof: prove that there must exist some node v, not equal to either s or t, such that deleting v from G destroys all s-t paths. (In other words, show that the graph obtained from G by deleting v contains no path from s to t.)

Solution. □

(c) Using your answer from part (b), give an algorithm with running time $O(m + n)$ to find such a node v.

Solution. □