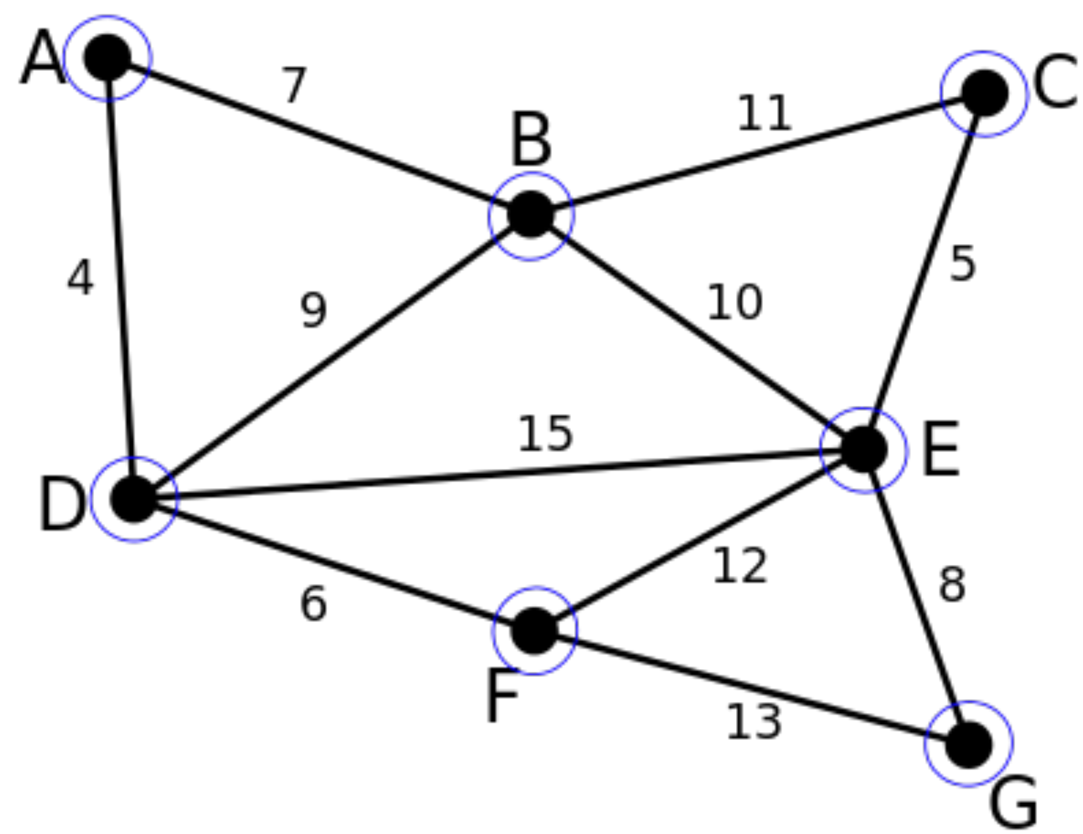
Greedy Graph Algorithms: Minimum Spanning Trees

Kruskal's Algorithm

Kruskal's Algorithm

Idea: Add the cheapest remaining edge **that does not create a cycle**.

- Initialize $T = \varnothing$, $H \leftarrow E$
- While $|T| < n - 1$:
    - Remove cheapest edge e from H
    - If adding e to T does not create a cycle
        - $T \leftarrow T \cup \{e\}$
    - $H \leftarrow H - \{e\}$



Union-Find Data Structure

Manages a **dynamic partition** of a set S

- Provides the following methods:
  - `MakeUnionFind()`: Initialize
  - `Find(x):` Return name of set containing x
  - `Union(X, Y)`: Replace sets `X, Y` with $X \cup Y$

Kruskal's Algorithm can then use

- `Find` for cycle checking
- `Union` to update after adding an edge to T

Union-Find: Any Ideas?

How can we get:

- $O(1)$ Find
- $O(n)$ Union

(Hint: we'll be maintaining labels)

Union-Find: First Attempt

Let $S = \{1, 2, \ \ldots, n\}$ be the set.

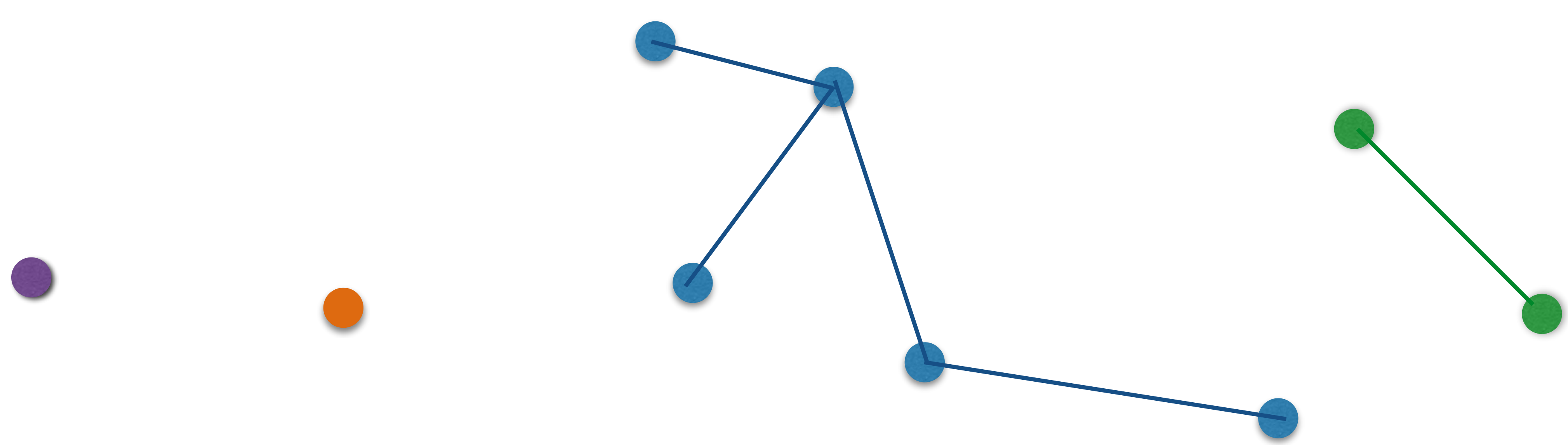
Idea: Each element stores the label of its set

- `Initialize()`: Set $L[x] = x$ for each $x \in S$: $O(n)$
- `Find(x)`: Return $L[x]$: $O(1)$
- `Union(X,Y)`:
  - For each $x \in X$, update $L[x]$ to label of set Y
  - $O(n)$ in the worst case (happens when we union two large sets)

Digging Deeper

Union-Find: Improving Union

- Let's perturb that idea just a little bit and analyze it a bit more carefully
- Think of a data structure with pointers instead of an array
- Each vertex points to a "head" node instead of a label; head points to itself



