Min Cut, Quicksort, and Quickselect

Admin

- Lecture at normal time on Monday, same zoom link
- Try to have your camera on when possible
- If I disconnect, please just hang on for a couple minutes; I’ll probably join again

Randomized Min Cut

- **Global min-cut problem.**
  Given an undirected, unweighted graph $G = (V, E)$, find a cut (A, B) of minimum cardinality (that is, min # of edges crossing it).
- **Applications.** Network reliability, network design, circuit design, etc.
- **Poly-time network-flow solution** (by reduction to min s-t cut).
  - Replace every undirected edge (u, v) with $u \to v$ and $v \to u$, each of capacity 1
  - Fix any $s \in V$ and compute min s-t cut for every other node $t \in V - \{s\}$
  - $(n - 1)$ executions of min s-t cut
- Gives impression that finding global min cut is harder than finding a min s-t cut, which is not true
- Deceptively simple and efficient randomized algorithm [Karger 1992]

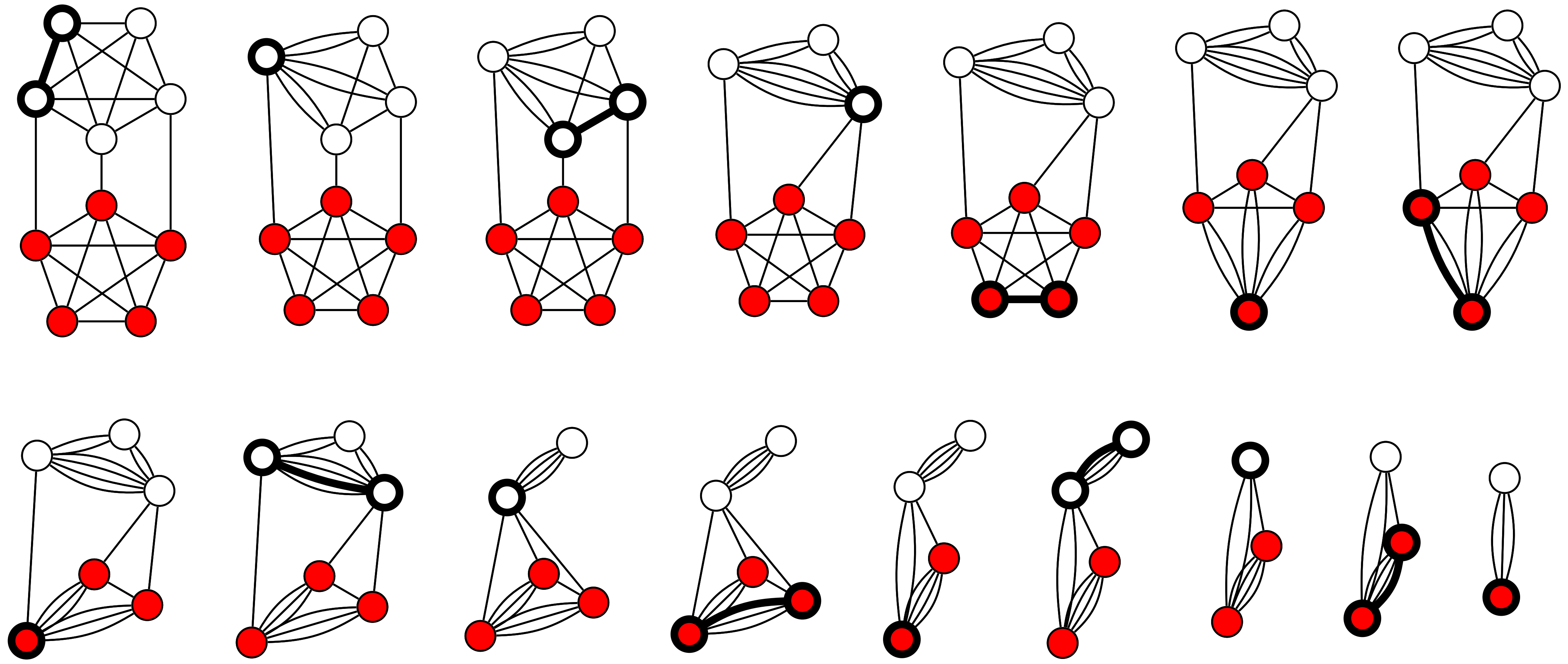
Karger's Min Cut

- Uses a primitive called *edge contraction*
- Contract edge e in G, denoted $G \leftarrow G/e$
    - Replace u and v by single new super-node w
    - Preserve edges, updating endpoints of u and v to w
    - Keep parallel edges, but delete self-loops
- An edge can be contracted in $O(n)$ time, assuming the graph is represented as an adjacency list

Karger's Min Cut

- Algorithm tries to *guess* the min cut by randomly contracting edges
- Running time $O(n^2)$ (why?)
- Correctness:
  How often, if ever, does it return the min cut?

<u>GuessMinCut(G):</u>
  for $i \leftarrow n$ downto 2
    pick a random edge e in G
    $G \leftarrow G/e$
  return the only cut in G



Reference: Thore Husfeldt

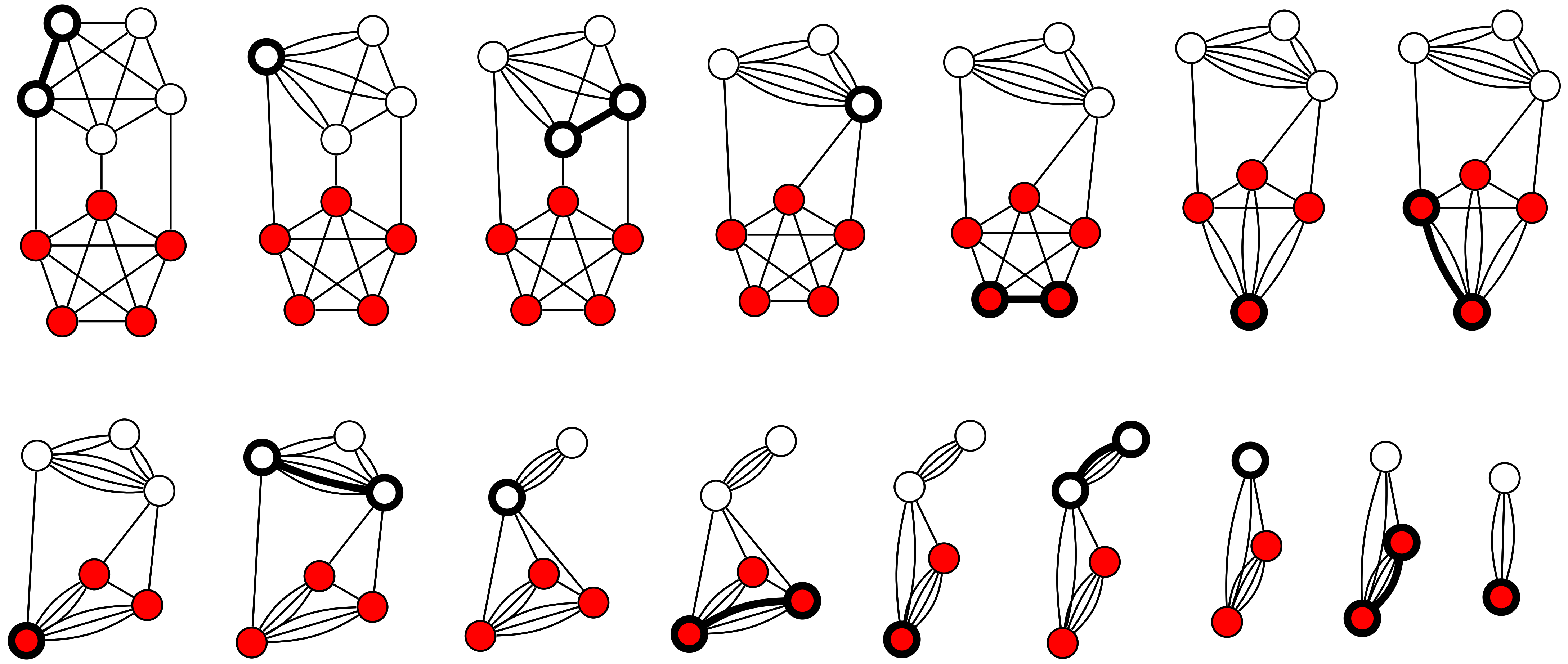
Observations:

- Any cut in the contracted graph is a cut in the original graph
- Let $C = (S, V - S)$ be any cut, if algorithm never contracts an edge crossing this cut, then it will produce the cut C

What can we say about how many edges there are?

If the minimum cut has size/cardinality k:

- Each vertex must have degree at least k, and thus the graph must have at least $nk/2$ edges



Reference: Thore Husfeldt

Karger's Analysis

- Let C be any arbitrary min cut of cardinality k
- If we pick an edge in G uniformly at random, what is the probability of picking an edge in C
    - $m \geq nk/2$
    - Pr(picking an edge in C) = $\dfrac{k}{m} \leq \dfrac{k}{nk/2} = \dfrac{2}{n}$
- The probability we don't contract a cut edge in the 1st step $\geq 1 - \dfrac{2}{n}$
- After the first edge is contracted, the algorithm proceeds recursively (with independent random choices) on the $(n-1)$-vertex graph

Karger's Analysis

- Let $P(n)$ denote the probability that the algorithm returns the correct min cut on an n-vertex graph, then
- $P(n) \geq \left(1 - \frac{2}{n}\right) \cdot P(n-1)$, with base case $P(2) = 1$

- Expanding the recurrence:
  - $P(n) \geq \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \frac{n-5}{n-3} \cdots\cdots \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3}$

- Terms cancel out to get: $P(n) \geq \frac{2}{n(n-1)} = \binom{n}{2}^{-1}$

Amplifying Success Probability

- Thus, a single execution of Karger's min cut algorithm finds the min cut with probability at least $1/\binom{n}{2}$, which is low
    - But, we can amplify our success probability!
- Run the algorithm R times (using independent random choices) and pick the best min-cut among them
- What is probability we don't find the min cut after R repetitions?
    - $\left(1 - 1/\binom{n}{2}\right)^R$

Amplifying Success Probability

- If we execute $R = \binom{n}{2}$ times, the probability of failure is
  - $\left(1 - 1/\binom{n}{2}\right)^{\binom{n}{2}} \le \frac{1}{e}$
- If we run the algorithm $R = \binom{n}{2} c \ln n$ times, we can make the failure probability polynomially small: $\left(\frac{1}{e}\right)^{c \ln n} = \frac{1}{n^c}$
- Karger's algorithm finds the min-cut with high probability (w.h.p.)

An algorithm is correct **with high probability (w.h.p.)** with respect to input size n if it fails with probability at most $\frac{1}{n^c}$ for any constant $c > 1$.

