More Randomized Algorithms

Randomization So Far

- Analyzed simple probabilistic processes
  - Birthday paradox
  - Pokemon collector problem
  - Random walks
- Designing and analyzed simple randomized algorithms
  - Karger's min cut
  - Randomized selection
  - Randomized quicksort
- Today: Use randomization to design approximation algorithms for NP complete problems:
  - Max-3-SAT and Max cut

Admin

- Next class in a week!
- Assignment 9 out Friday
- Any questions?

Randomized Approximation Algorithms

- Sometimes it's hard to get the correct answer to a problem using an efficient algorithm
- Can we give guarantees on the algorithm's performance, even if they fall short of giving the correct answer?
- **Approximation Algorithm**: gives an answer with a guarantee of the quality of that answer compared to the optimal answer

- First example:
- We can't satisfy all clauses of a 3-SAT instance in polynomial time. If the optimal algorithm satisfies k clauses in the 3SAT instance, how many can we satisfy in polynomial time?
    - $7k/8$
- We'll use a randomized algorithm to get this bound

Randomized Approximation:
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- Widespread principle in combinatorics:
    - **Probabilistic method**. [Paul Erdös] Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability!

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- As long as p is polynomial, expected running time is polynomial

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- Rewrite the expectation as:
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- There is a randomized algorithm with polynomial running time that is $7/8$th approximation algorithm to MAX 3-SAT
- Fun fact: It is **NP hard** to approximation MAX 3-SAT with an approximation factor $7/8 + \varepsilon$, for any $\epsilon > 0$ [Håstad 97]

Randomized Approximation: Max Cut

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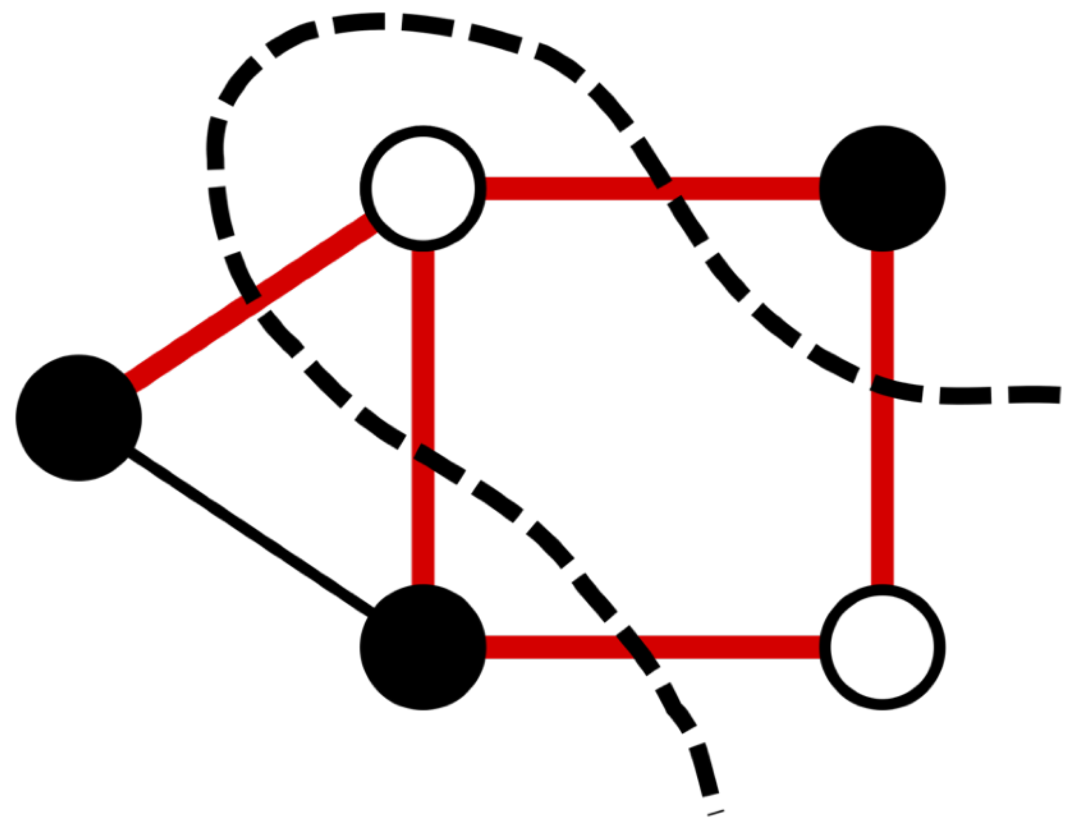
- A 1/2-approximation to max-cut will produce a cut whose size is at least $1/2$ of the optimal (largest cut in the graph)