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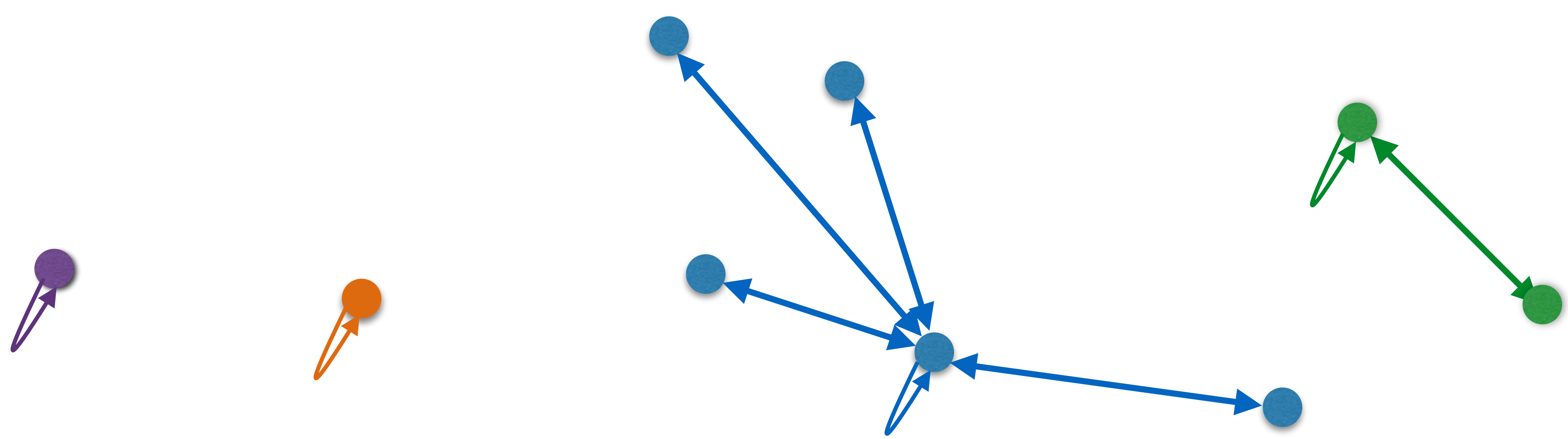
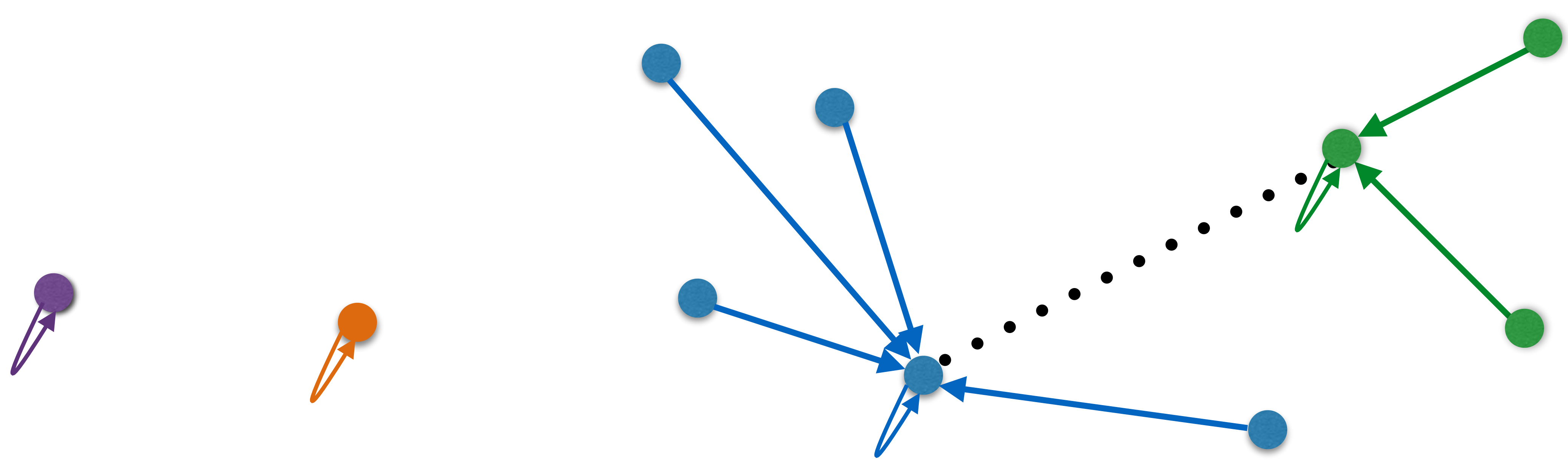


Union-Find: Improving Union

- Let’s perturb that idea just a little bit and analyze it more tightly
- Each vertex points to a “head” node instead of a label; head points to itself
- Also store size of each set in the head
- Now, to do a union, make every element in the smaller set point at the head of the larger set
    - Update the size

Union-Find: Improving Union

- Let’s say we have an edge between the blue tree and the green tree
- Update the green tree!
- Follow back pointers from the head of the tree so we get every node



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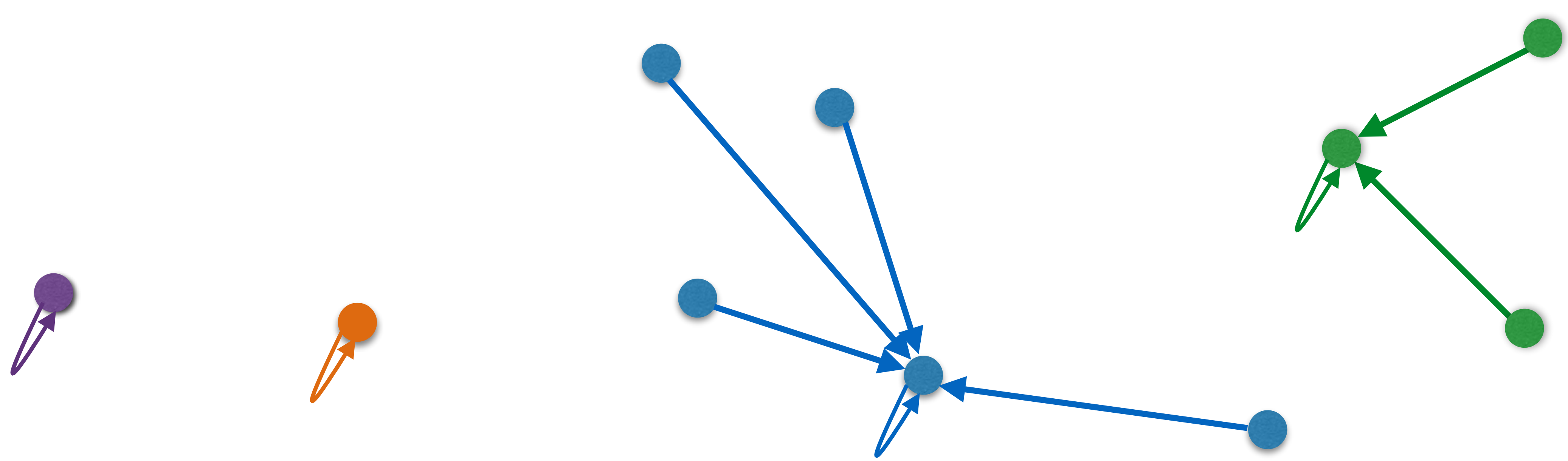

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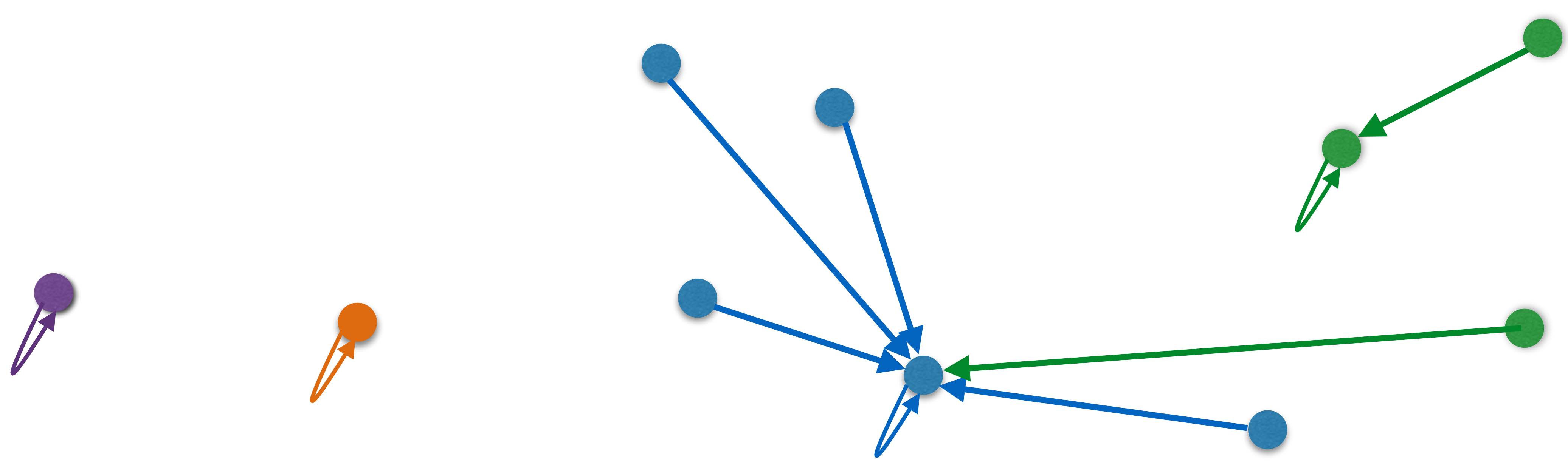