Example Execution

trial 1

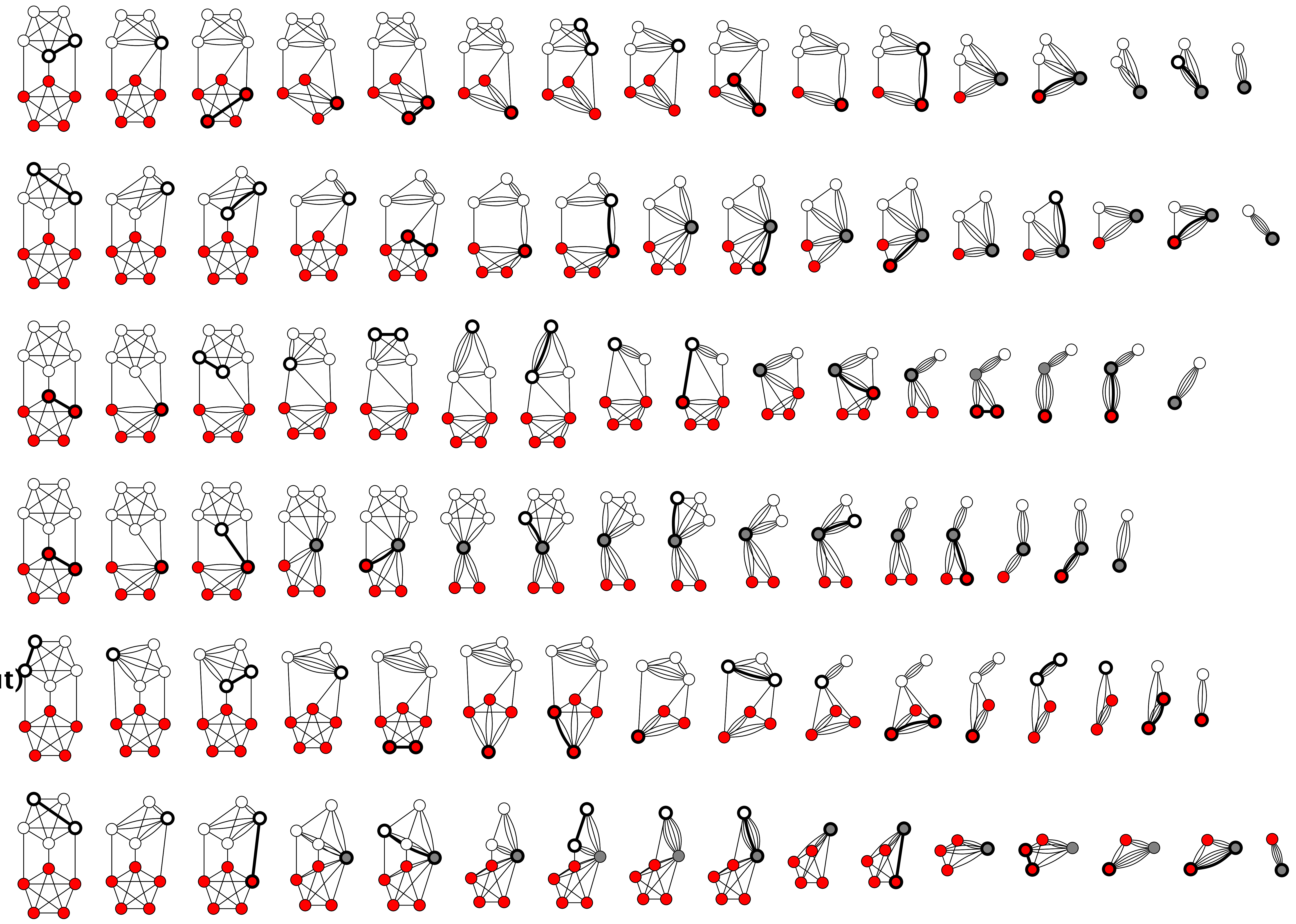
trial 2

trial 3

trial 4

trial 5
(finds min cut)

trial 6



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Reference: Thore Husfeldt

Karger’s Running Time

- Thus, Karger’s algorithm finds the min-cut with high probability (w.h.p.)
- Running time: we perform $\Theta(n^2 \log n)$ iterations, each $O(n^2)$ time
    - $O(n^4 \log n)$ time
    - Faster than naive-flow-techniques, nothing to get excited about
- Improves to $O(n^2 \log^3 n)$ by guessing cleverly! [Karger-Stein 1996]
- **Idea**: Improve the guessing algorithm using the observation:
    - As the graph shrinks, the probability of contracting an edge in the minimum cut increases
    - At first the probability is very small: $2/n$ but by the time there are three nodes, we have a $2/3$ chance of screwing up!

