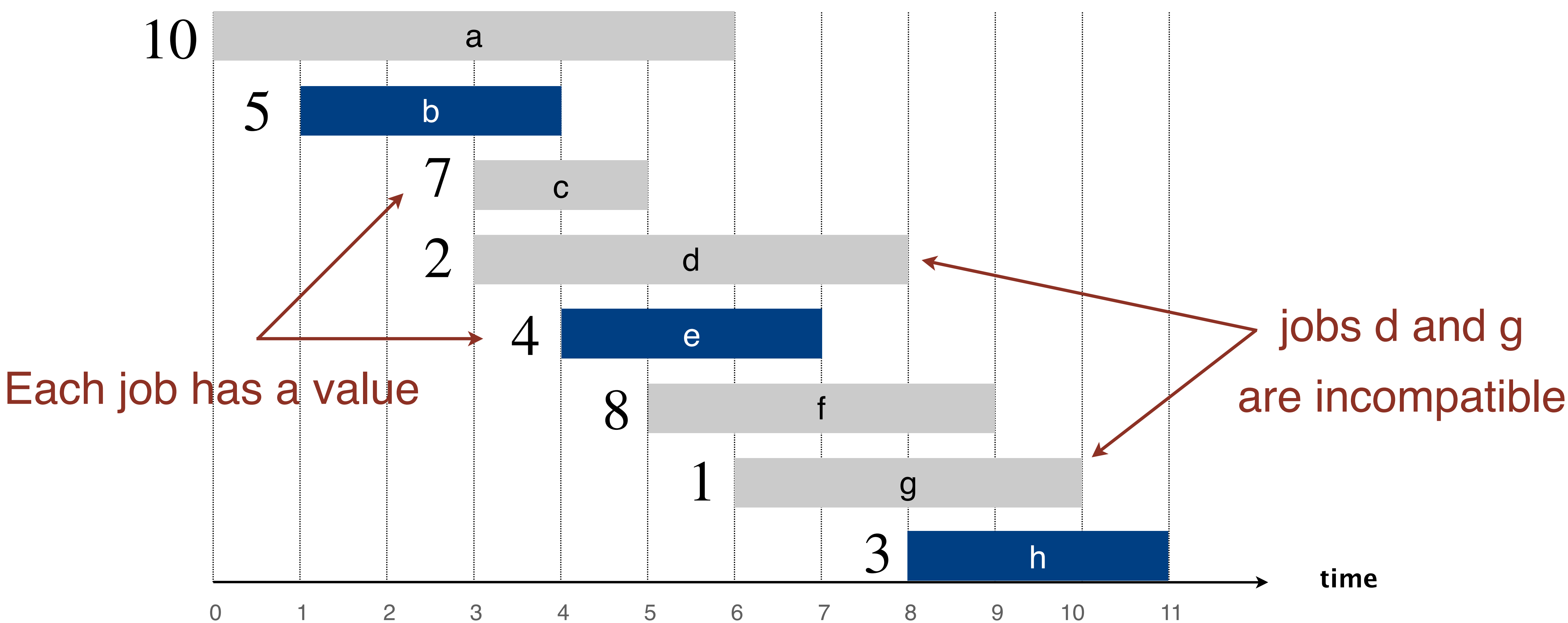
Weighted Scheduling

Weighted Scheduling

Job scheduling. Suppose you have a machine that can run one job at a time; n job requests, where each job i has a start time s_i, finish time f_i and weight $v_i \geq 0$.



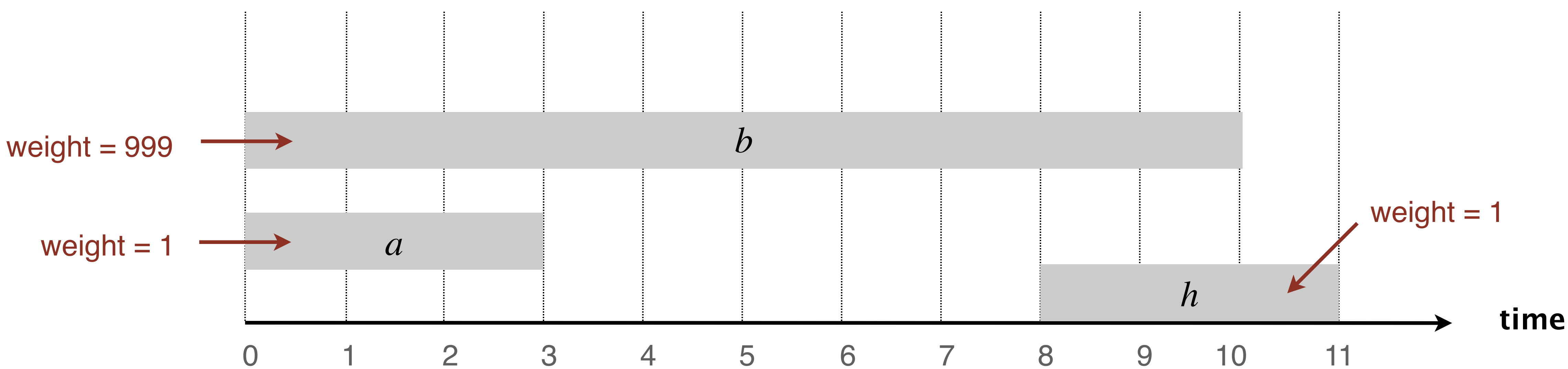
Weighted Scheduling

- **Input**. Given n intervals labeled $1,\ldots,n$ with starting and finishing times $(s_1, f_1), \ldots, (s_n, f_n)$ and each interval has a non-negative value or weight v_i

- **Goal**. We must select non-overlapping (compatible) intervals with the maximum weight. That is, our goal is to find $I \subseteq \{1,\ldots,n\}$ that are pairwise non-overlapping that maximize $\sum_{i \in I} v_i$

Remember Greedy?