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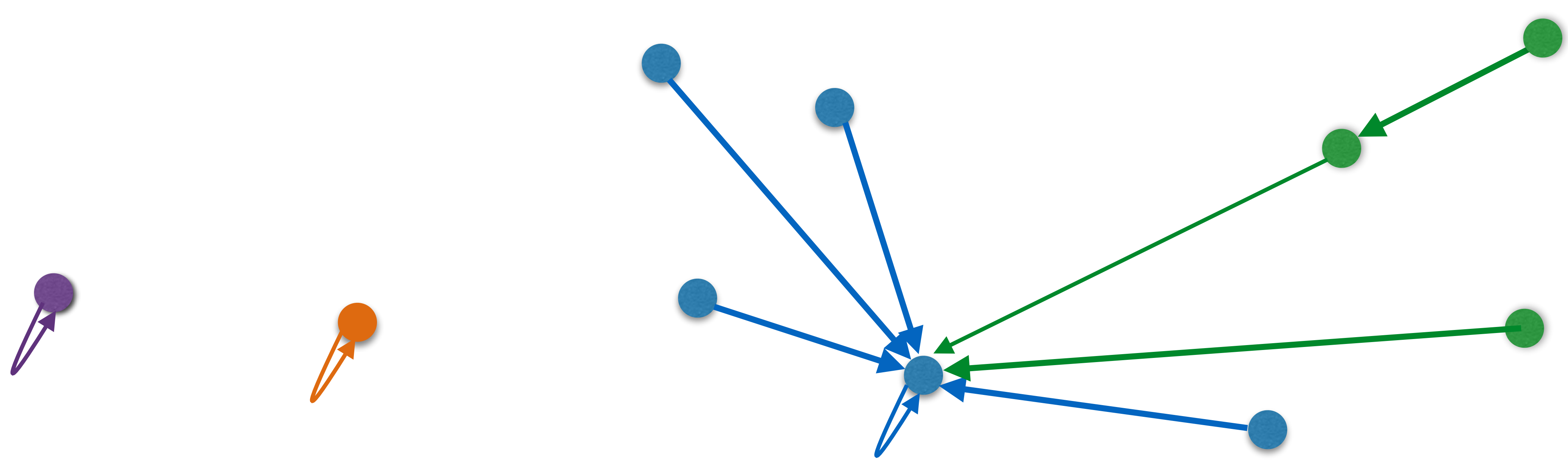

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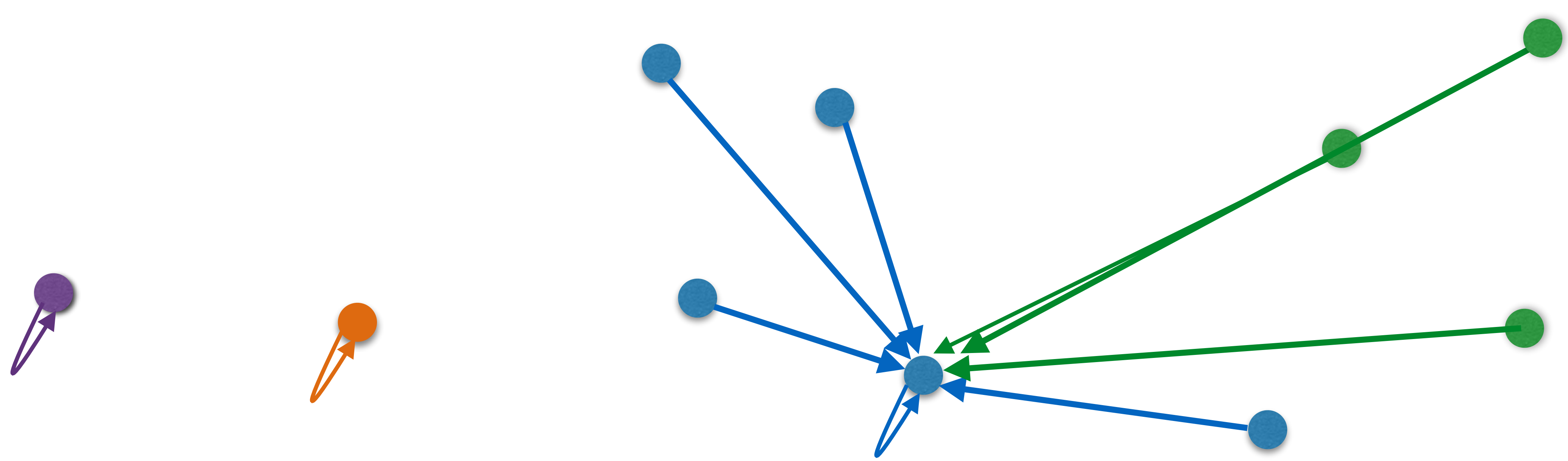

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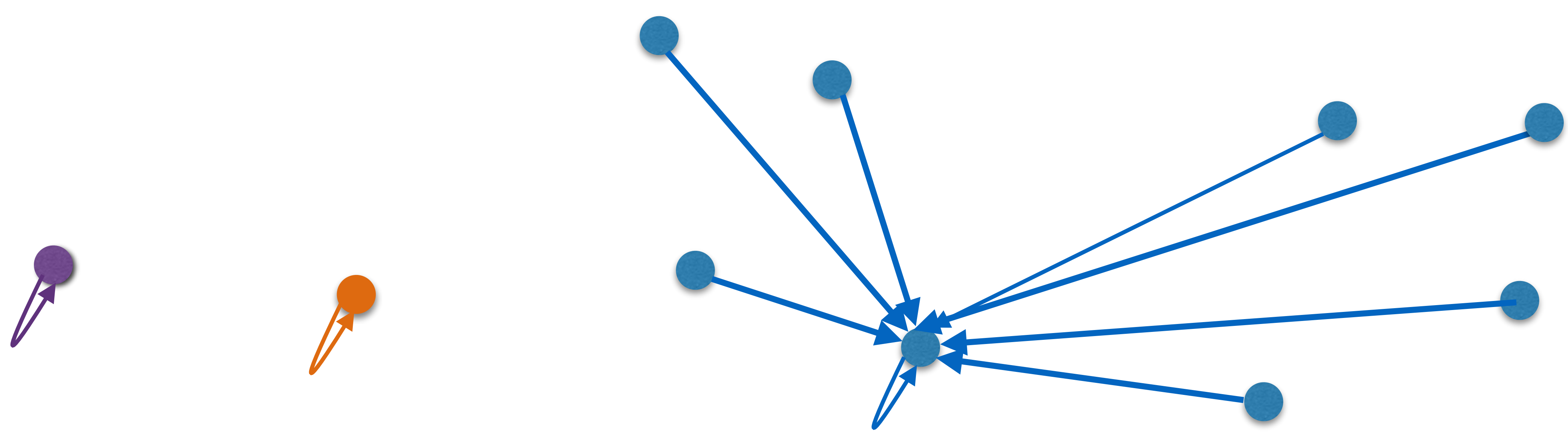


Union Find: Amortized Analysis

- Find $O(1)$ (how?)
- Union?
  - Worst case is $O(n)$ but that's not the whole story
  - Every time we change the label ("head" pointer) of a node, the size of its set at least doubles
  - Each node's head pointer only changes $O(\log n)$ times

Union Find: Amortized Analysis

- Starting with sets of size 1, any n Union operations will take $O(n \log n)$ time
- We say $O(\log n)$ amortized time for a Union operation

Definition. If n operations take total time $O(t \cdot n)$, then the amortized time per operation is $O(t)$.