Motivation



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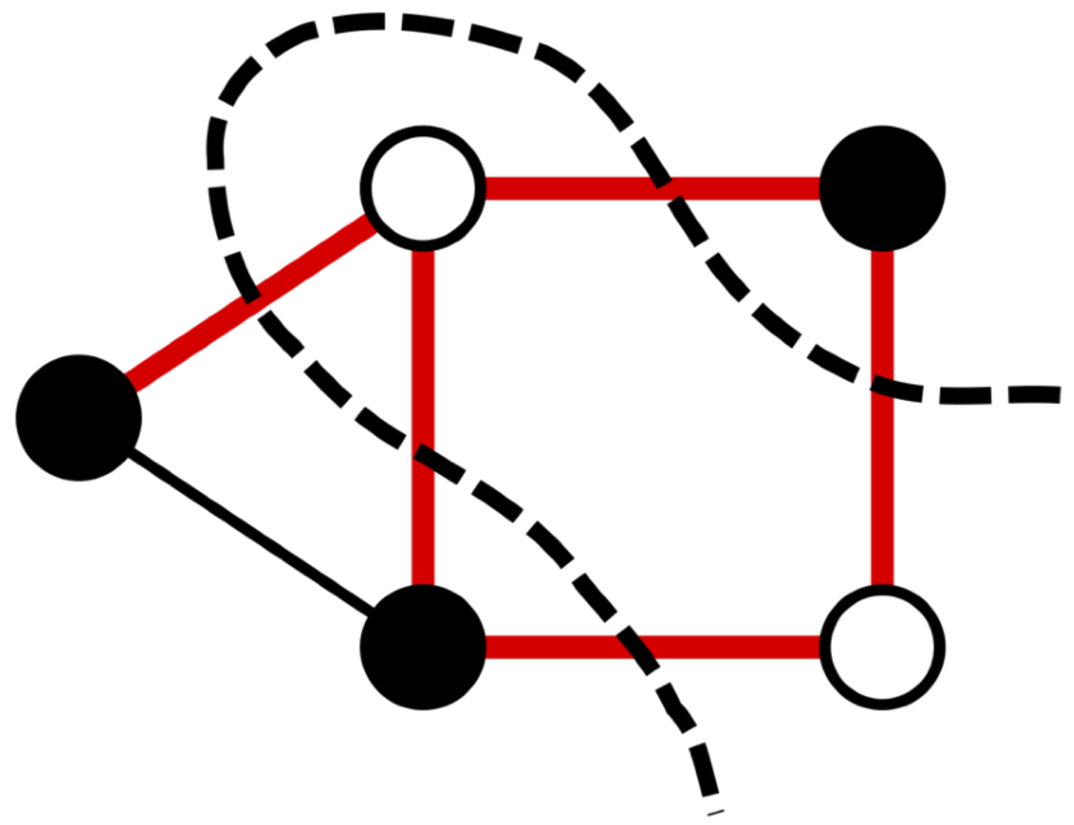

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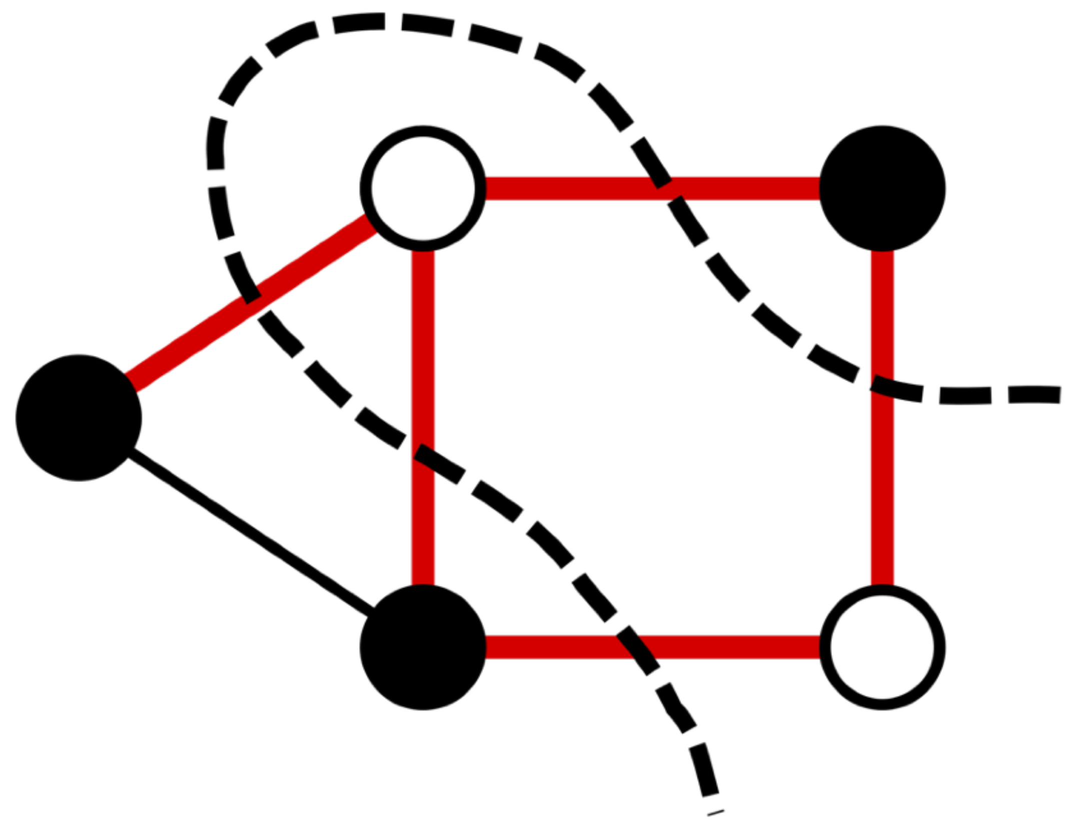