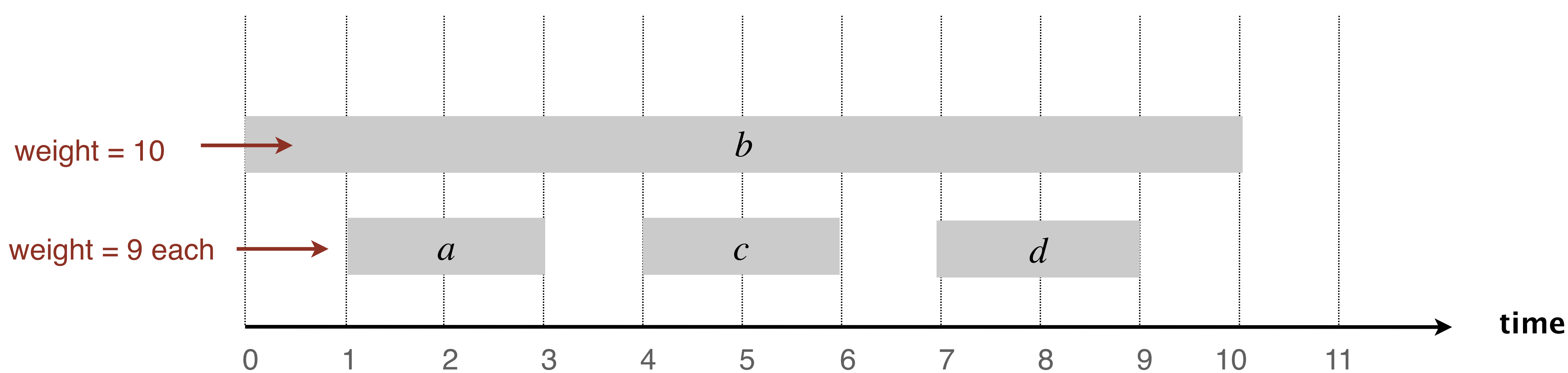
- Greedy algorithm earliest-finish-time first
    - Considers jobs in order of finish times
    - Greedily picks jobs that are non-overlapping
- We proved greedy is optimal when all weights are one
- How about the weighted interval scheduling problem?



Greedy fails spectacularly

Different Greedy?

- A different greedy algorithm: greedily select intervals with the maximum weights, remove overlapping intervals
- Does that work?



Greedy fails spectacularly

Let's Think Recursively

- The heart of dynamic programming is recursively thinking
- Coming up with a **smaller subproblem** which has the same optimal structure as the original problem
- First, to make things easy, we will focus on the total value of the optimal solution, rather than the actual optimal set, that is,
- **Optimal value.**
  Find the largest $\sum_{i \in I} v_i$ where intervals in I are compatible.
- Opt-Schedule(n): the value of the optimal schedule of n intervals

Let's Think Recursively

- Consider the last interval: either it is in the optimal solution or not
- Whatever the overall optimal solution is, we can find it by considering both cases and taking the maximum over them
- **Case 1.** Last interval is not in the optimal solution
    - Remove it, we now have a smaller subproblem!
- **Case 2.** Last interval is in the optimal solution
    - Means anything overlapping with this interval cannot be in the solution, remove them
    - We have a smaller subproblem!

Formalize the Subproblem

Opt-Schedule(i): value of the optimal schedule that only uses intervals $\{1,\ldots,i\}$, for $0 \leq i \leq n$

Intervals sorted by *finishing time*, so:

Opt-Schedule(i): value of the optimal schedule that finishes by the time i finishes

Base Case & Final Answer

Opt-Schedule(i): value of the optimal schedule that only uses intervals $\{1,\ldots,i\}$, for $0 \leq i \leq n$

Base Case. Opt-Schedule$(0) = 0$

Goal (Final answer.) Opt-Schedule(n)

Recurrence

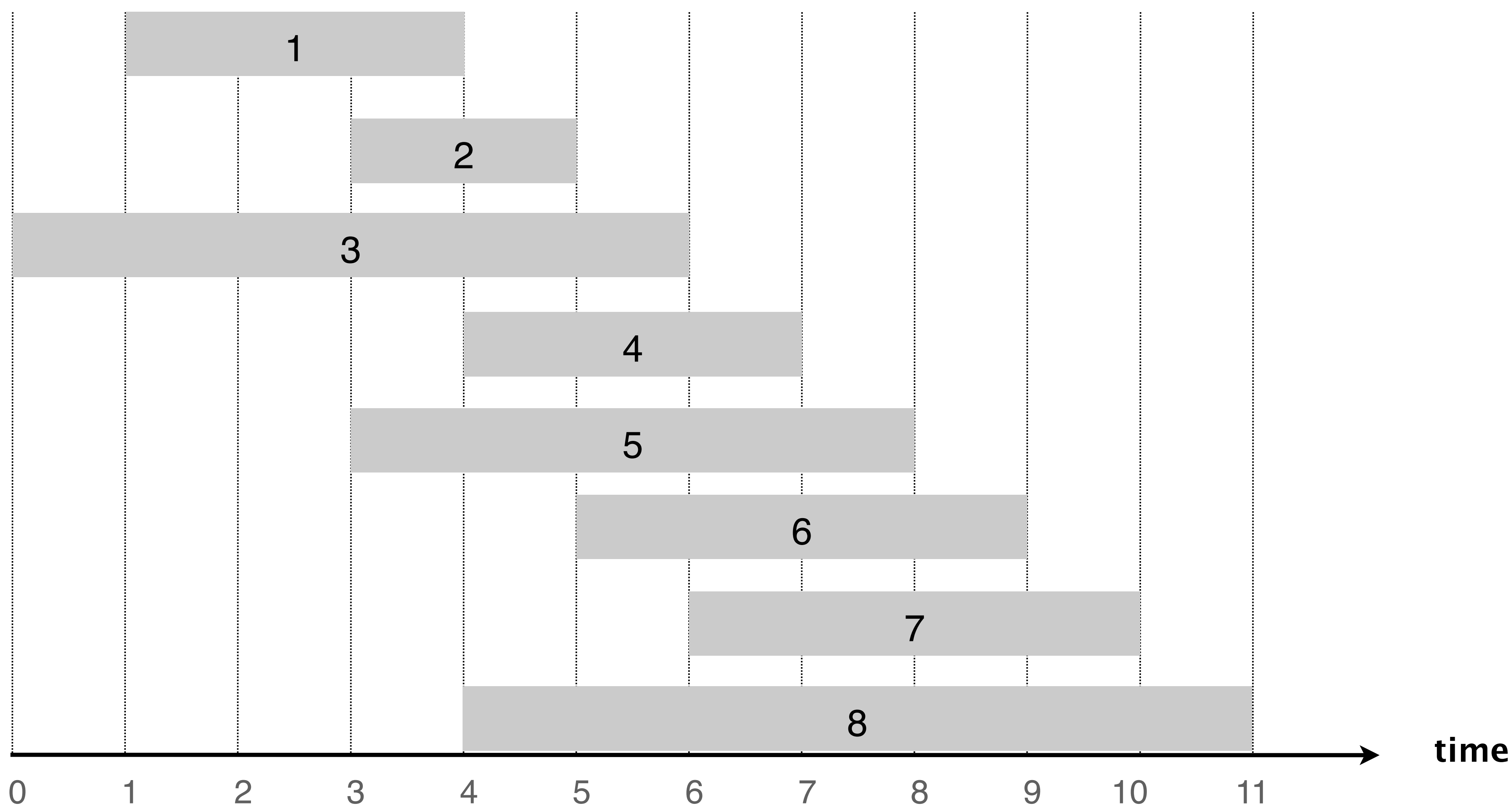
- How do we go from one subproblem to the next?
- The recurrence says how we can compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$
- **Case 1.** Say interval i is not in the optimal solution, can we write the recurrence for this case?
    - Opt-Schedule$(i) =$ Opt-Schedule$(i - 1)$
- **Case 2.** Say interval i **is** in the optimal solution, what is the smaller subproblem we should recurse on in this case?