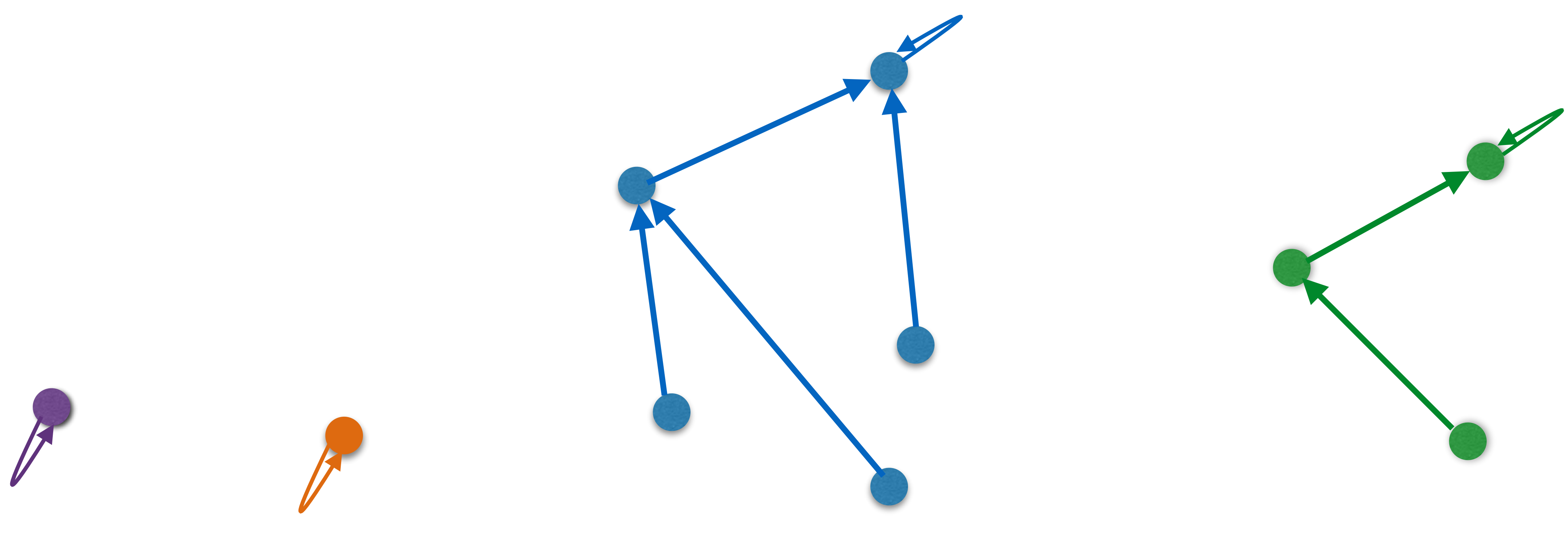
Can We Make Union faster?

- What if, instead of
  - $O(1)$ Find and $O(\log n)$ Union,
  - We want $O(\log n)$ Find and $O(1)$ Union?
- Any ideas?

Fast Union with "Trees"

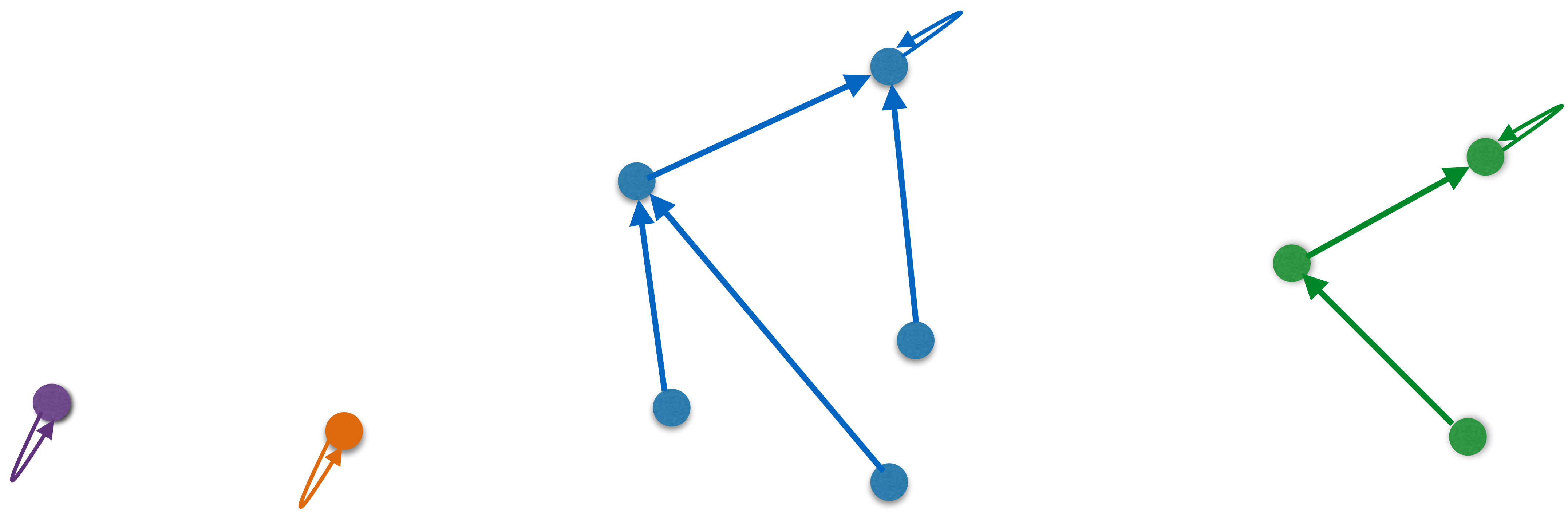
- Let's keep a head node as before

- Now, let's have our pointers act like a tree, but pointing up ("up tree")



Fast Union with “Trees”

- Let’s keep a head node as before
- Now, let’s have our pointers act like a tree, but pointing up
- How can we Find?



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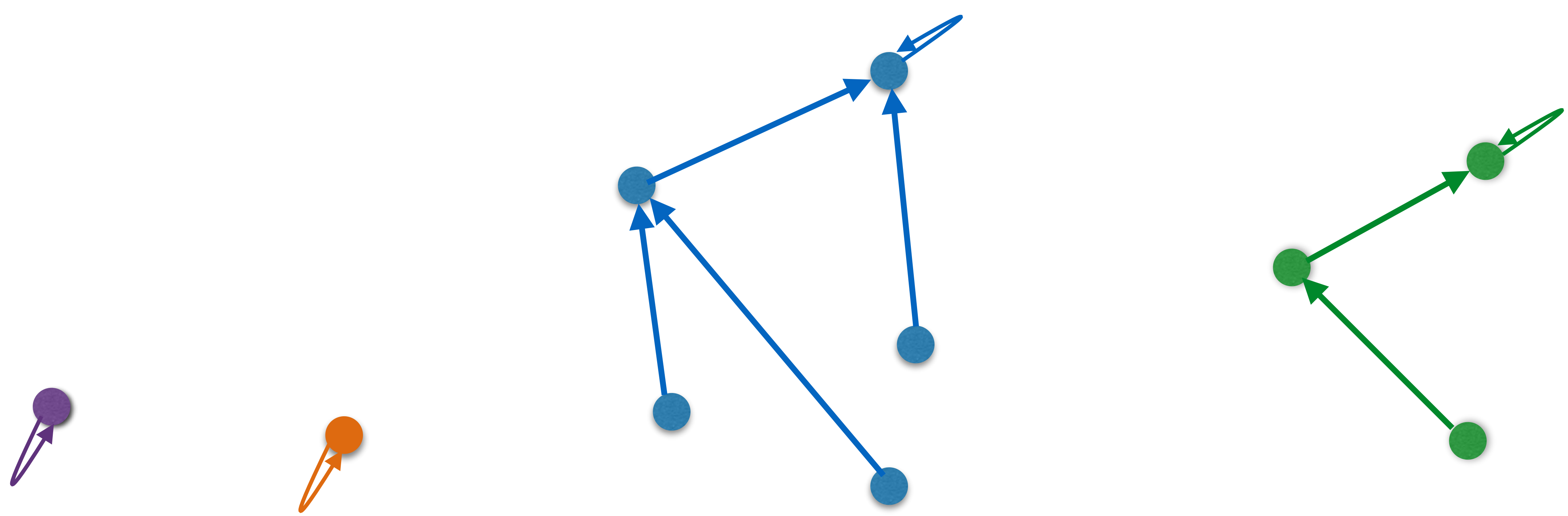

