CS256: Algorithm Design and Analysis **Spring 2024**

Assignment 6 (due 04/24/2024 at 9:59pm)

Instructor: Sam McCauley

Note. This assignment is to be done individually.

Problem 1. (KT 7.5) Are the following statements true or false? If true, you must give a justification.; if false, you must give a counterexample.

(a) Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e. If f is a maximum $s-t$ flow in G, then f saturates every edge out of s with flow (i.e., for all edges e out of s, we have $f(e) = c_e$).

(b) Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e. Let (A, B) be a minimum s-t cut with respect to the capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum s-t cut with respect to the new capacities $\{1 + c_e : e \in e\}$.

Solution.

□

Problem 2. (Modified KT 7.23 and 7.24) Suppose you're looking at a flow network G with source s and sink t, and you want to be able to express something like the following intuitive notion: *Some nodes are clearly on the "source side" of the main bottlenecks; some nodes are clearly on the "sink side" of the main bottlenecks; and some nodes are in the middle.* However, G can have many minimum cuts, so we have to be careful in how we try making this idea precise. Here's one way to divide the nodes of G into three categories of this sort.

- We say a node v is *upstream* if, for all minimum s-t cuts (A, B), we have $v \in A$—that is, v lies on the source side of every minimum cut.
- We say a node v is *downstream* if, for all minimum s-t cuts (A, B), we have $v \in B$—that is, v lies on the sink side of every minimum cut.
- We say a node v is *central* if it is neither upstream nor downstream; there is at least one minimum s-t cut (A, B) for every $v \in A$, and at least one minimum s-t cut (A', B') for which $v \in B'$.

In this question, we design an algorithm to classify vertices of G into these categories and use the classification to characterize graphs that have a unique minimum cut. Let f be the maximum flow in G. Consider the cut (A^*, B^*), where $A^* = \{u \mid u \text{ is reachable from } s \text{ in } G_f\}$ and let $B^* = V - A^*$. Thus, $v(f) = cap(A^*, B^*)$ and (A^*, B^*) is a minimum cut of G.

(a) Show that the set A^* is the set of upstream vertices of G, that is, v is upstream if and only if $v \in A^*$.

(b) Using part (a), describe an efficient algorithm to find the downstream vertices in G. (*Hint.* Consider the graph G^R, with all direction of edges in G reversed. G^R now has source t and sink s.)

(c) Show that G has a unique minimum cut if and only if G has no central vertices, that is, the union of upstream and downstream vertices is the set V.

Solution.

$\square$

Note. This question is about network flow reductions. We will begin those Thursday, but you'll see much more about them on Monday—you can probably start thinking about this problem before Monday's class, but you'll have an easier time solving it completely after the lecture.

Problem 3. (From Dave Mount's Algorithms Class) The computer science department at a major university has a tutoring program. There are m tutors, $\{t_1, \ldots, t_m\}$ and n students who have requested the tutoring service $\{s_1, \ldots, s_n\}$. Each tutor t_i has a set T_i of topics that they know, and each student s_j has a set of topics S_j that they want help with. We say that tutor t_i is suitable to work with student s_j if $S_j \subseteq T_i$ (That is, the tutor t_i knows all the topics of interest to student s_j.) Finally, each tutor t_i has a range $[a_i, b_i]$, indicating that t_i would like to work with at least a_i students (so their time is well-utilized) and at most b_i students (so they are not overburdened).

Given a list of students, a list of tutors, the ranges $[a_i, b_i]$ for the tutors, and a list of suitable tutors for each student, describe an efficient algorithm that determines whether it is possible to generate a pairing of tutors to students such that:

- Each student is paired with exactly one tutor.
- Each tutor t_i is paired with at least a_i and at most b_i students.
- Each student is paired only with a suitable tutor.

Solution.

□