Recurrence

- The recurrence says how we can compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$
- **Case 1.** Say interval i is not in the optimal solution:
  - Opt-Schedule$(i) =$ Opt-Schedule$(i - 1)$
- **Case 2.** Say interval i **is** in the optimal solution:
  - No interval $j < i$ that overlaps with i can be in solution
  - Need to remove all such intervals to get our smaller subproblem
  - How do we do that?

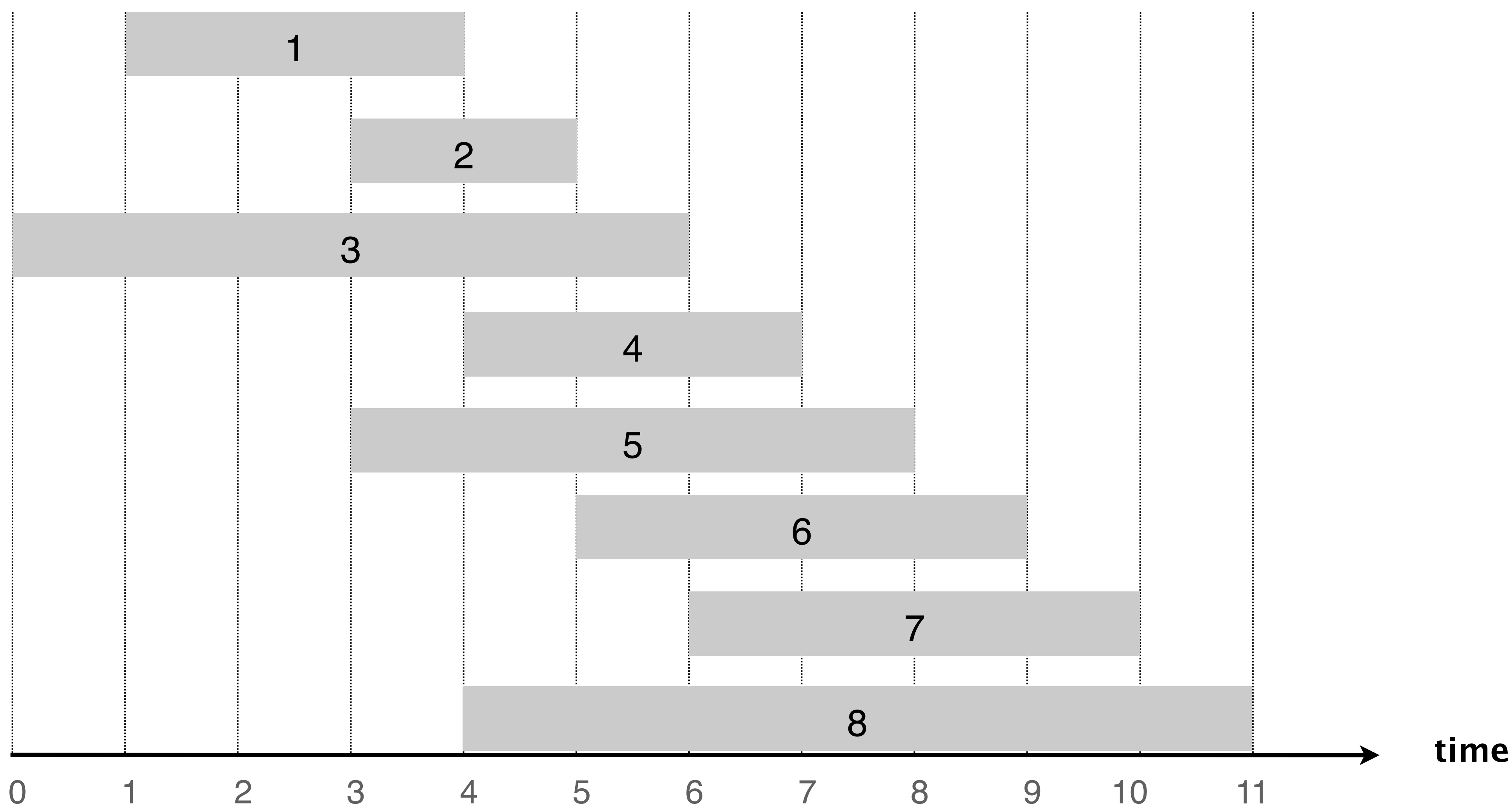
Helpful Information

- Suppose the intervals are sorted by finish times
- Let $p(j)$ be the predecessor of j that is, largest index $i < j$ such that intervals i and j are not overlapping
- Define $p(j) = 0$ if all intervals $i < j$ overlap with j