Takeaways

- Notice: Karger's algorithm had *one-sided error*:
  - Might produce a cut that is not min cut
- You can increase the success rate of a "Monte Carlo" algorithm with one-sided errors by iterating it multiple times and taking the best solution
  - If the probability of success is $1/f(n)$, then running it $O(f(n)\log n)$ times gives a high probability of success
- If you're more intelligent about how you iterate the algorithm, you can often do much better than this
- Next, we'll see an example of a "Las Vegas" algorithm
  - Randomized selection and quick sort

Randomized Algorithm II

Randomized Selection

Randomized Selection

- **Problem.** Find the kth smallest/largest element in an unsorted array
- Recall our selection algorithm

Select (A, k):

If $|A| = 1$: return $A[1]$

Else:

- Choose a pivot $p \leftarrow A[1,\ldots,n]$; let r be the rank of p
- $r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p))
- If $k == r$, return p
- Else if $k < r$: Select $(A_{<p}, k)$
- Else: Select $(A_{>p}, k - r)$

Selection with a Good Pivot

- Recall: we called the pivot “good” if it reduced the array size by at least a constant
  - Which would give a recurrence $T(n) \leq T(\alpha n) + O(n)$ for some constant $\alpha < 1$
  - Expands to a decreasing geometric series $T(n) = O(n)$
- In the deterministic algorithm, how did we find a good pivot?
  - Split array into groups of 5
  - And computed the median of group medians
  - The pivot guaranteed that $n \rightarrow 7n/10$
- **Here is a silly idea:** What if we pick the pivot uniformly at random?
- Seems like the pivot is “usually” around the midpoint
- What is the expected running time?

Randomized Selection

- **Problem.** Find the kth smallest/largest element in an unsorted array
- Recall our selection algorithm

Select (A, k):

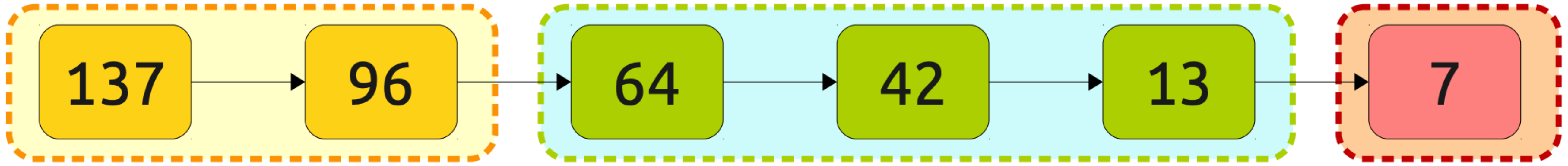
If $|A| = 1$: return $A[1]$

Else:

- Choose a pivot $p \leftarrow A[1,\ldots,n]$ at random; let r be the rank of p
- $r, A_{<p}, A_{>p} \leftarrow$ Partition((A, p)
- If $k == r$, return p
- Else if $k < r$: Select $(A_{<p}, k)$
- Else: Select $(A_{>p}, k - r)$

Analyzing Rand. Selection

- Normally, we'd write a recurrence relation for a recursive function
- But the array size in later recursive call depends on the random choice of pivots in earlier calls
- We use a different accounting trick for running time
- Randomized selection makes at most one recursive call each time:
    - Group multiple recursive call in "phases"
    - Sum of work done by all calls is equal to the sum of the work done in all the phases