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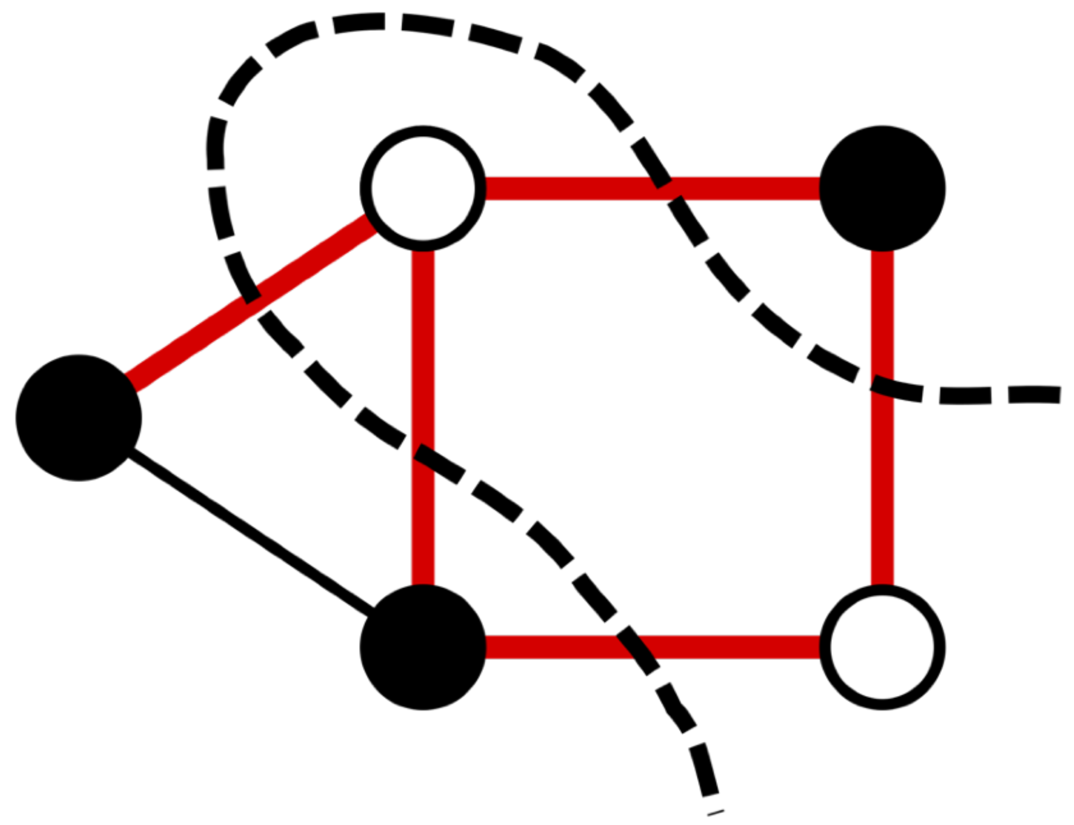