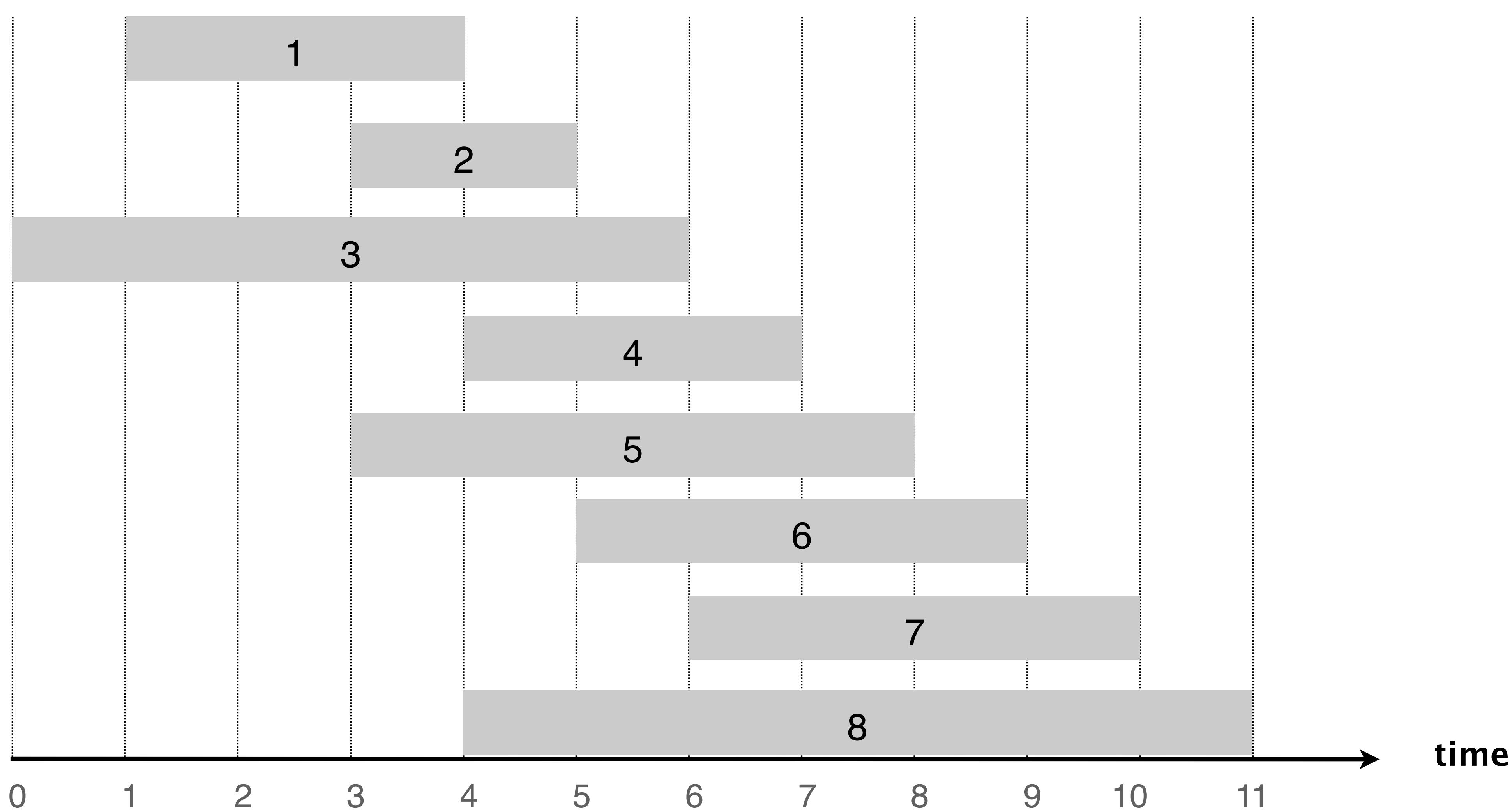
Helpful Information

- Let $p(j)$ be the predecessor of j that is, largest index $i < j$ such that intervals i and j are not overlapping
  - $p(8) = ?, \;\; p(7) = ?, \;\; p(2) = ?$



Helpful Information

- Let $p(j)$ be the predecessor of j that is, largest index $i < j$ such that intervals i and j are not overlapping
  - $p(8) = 1, \;\; p(7) = 3, \;\; p(2) = 0$



Recurrence

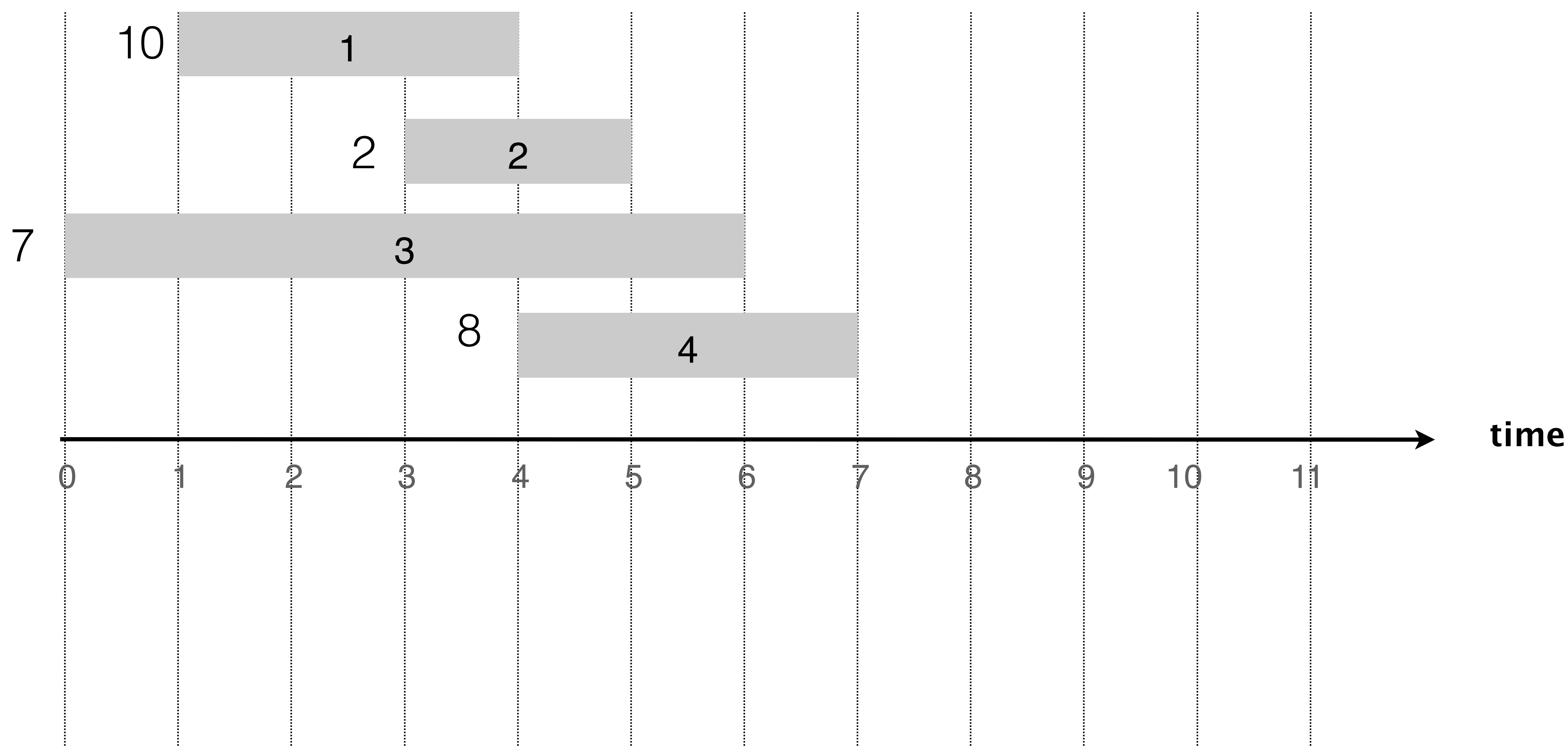
- The recurrence says how we can compute Opt-Schedule(i) by using values of Opt-Schedule(j) where $j < i$
- **Case 1.** Say interval i is not in the optimal solution:
  - Opt-Schedule$(i) =$ Opt-Schedule$(i - 1)$
- **Case 2.** Say interval i **is** in the optimal solution:
  - Suppose I know $p(i)$ predecessor of i, how can I write the recurrence for this case?
  - Opt-Schedule$(i) =$ Opt-Schedule$(p(i))$ + v_i