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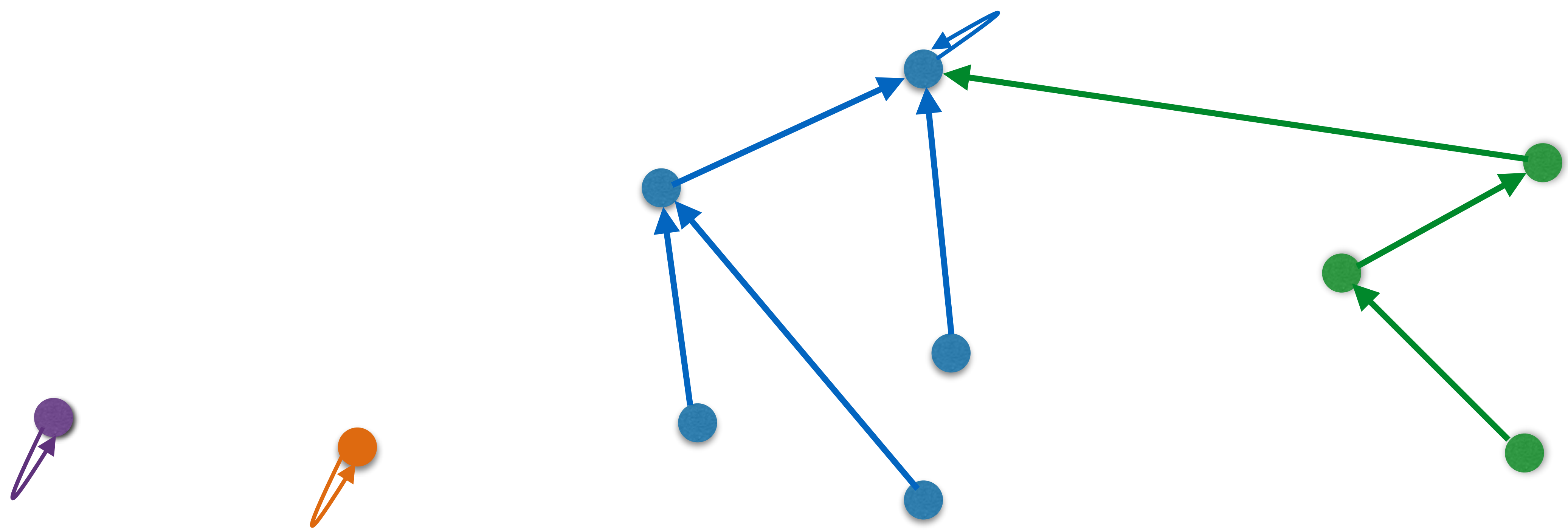

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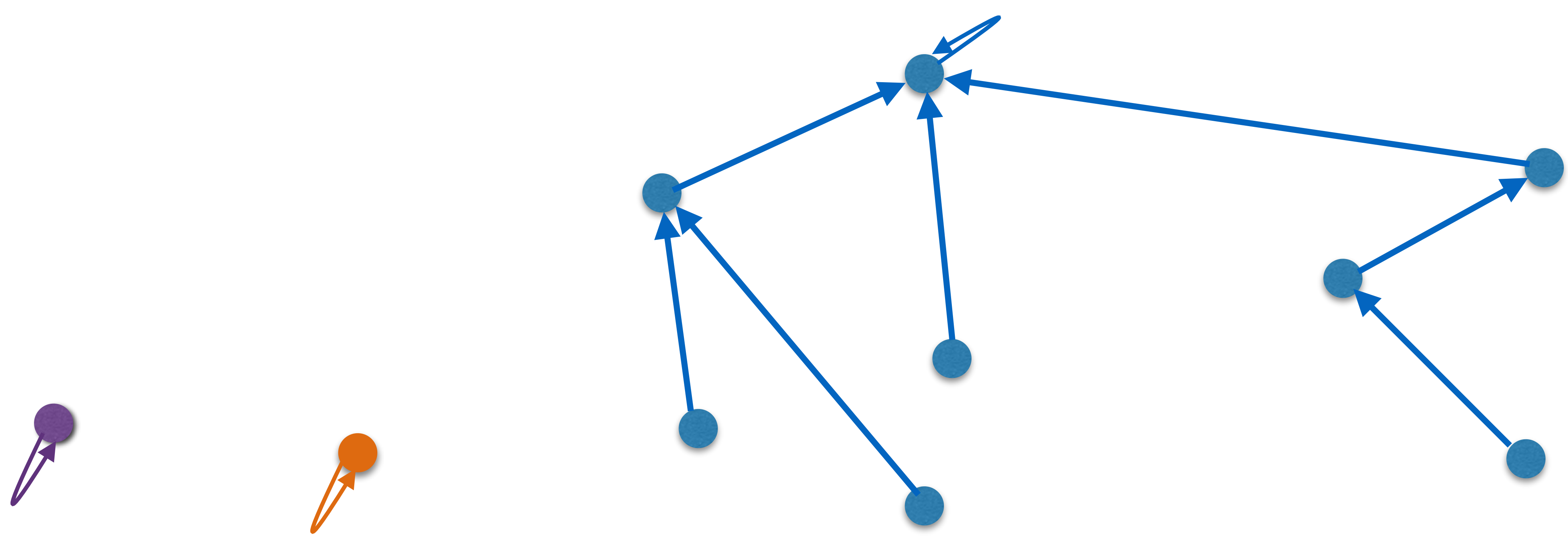


Fast Union with "Trees"

- Let's keep a head node as before
- Now, let's have our pointers act like a tree, but pointing up
- How can we Union?
    - Keep height of each up tree
    - Up tree with smaller height points to up tree of bigger height
    - At home: show that a set of size k is represented by an up tree of height at most $O(\log k)$

How Fast Is This?

- "Up tree" method:
  - $O(1)$ Union, $O(\log n)$ Find
- "Point to head" method:
  - $O(\log n)$ amortized Union, $O(1)$ Find

Class poll!

Do you think we can do better? Which of the following do you think is the case?

A. Either Union or Find take $\Omega(\log n)$

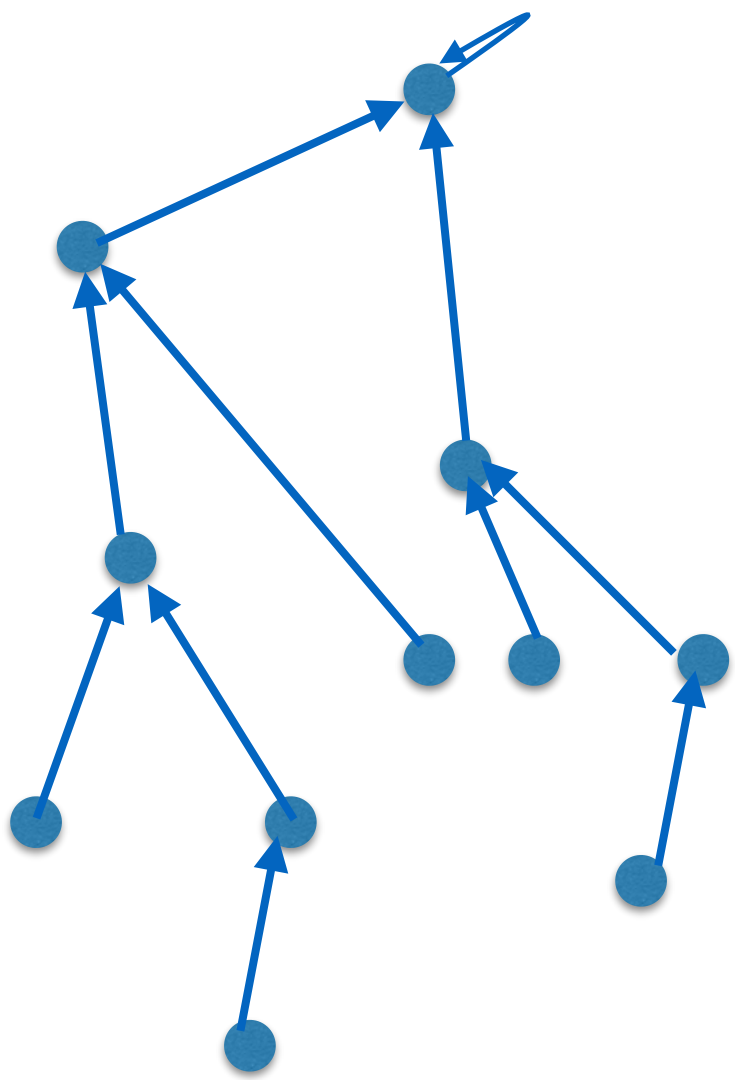
B. If you multiply Union and Find, the product of their times must be $\Omega(\log n)$

C. Both can be $O(1)$

D. Something in the middle

Let’s make things work a little faster in practice

- Think about the “up trees”
- When we’re doing a Find, is there work we can do to make future finds faster?