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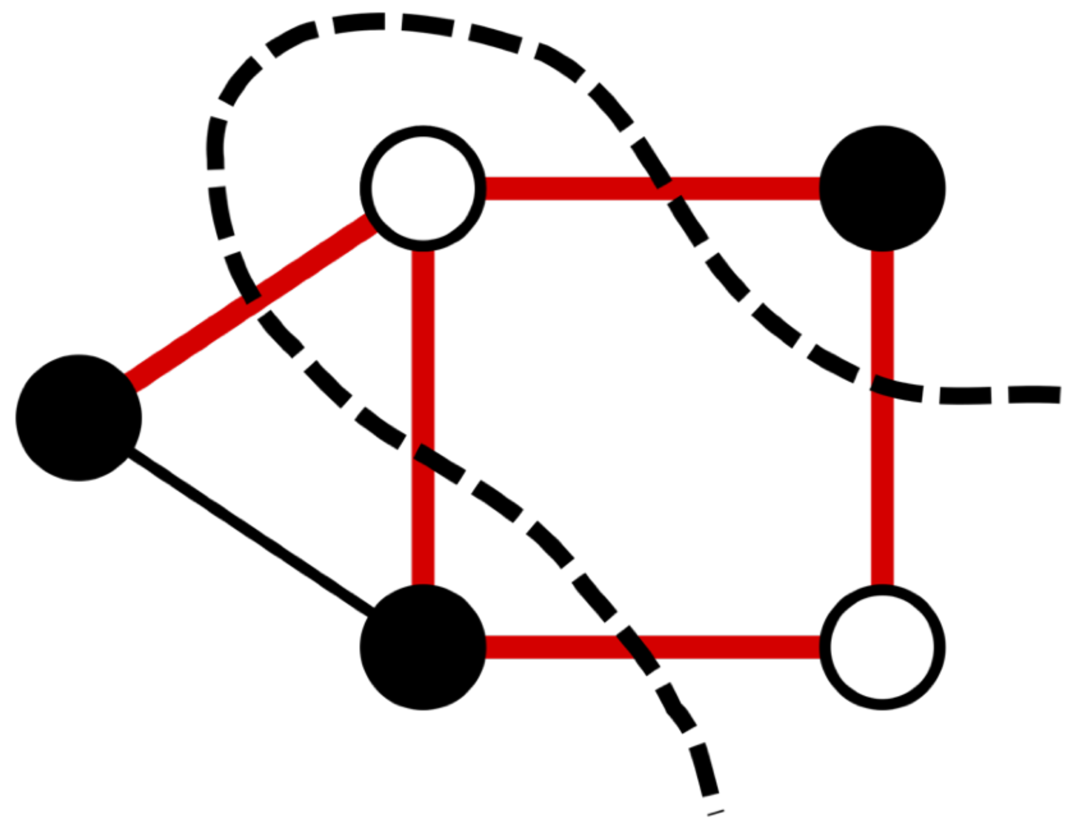

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- It's a huge pain that this is NP-hard. But, at least we can approximate it!

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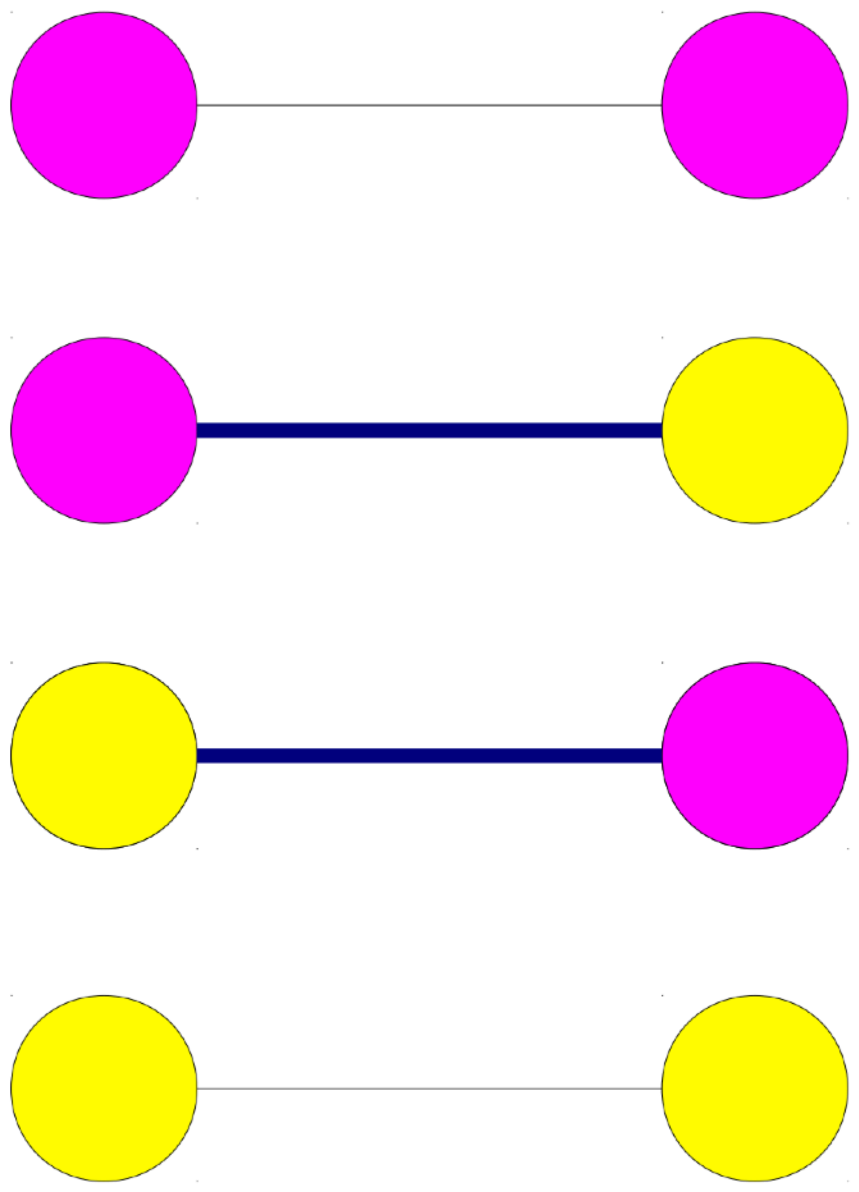