DP Recurrence

$$\text{Opt-Schedule}(i) =$$
$$\max\{\text{Opt-Schedule}(i-1),\ v_i + \text{Opt-Schedule}(p(i))\}$$

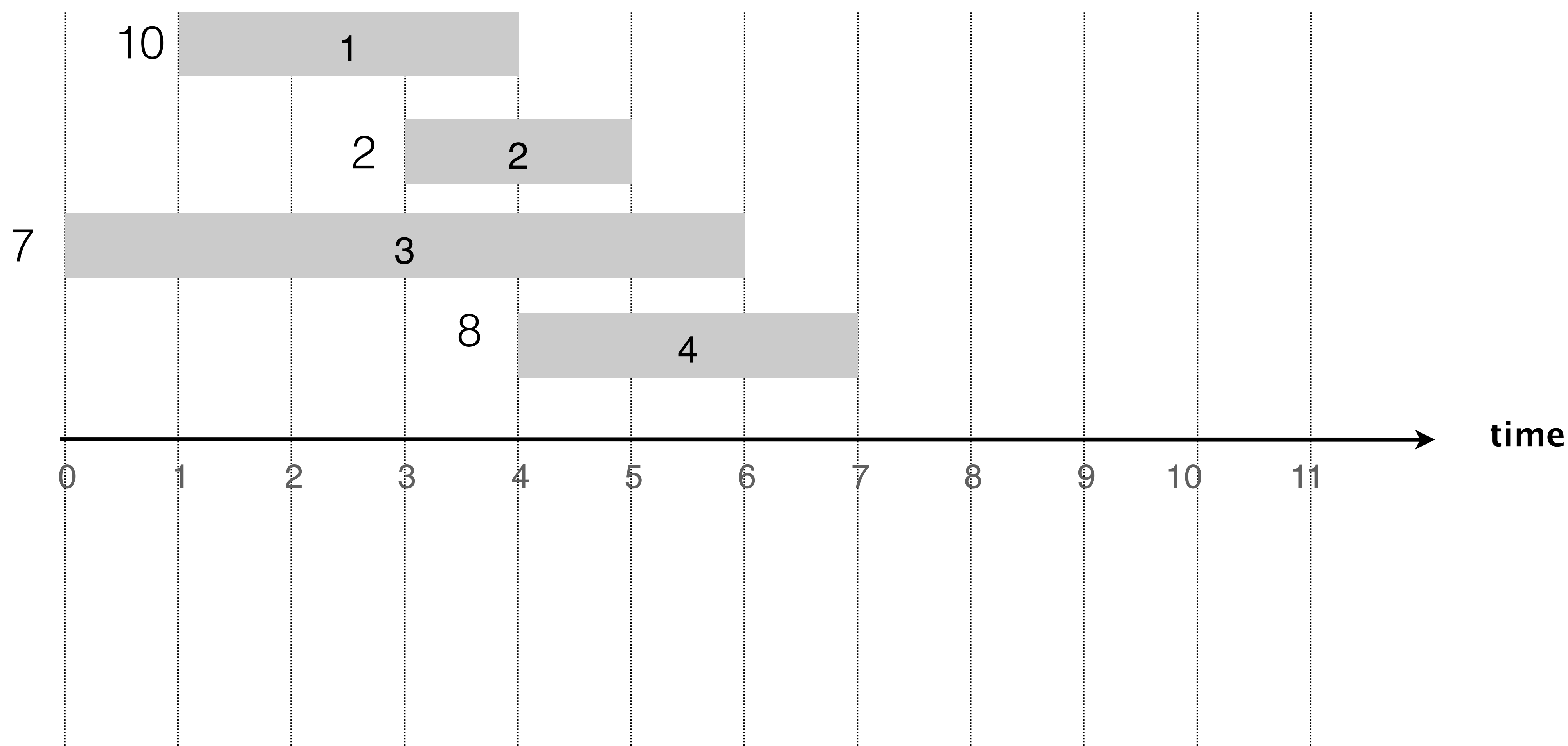
Filling Out the DP Table

0				
0	1	2	3	4



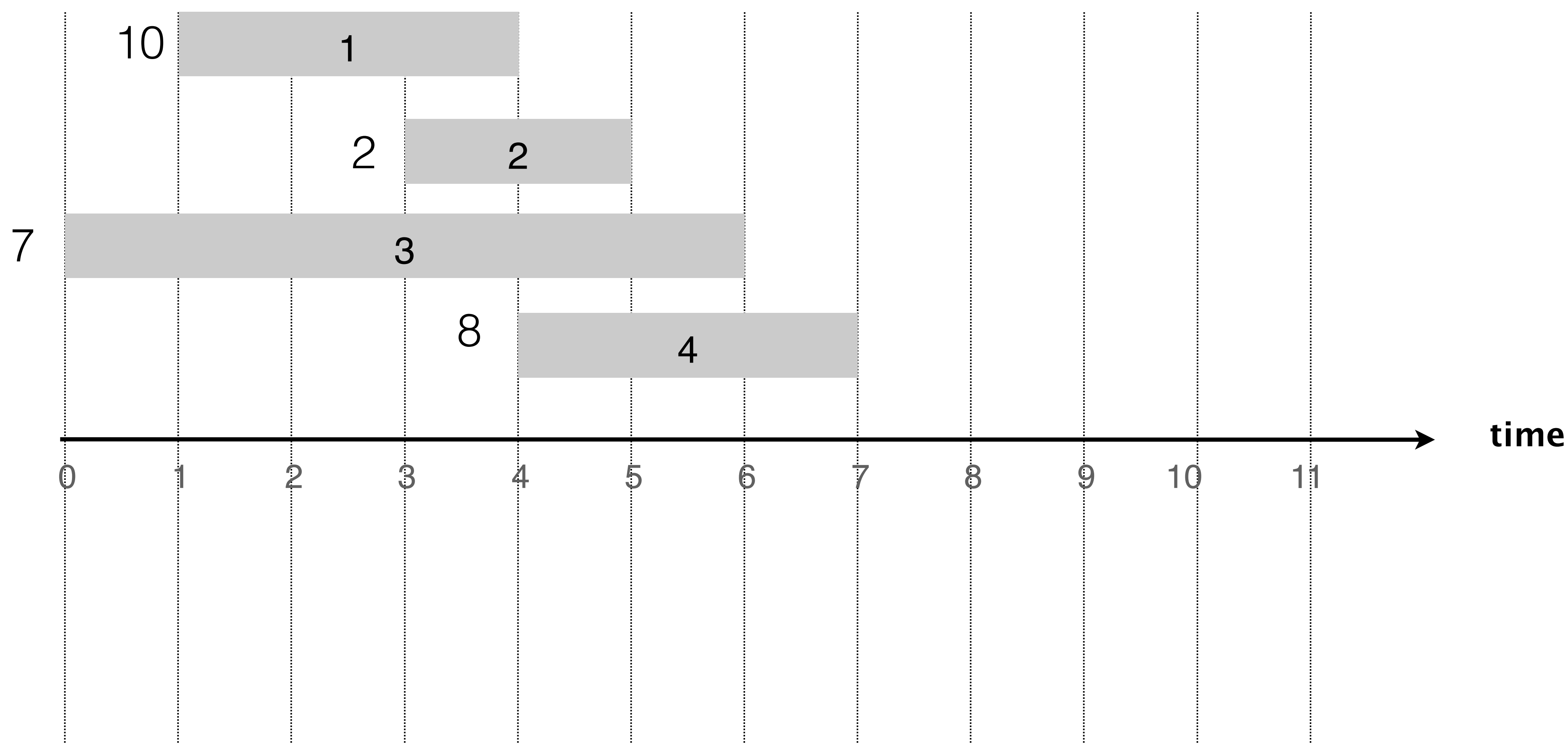
Filling Out the DP Table

0	10			
0	1	2	3	4



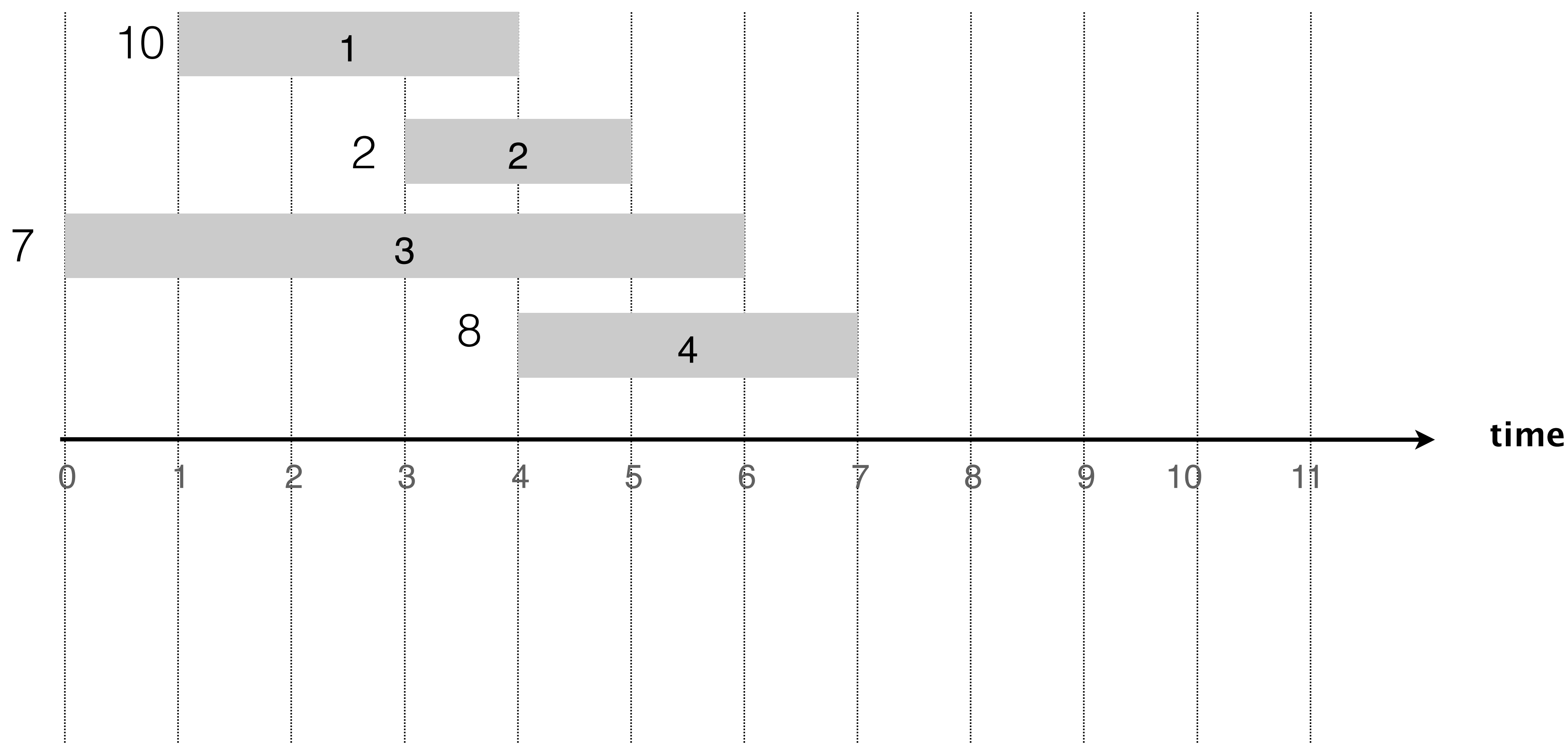
Filling Out the DP Table

0	10	10		
0	1	2	3	4



Filling Out the DP Table

0	1	2	3	4
0	10	10	10	18



Summary of DP

- **Subproblem**.
  - For $0 \leq i \leq n$, let Opt-Schedule(i) be the value of the optimal schedule that only uses intervals $\{1,\ldots,i\}$
  - Notice the optimal substructure
- **Recurrence.** Going from one subproblem to the next
  - Opt-Schedule$(i) =$ $\max\{$Opt-Schedule$(i-1)$, $v_i +$ Opt-Schedule$(p(i))\}$
- **Base case**.
  - Opt-Scheduler$(0) = 0$ (no intervals to schedule)

Remaining Pieces

- Final answer in terms of subproblem?