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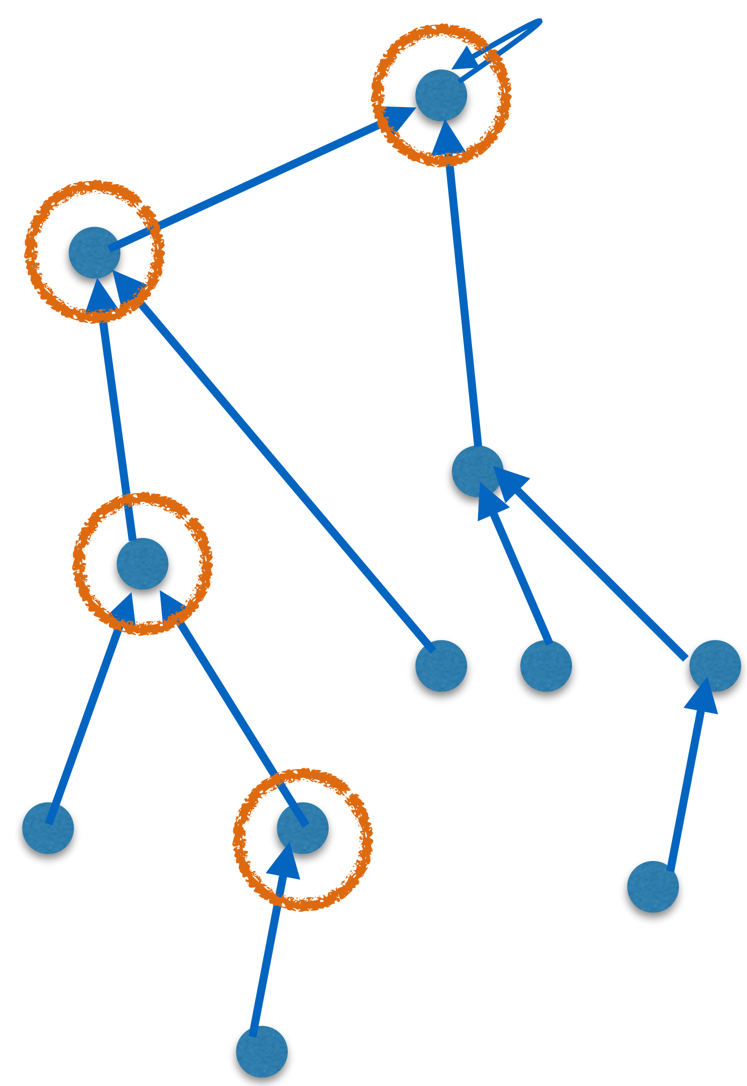
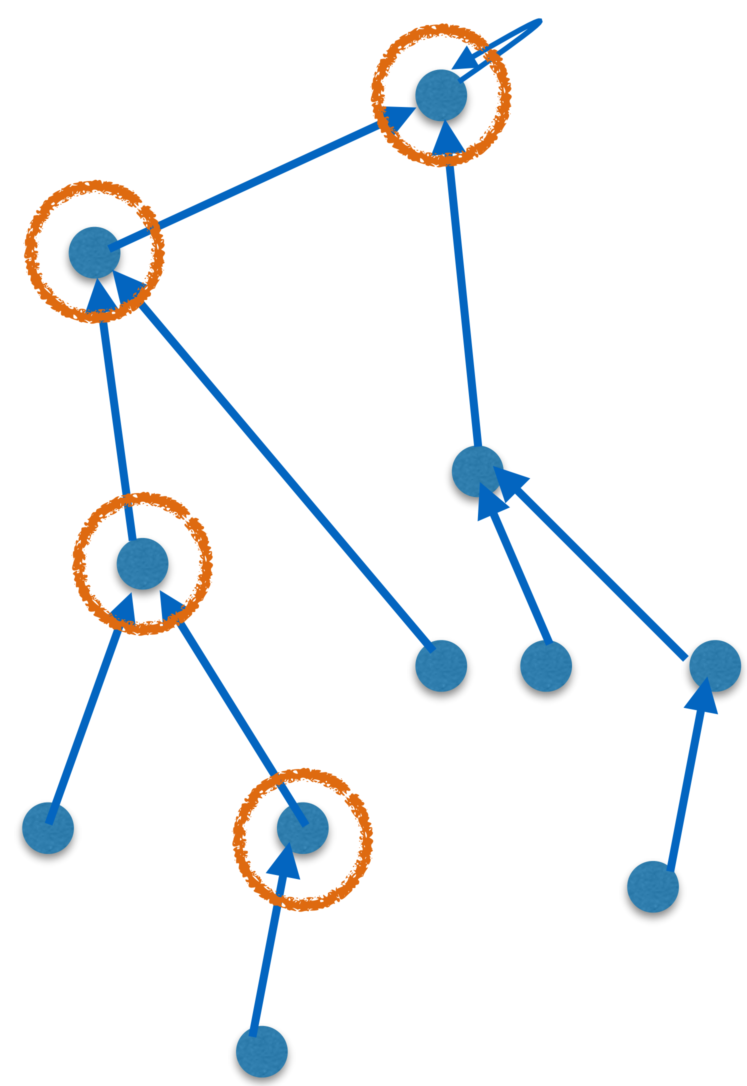


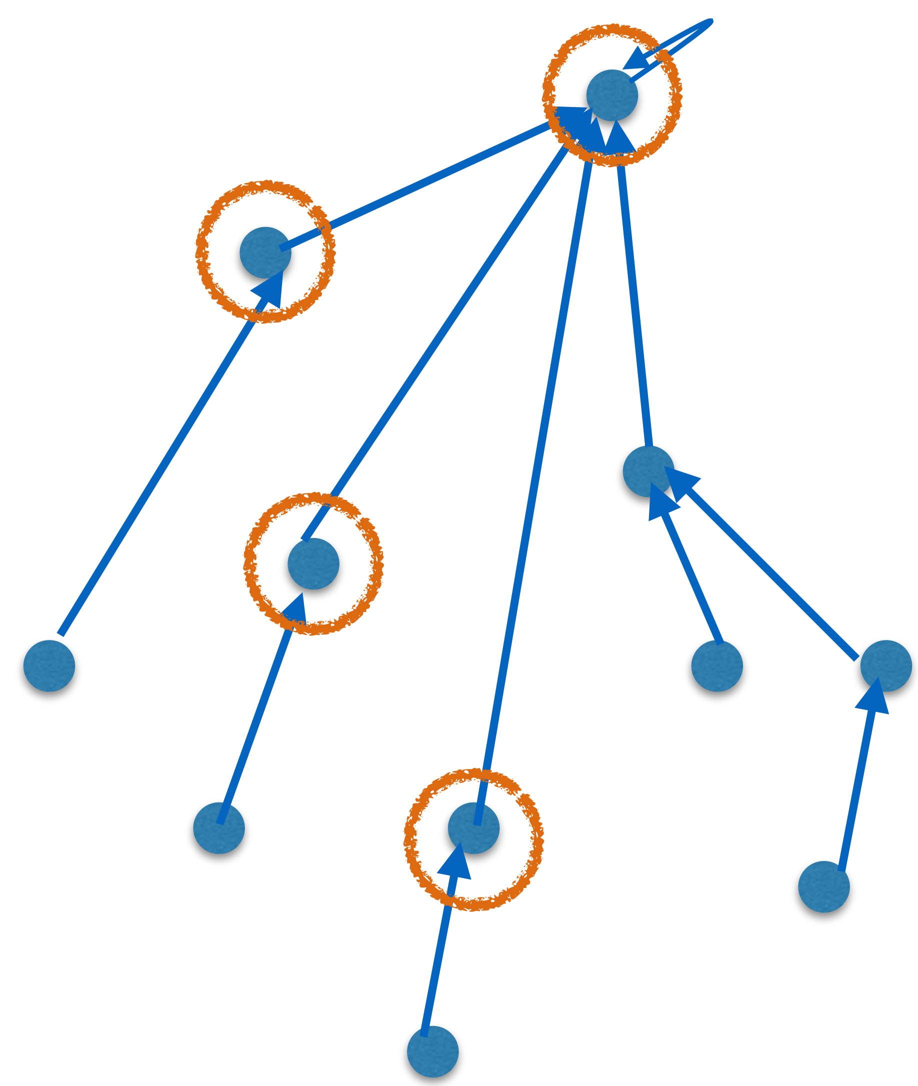
Let’s make things work a little faster in practice

- When we’re doing a Find, is there work we can do to make future finds faster?
- We really want all of these to point right to the head
- So…let’s do that!