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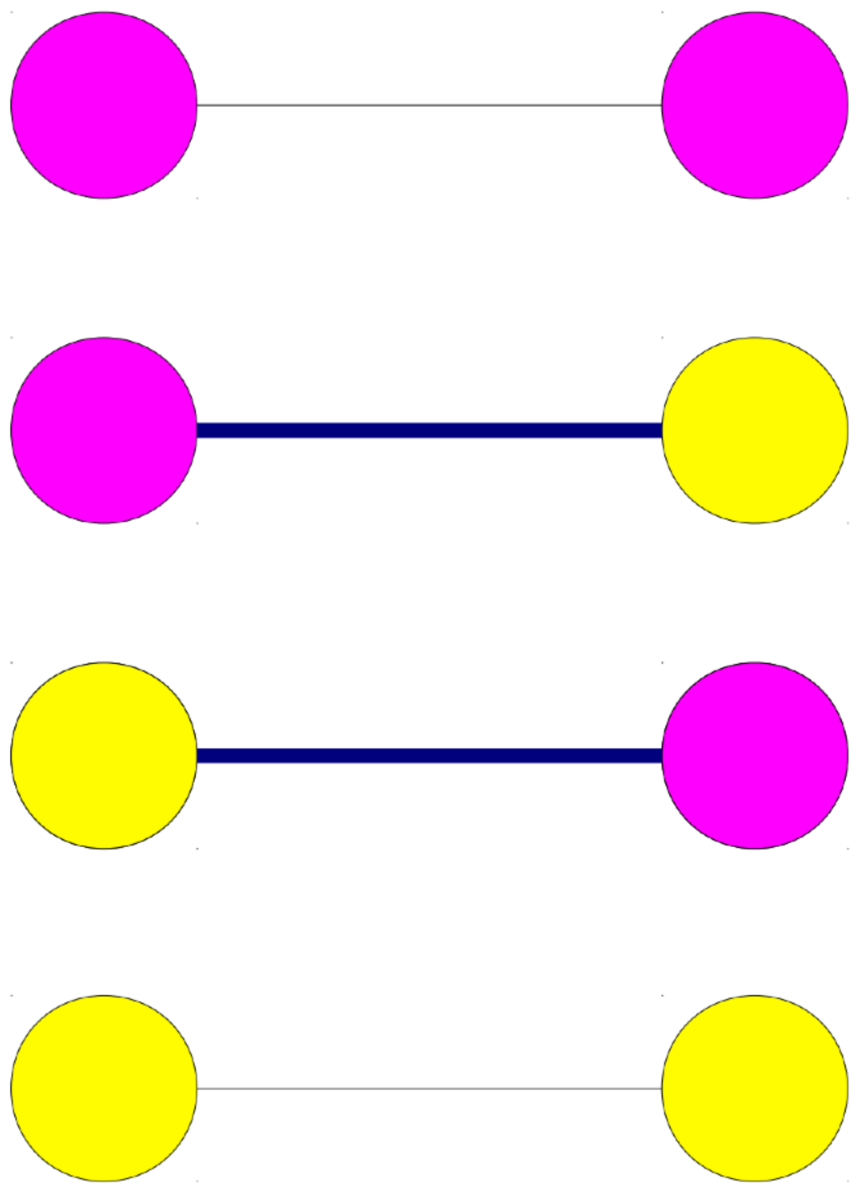