  - Opt-Schedule(n)
- Evaluation order (in what order can be fill the DP table)
  - $i = 0 \rightarrow n$, start with base case and use that to fill the rest
- Memoization data structure: 1-D array
- Final piece:
  - Running time and space
  - Space: $O(n)$
  - Time: preprocessing + time to fill array

Computing $p[i]$

- How quickly can we compute $p[i]$?
  - Can do a linear scan for each i: $O(i)$ per interval
  - Would be $O(n^2)$ overall
- We have intervals sorted by their finish time $F[1,\ldots,n]$
  - Can we use this?
  - For each interval, we can binary search over $F[1,\ldots,n]$, to need to find the first $j < i$ such that $f_j \leq s_i$
  - $O(\log n)$ for each interval
- Time $O(n \log n)$ to compute the array $p[]$

Running Time

- How many subproblems do we need to solve?
  - $O(n)$
- How long does it take to solve a subproblem?
  - $O(1)$ to take the max
- Preprocessing time:
  - Need to sort; $O(n \log n)$
  - Need to find $p(i)$ for all each i: $O(n \log n)$
- Overall: $O(n \log n) + O(n) = O(n \log n)$
- Space: $O(n)$

Recreating Chosen Intervals

- Suppose we have $M[]$ of optimal solutions
- How can we reconstruct the optimal set of intervals?
- When should an interval be included in the optimal?
- Depending on which of the two cases results in max tells us whether or not interval i is include:
  - Opt-Schedule$(i) =$ $\max\{$Opt-Schedule$(i-1)$, $v_i +$ Opt-Schedule$(p(i))\}$

This value is bigger: i not in OPT

This value is bigger: i is in OPT

Recursive Solution?

Suppose for now that we do not memoize: just a divide and conquer recursion approach to the problem.

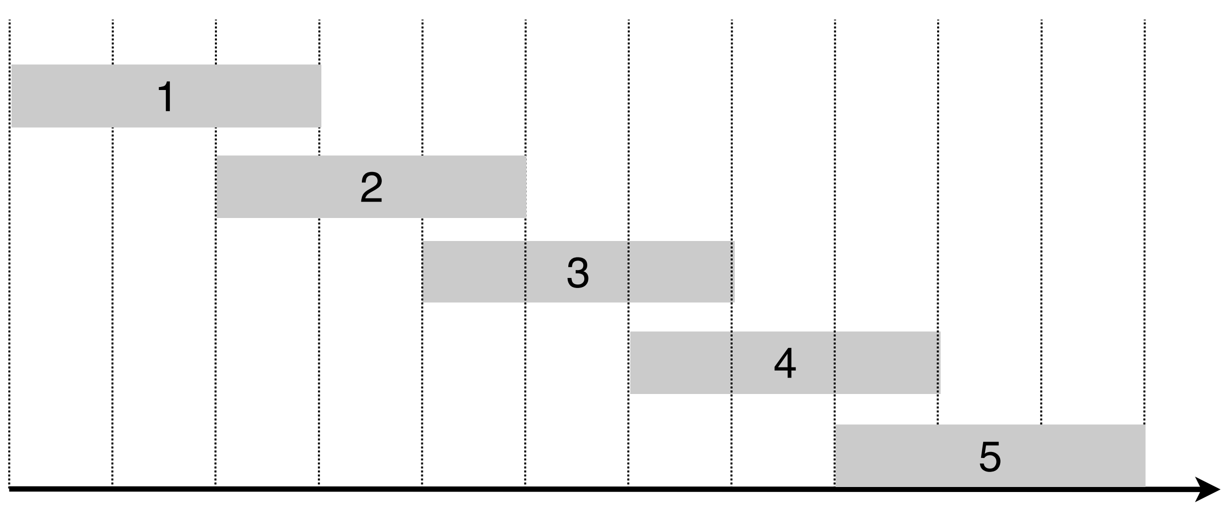
Opt-Schedule(i):

- If $j = 0$, return 0
- Else
  - Return $\max($Opt-Schedule$(j-1), v_j +$ Opt-Schedule$(p(j)))$