Let’s make things work a little faster in practice

- When we’re doing a Find, is there work we can do to make future finds faster?
- We really want all of these to point right to the head
- So…let’s do that!
- Wait, I’ve broken the data structure!
  - I can’t maintain “height”



Maintaining “Height”

- We can’t maintain the exact height. What if we pretend we can? Just do the same bookkeeping:
- Keep a “rank”
- Always point the head of smaller rank to the head of larger rank; keep rank the same
- If both ranks are the same, point one to the other, and increment the rank

What do we get?

- Every time I have an expensive Find, I get a lot of great work done for the future by shrinking the tree
    - Called “path compression”
- Now I have an inaccurate “rank” instead of an actual “height”
- First: did this make things worse? Union is still $O(1)$, is Find $O(\log n)$?
    - We did not make things worse, Find is $O(\log n)$
    - Proof idea: our rank is never higher than the actual height
- Can we show that we made things better?

Surprising Result: Hopcroft Ulman'73

- Amortized complexity of union find with path compression improves significantly!
- Time complexity for n union and find operations on n elements is $O(n \log^* n)$
- $\log^* n$ is the number of times you need to apply the log function before you get to a number <= 1
- Very small! **Less than 5 for all reasonable values**

$$\log^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log n) & \text{if } n > 1 \end{cases}$$

n	1	2	$4 = 2^2$	$16 = 2^4$	$65,536 = 2^{16}$	$2^{65,536}$
$\log^*(n)$	0	1	2	3	4	5

Digging Deeper

Surprising Result: Tarjan '75

- Improved bound on amortized complexity of union-find with path compression
- Time complexity for n union and find operations on n elements is $O(n\alpha(n))$, where
    - $\alpha(n)$ is extremely slow-growing, **inverse-Ackermann function**
    - Essentially a constant
- Grows much **muuchch morrree** slowly than $\log^*$
- $\alpha(n) \leq 4$ for all values in practice
- **Result.** Union and Find become (essentially) amortized constant time in practice (just short of $O(1)$ in theory) !

Digging Deeper

Inverse Ackermann

- **Inverse Ackerman:** The function $\alpha(n)$ grows much more slowly than $\log^{*^c} n$ for **any fixed c**

- With $\log*$, you count how many times does applying $\log$ over and over gets the result to become small
- With the inverse Ackermann, essentially you count how many times does applying $\log*$ (not log!) over and over gets the result to become small

- $\alpha(n) = \min\{k \mid \log^{\overbrace{****\cdots*}^{k}}(n) \leq 2\}$

- $\alpha(n) = 4$ for $n = 2^{2^{2^{2^{16}}}}$

Digging Deeper

Can we do better?

- OK, so that's “basically constant”. Can we get constant?
- No. *Any data structure* for union find requires $\Omega(\alpha(n))$ amortized time (Fredman, Saks ’89)
- So up trees with path compression are optimal(!)

Union-Find: Applications

- Good for applications in need of clustering
  - cities connected by roads
  - cities belonging to the same country
  - connected components of a graph
- Maintaining equivalence classes
- Maze creation!

Digging Deeper

Back to MST

- Prim's algorithm: $O(m + n \log n)$ using a Fibonnacci tree
- Kruskal's algorithm:
  $O(m \log m + m\alpha(m)) = O(m \log m)$
- Which is better in practice?
  - Usually Kruskal's: a single sort is much better than Prim's repeated priority queue removals
- Is sorting time $\Omega(n \log n)$ required?

Digging Deeper

Can we do better?

Best known algorithm by Chazelle (1999)

A Minimum Spanning Tree Algorithm with Inverse-Ackermann Type Complexity*

BERNARD CHAZELLE†

NECI Research Tech Report 99-099 (July 1999)
Journal of the ACM, 47(6), 2000, pp. 1028–1047.

Abstract

A deterministic algorithm for computing a minimum spanning tree of a connected graph is presented. Its running time is $O(m\alpha(m,n))$, where α is the classical functional inverse of Ackermann's function and n (resp. m) is the number of vertices (resp. edges). The algorithm is comparison-based: it uses pointers, not arrays, and it makes no numeric assumptions on the edge costs.

1 Introduction

The history of the minimum spanning tree (MST) problem is long and rich, going a as Borůvka's work in 1926 [1, 9, 13]. In fact, MST is perhaps the oldest open p computer science. According to Nešetřil [13], "this is a cornerstone problem of com optimization and in a sense its cradle." Textbook algorithms run in $O(m \log n)$ time, where n

Digging Deeper

Can we do better?

Using randomness, can get $O(m)$ time!

A Randomized Linear-Time Algorithm to Find Minimum Spanning Trees

DAVID R. KARGER

Stanford University, Stanford, California

PHILIP N. KLEIN

Brown University, Providence, Rhode Island

AND

ROBERT E. TARJAN

Princeton University and NEC Research Institute, Princeton, New Jersey

Abstract. We present a randomized linear-time algorithm to find a minimum spanning tree in a connected graph with edge weights. The algorithm uses random sampling in combinatic recently discovered linear-time algorithm for verifying a minimum spanning tree. Our c tional model is a unit-cost random-access machine with the restriction that the only op allowed on edge weights are binary comparisons.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Com** Nonnumerical Algorithms and Problems—*computations on discrete structures*; G.2.2 [**Discrete**

Optimal MST Algorithm?

Has been discovered but don't know its running time!

An Optimal Minimum Spanning Tree Algorithm

SETH PETTIE AND VIJAYA RAMACHANDRAN

The University of Texas at Austin, Austin, Texas

Abstract. We establish that the algorithmic complexity of the minimum spanning tree problem is equal to its decision-tree complexity. Specifically, we present a deterministic algorithm to find a minimum spanning tree of a graph with n vertices and m edges that runs in time $O(\mathcal{T}^*(m,n))$ where $\mathcal{T}^*$ is the minimum number of edge-weight comparisons needed to determine the solution. The algorithm is quite simple and can be implemented on a pointer machine.

Although our time bound is optimal, the exact function describing it is not known at present. The current best bounds known for $\mathcal{T}^*$ are $\mathcal{T}^*(m,n) = \Omega(m)$ and $\mathcal{T}^*(m,n) = O(m \cdot \alpha(m$ … a certain natural inverse of Ackermann's function.