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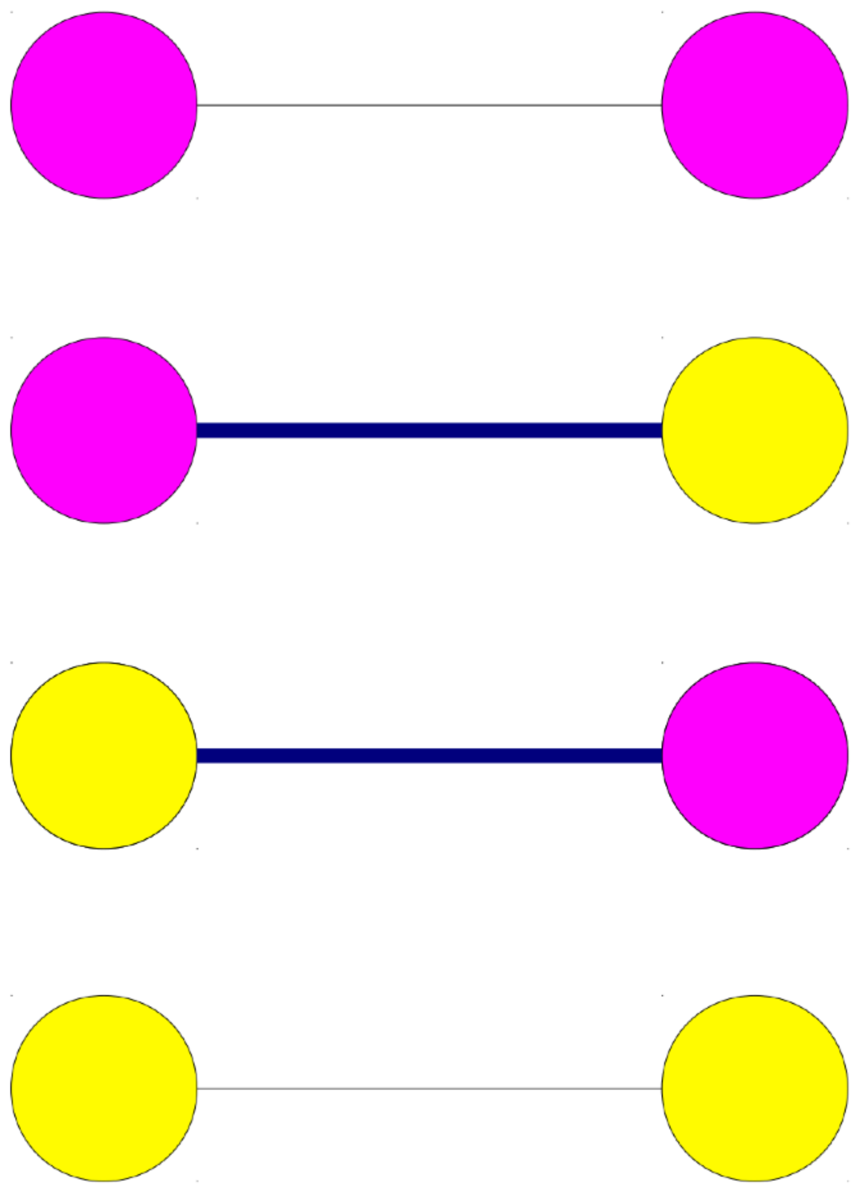