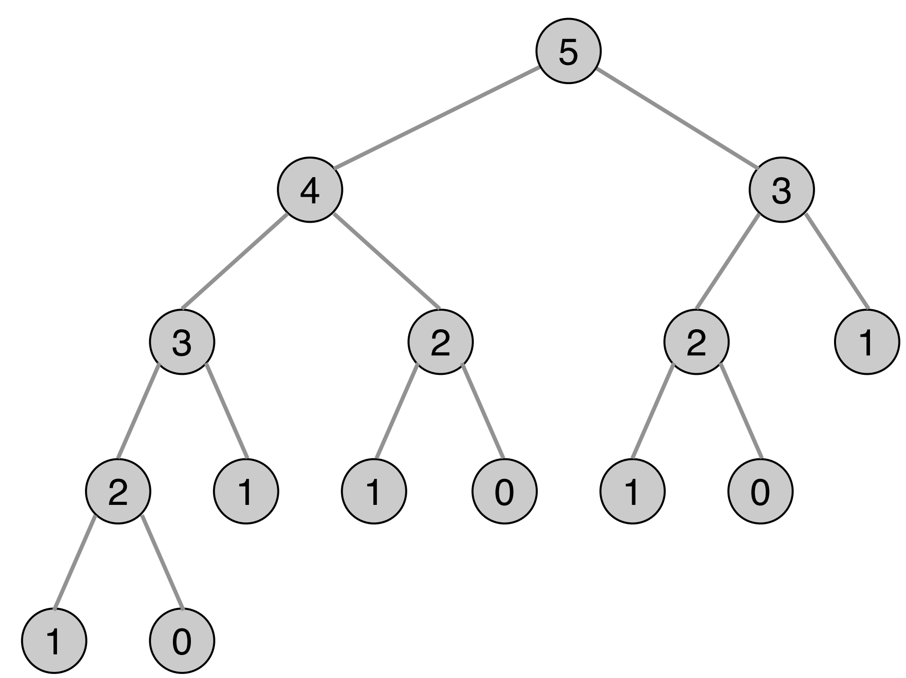
- How many recursive calls in the worst case?
  - Depends on $p(i)$
- Can we create a bad instance?

Recursive Solution: Exponential

- For this example, asymptotically how many recursive calls?
- Grows like the Fibonacci sequence (exponential):
  $T(n) = T(n-1) + T(n-2) + O(1)$
- Lots of redundancy!
  - How many distinct subproblems are there to solve?
  - Opt-Schedule(i) for $1 \leq i \leq n+1$



p(1) = 0, p(j) = j-2



recursion tree

Dynamic Programming Tips

- Recurrence/subproblem is the key!
  - DP is a lot like divide and conquer, while writing extra things down
  - When coming to a new problem, ask yourself what subproblems may be useful? How can you break that subproblem into smaller subproblems?
  - Be clear while writing the subproblem and recurrence!
- In DP we usually keep track of the *cost* of a solution, rather than the solution itself

Longest Increasing Subsequence

Longest Increasing Subsequence

- Given a sequence of integers as an array $A[1,\ldots n]$, find the longest subsequence whose elements are in increasing order
- Find the longest possible sequence of indices $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$ such that $A[i_k] < A[i_{k+1}]$

1	**2**	10	**3**	7	6	**4**	**8**	**11**

LIS: Length 6

1	**2**	**10**	3	7	6	4	8	**11**

A different increasing subsequence that is length 4

Longest Increasing Subsequence

- Given a sequence of integers as an array $A[1,\ldots n]$, find the longest subsequence whose elements are in increasing order
- Find the longest possible sequence of indices $1 \leq i_1 < i_2 < \ldots < i_\ell \leq n$ such that $A[i_k] < A[i_{k+1}]$

1	**2**	10	**3**	7	6	**4**	**8**	**11**

- Length of the longest increasing subsequence above is 6
- To simplify, we will only compute **length of the LIS**

Formalize the Subproblem

$L[i]$: length of the longest increasing subsequence in $A[1,\ldots,i]$ that ends at (and includes) $A[i]$

Identify the Base Case

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

Base Case. $L[1] = ?$

Identify the Final Answer

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

Base Case. $L[1] = 1$

Final answer. $?$

Base Case & Final Answer

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

Base Case. $L[1] = 1$

Final answer. $\max_{1 \leq i \leq n} L[i]$

Recurrence

- How do we go from one subproblem to the next?
- That is, how do we compute $L[i]$ assuming I know the values of $L[1],\ \ldots,\ L[i-1]$

1 **2** 10 **3** 7 6 **4** **8** **11**

Length of the LIS ending at 2?

Length of the LIS ending at 10?

Recurrence

- Let’s say we know the length of the longest subsequence ending at $A[1], A[2], \ldots A[i-1]$
- What is the longest subsequence ending at $A[i]$?
- $A[i]$ could potential extend an earlier subsequence:
    - Can extend a longest subsequence ending at some $A[k]$, with $A[k] < A[i]$, but which k?
    - OK, let’s try all k to get the answer
- Or it doesn’t extend any earlier increasing subsequence

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	1	2	10	3	7	6	4	8	11

L	1							

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	**1**	**2**	10	3	7	6	4	8	11
L	1	2							

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	**1**	**2**	**10**	3	7	6	4	8	11

L	1	2	3					

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	**1**	**2**	10	**3**	7	6	4	8	11

L	1	2	3	3				

How do we know 3 extends a past LIS?

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A									
	1	**2**	10	**3**	7	6	4	8	11

L								
	1	2	3	3				

$L[j]$ extends an LIS ending at $L[i]$ if $A[j] > A[i]$

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	**1**	**2**	10	**3**	**7**	6	4	8	11

L	1	2	3	3	4					

$L[j]$ extends an LIS ending at $L[i]$ if $A[j] > A[i]$

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A

1	**2**	10	**3**	7	**6**	4	8	11

L

1	2	3	3	4	4				

$L[j]$ extends an LIS ending at $L[i]$ if $A[j] > A[i]$

Example: Building a Recurrence

$L[i]$: length of the longest increasing subsequence in A that ends at (and includes) $A[i]$

A	**1**	**2**	10	**3**	7	6	**4**	8	11

L	1	2	3	3	4	4	4			

$L[j]$ extends an LIS ending at $L[i]$ if $A[j] > A[i]$

LIS: Recurrence

$$\boxed{\begin{array}{l} L[j] = 1 + \max\{L[i] \mid i < j \text{ and } A[i] < A[j]\} \\ \text{Assuming } \max \varnothing = 0 \end{array}}$$

Recursion → DP

- If we used recursion (without memoization) we'll be inefficient—we'll do a lot of repeated work
- Once you have your recurrence, the remaining pieces of the dynamic programming algorithm are
  - **Evaluation order.** In what order should I evaluate my subproblems so that everything I need is available to evaluate a new subproblem?
    - For LIS we just left-to-right on array indices
  - **Memoization structure.** Need a table (array or multi-dimensional array) to store computed values
    - For LIS, we just need a one dimensional array
    - For others, we may need a table (two-dimensional array)

LIS Analysis

- Correctness
    - Follows from the recurrence using induction
- Running time?
    - Solve $O(n)$ subproblems
    - Each one requires $O(n)$ time to take the min
    - $O(n^2)$
    - An Improved DP solution takes $O(n \log n)$
- Space?
    - $O(n)$ to store array $L[]$