Even under the assumption that $\mathcal{T}^*$ is superlinear, we show that if the input graph … $G_{n,m}$, our algorithm runs in linear time with high probability, regardless of n, m, or th… edge weights. The analysis uses a new martingale for $G_{n,m}$ similar to the edge-exp… for G…

Digging Deeper

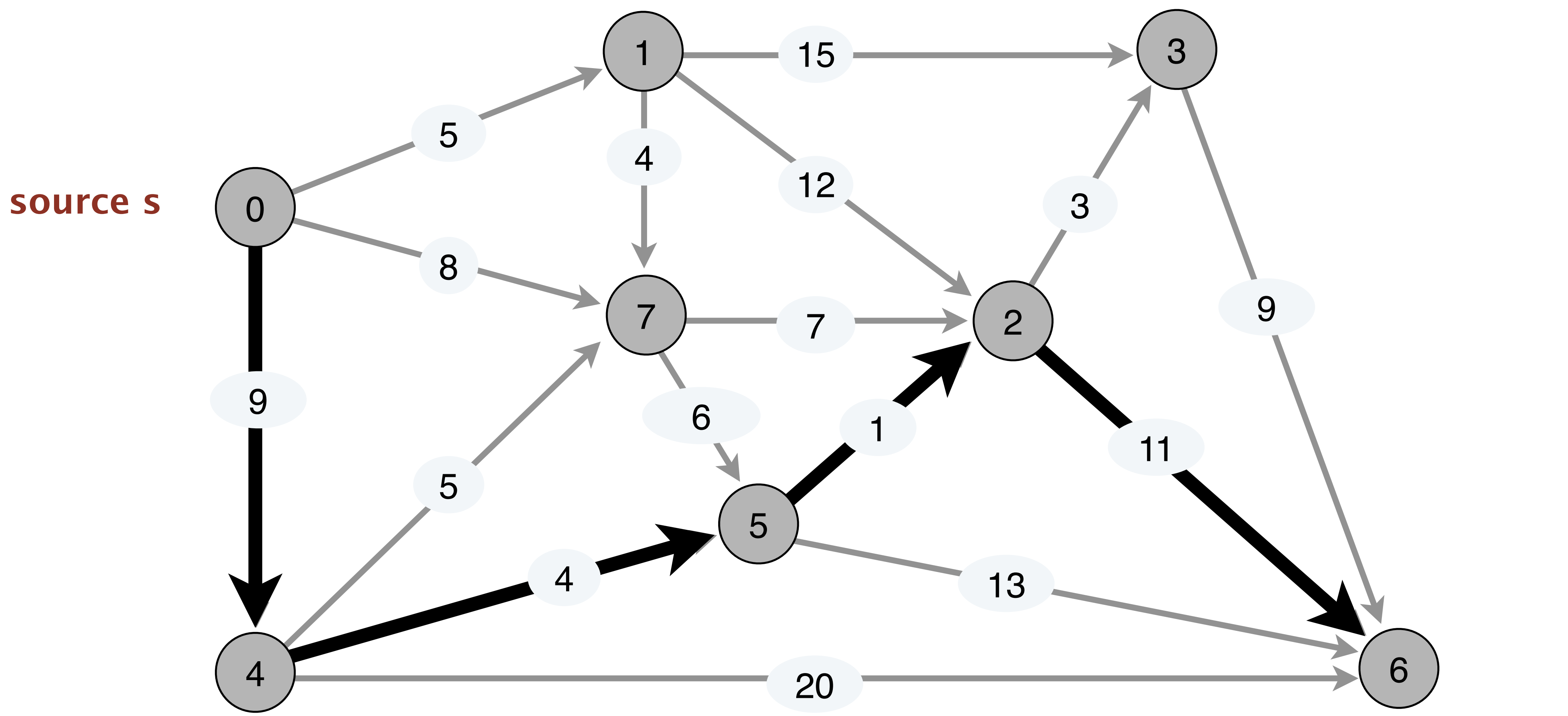
MST Algorithms History

- **Borůvka's Algorithm** (1926)
  - The Borvka / Choquet / Florek-ukaziewicz-Perkal-Steinhaus-Zubrzycki / Prim / Sollin / Brosh algorithm
  - Oldest, most-ignored MST algorithm, but actually very good
- **Jarník's Algorithm** ("Prim's Algorithm", 1929)
  - Published by Jarník, independently discovered by Kruskal in 1956, by Prim in 1957
- **Kruskal's Algorithm** (1956)
  - Kruskal designed this because he found Borůvka's algorithm "unnecessarily complicated"

Next class:

Greedy Algorithms: Shortest Path

Shortest Paths in Weighted Graph



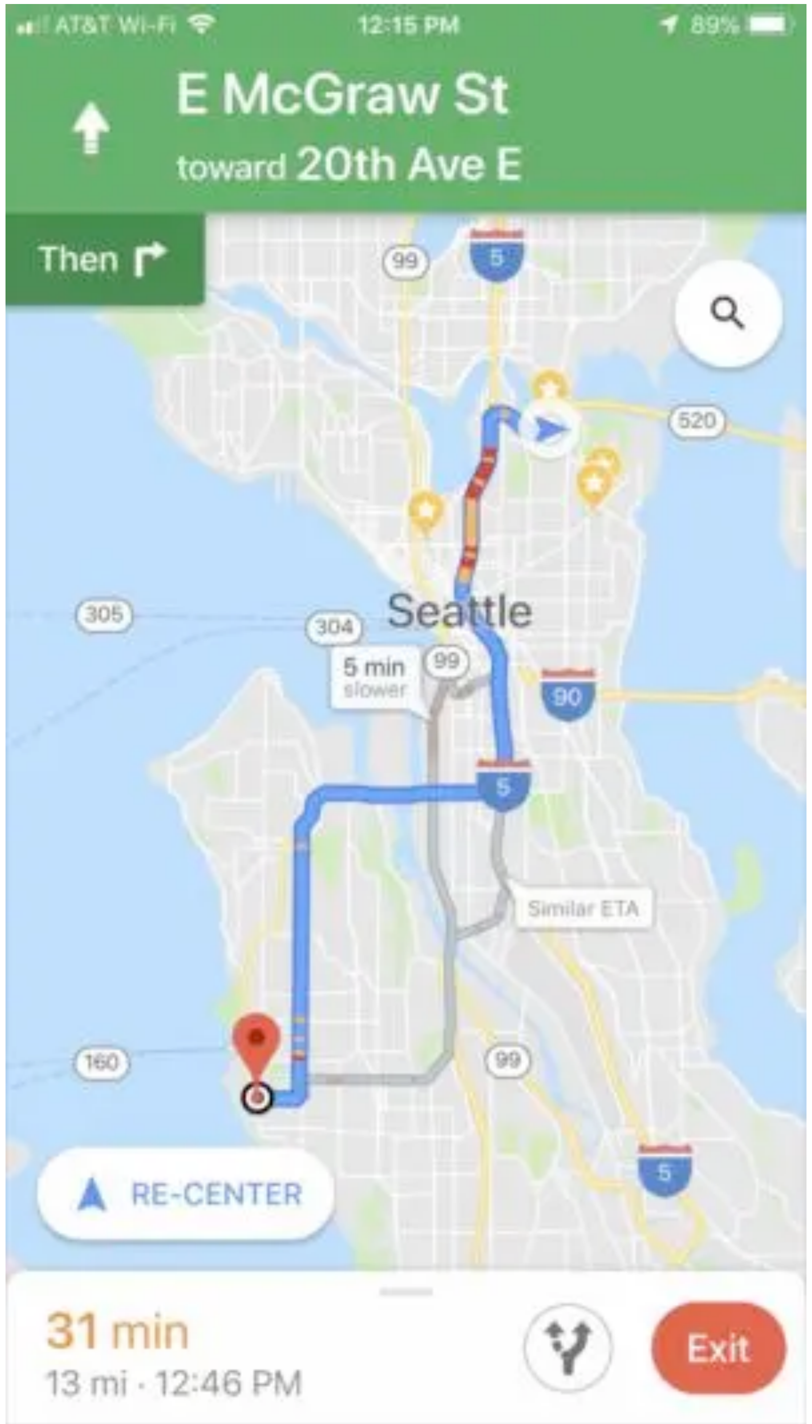
length of path = 9 + 4 + 1 + 11 = 25

Shortest Paths in Weighted Graph

Problem.

Given a directed graph $G = (V, E)$ with positive edge weights: that is, each edge $e \in E$ has a positive weight $w(e)$ and vertices s and t, find the shortest path from s to t.

Definition. The shortest path from s to t in a weighted graph is a path P from s to t (or a s-t path) with minimum weight $w(P) = \sum_{e \in P} w(e)$.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Exit

Midterm Questions?

Assignment questions (from any assignment)

Practice midterm questions

- I won't ask you to "analyze space" of an algorithm on the midterm

Acknowledgments

- The pictures in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)