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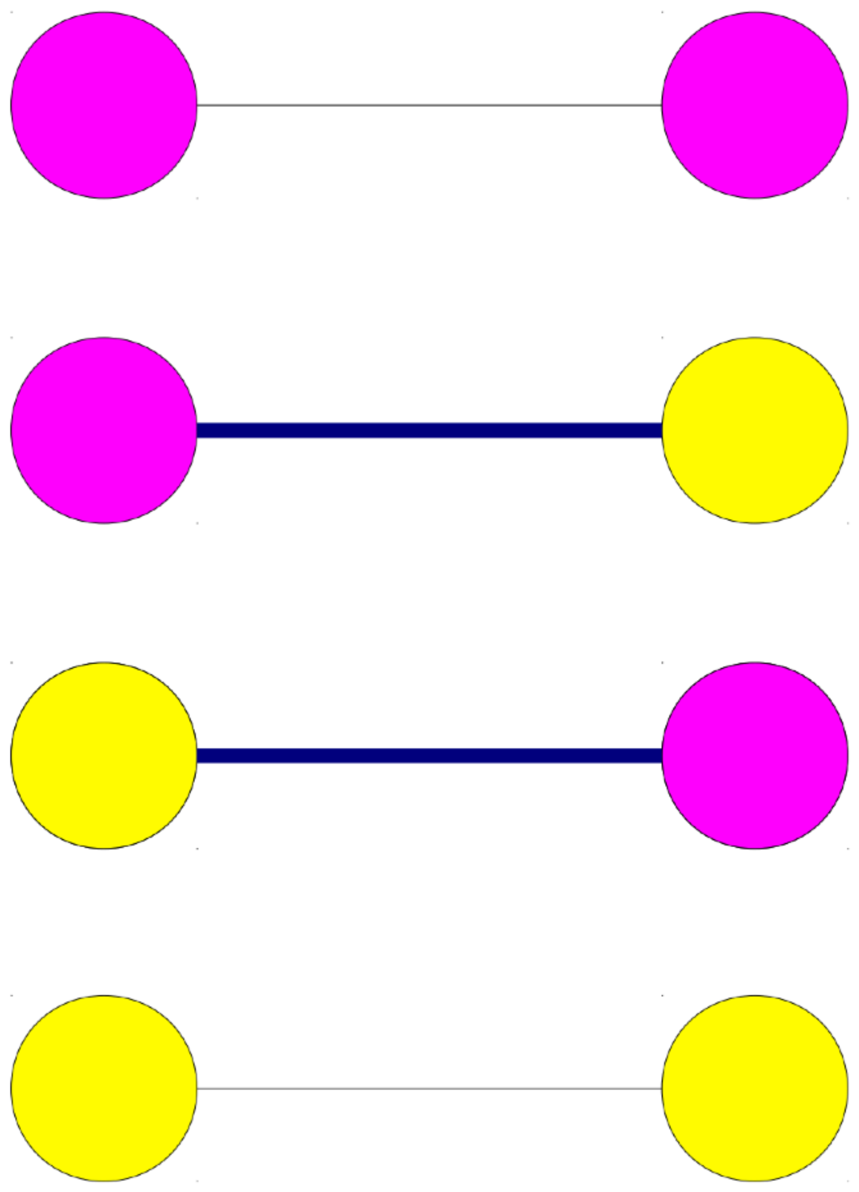

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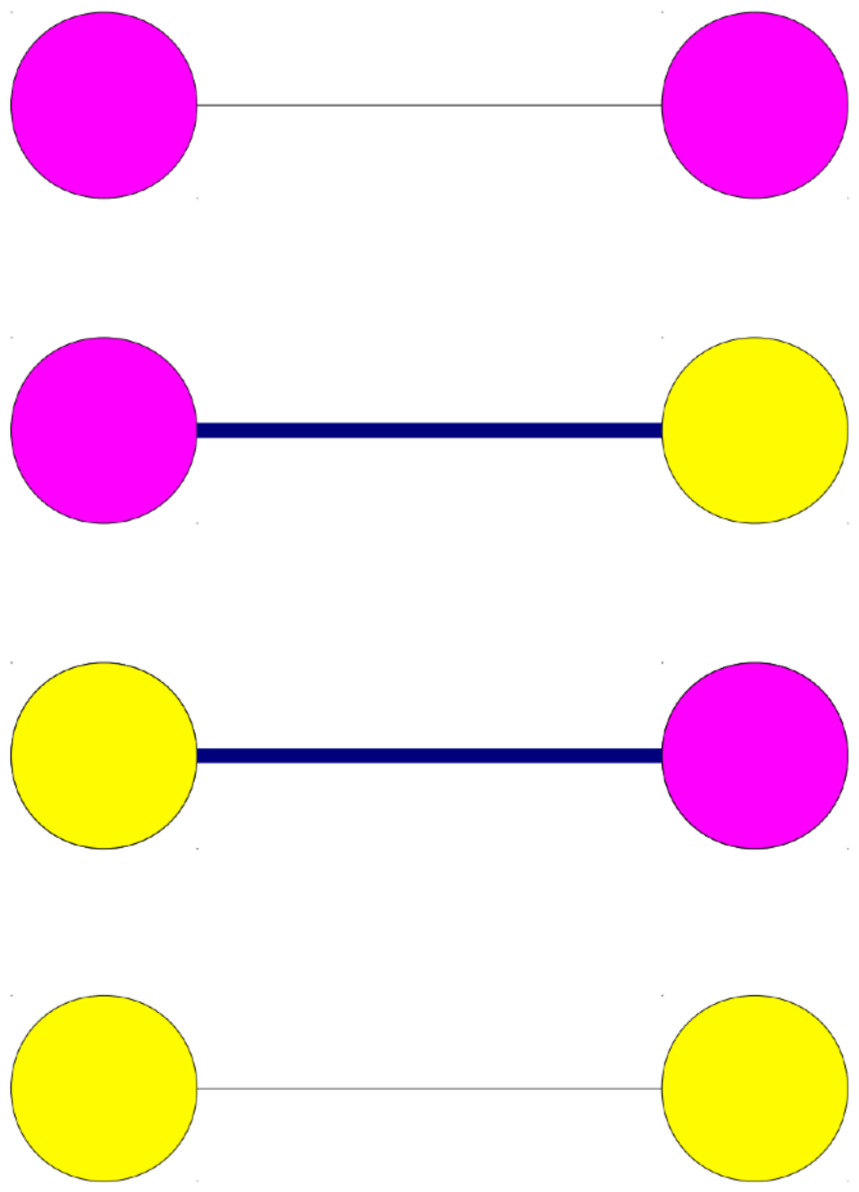

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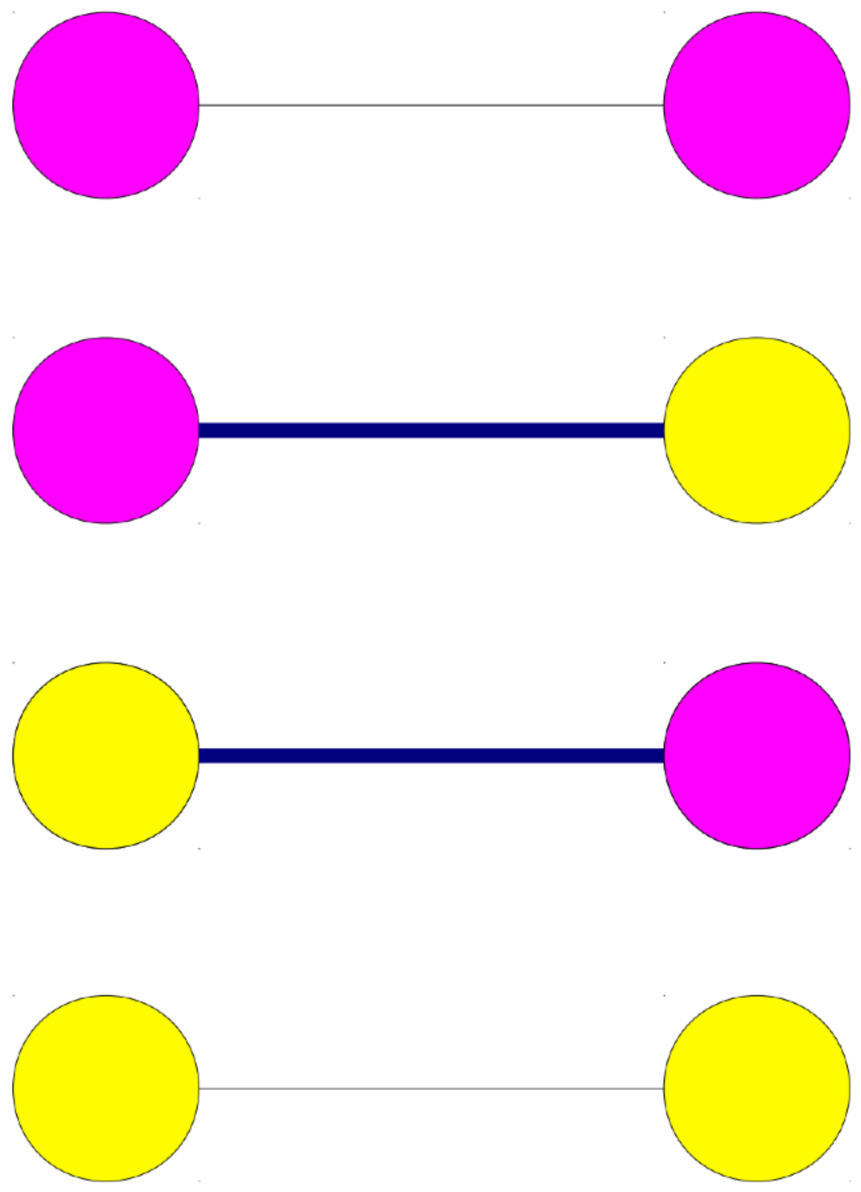

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- Thus, our randomized algorithm has
  an expected approximation ratio at least $1/2$, as it produces a cut
  of size at least $1/2$ of OPT in expectation

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  - Might be optimal. Better than $.941$ is NP-hard

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)
  - MIT course notes, 6.042/18.062J Mathematics for Computer Science April 26, 2005, Devadas and Lehman