Analyzing in Phases

- Idea: let a "phase" of the algorithm be the time it takes for the array size to drop by a constant factor (say $n \to (3/4) \cdot n$)
- If array shrinks by a constant factor in each phase and linear work done in each phase, what would be the running time?
- $T(n) = c(n + 3n/4 + (3/4)^2 n + \ldots + 1) = O(n)$
- If we want a $1/4$th, $3/4$th split, what range should our pivot be in?
  - Middle half of the array (if n size array, then pivot in $[n/4, 3n/4]$)
  - What is the probability of picking such a pivot?
    - $1/2$
  - Phase ends as soon as we pick a pivot in the middle half
    - Expected # of recursive calls until phase ends? 2

Expected Running Time

- Let the algorithm be in phase j when the size of the array is
    - At least $n\left(\frac{3}{4}\right)^{j}$ but not greater that $n\left(\frac{3}{4}\right)^{j+1}$
- Expected number of iterations within a phase: 2
- Let X_j be the expected number of steps spent in phase j
- $X = X_0 + X_1 + X_2 \ldots$ be the total number of steps taken by the algorithm
- Within a phase, the algorithm does work linear in the size of the array in one iterations and thus, $E[X_j] \leq 2cn\left(\frac{3}{4}\right)^{j}$
- Expected running time:

$$E[X] = \sum_j E[X_j] \leq \sum_j 2cn\left(\frac{3}{4}\right)^{j} = 2cn\sum_j\left(\frac{3}{4}\right)^{j} \leq 8cn = O(n)$$

Pivot Selection

- Deterministic and random both take $O(n)$ time
    - What's the advantage of the deterministic algorithm?
    - Worst-case guarantee—the random algorithm could be very slow sometimes
    - What's the advantage of the random algorithm?
    - Much much simpler
    - Better constants
- Which should you use?
    - Pretty much always random
    - Question to ask yourself: how often is the randomized algorithm going to be much worse than $O(n)$?

Monte Carlo vs Las Vegas

- Monte Carlo algorithm: run a certain number of times; algorithm succeeds with some probability
- Las Vegas algorithm: the algorithm always succeeds, but the running time is probabilistic

Randomized Algorithm III

Randomized QuickSort

Randomized Quicksort

- Recall deterministic Quicksort
- Depending on the choice pivot, could be $O(n^2)$
- What if we pick the pivot uniformly at random?
  - Can get expected running time as $O(n \log n)$

Quicksort(A)**:**

If $|A| < 3$: Sort(A) directly

Else: choose a pivot element $p \leftarrow A$

$A_{<p}, A_{>p} \leftarrow$ Partition around p

Quicksort$(A_{<p})$

Quicksort$(A_{>p})$

Modified Rand. Quicksort

- Before we analyze quick sort with uniform random pivot
- Consider the following modification
  - Pick pivot p randomly
  - Partition array around p
  - If p is a bad pivot (say, $\max\{|A_{<p}|, |A_{>p}|\} > (3/4)|A|$), we throw it out and pick another pivot
  - Else, we recursively call Quicksort on the partitions
- We know that expected number of trials before we get a good pivot is 2 and a good pivot gives a $1/4,3/4$ split
- This immediately gives us expected running time as $O(n \log n)$

Randomized Quicksort

- Suppose we don't throw out bad pivots (its wasteful anyway)
- Can we still show the expected running time is the same
    - Intuitively bad pivots don't hurt asymptotically, because they only occur $1/2$ the time
- We analyze quicksort using another accounting trick
    - Only two types of work:
        - Work making recursive calls (lower order term, turns out)
        - Work partitioning the elements
- How many recursive calls in the worst case?
    - $O(n)$

Randomized Quicksort

- We thus need to bound the work partitioning elements
- Partitioning an array of size n around a pivot element p takes exactly $n-1$ comparisons
- We won't look at partitions made in each recursive calls, which depend on the choice of random pivot
- **Idea:** Account for the total work done by the partition step by summing up the total number of comparisons made
- Two ways to count total comparisons:
  - Look at the size of arrays across recursive calls and sum
  - Look at all pairs of elements and count total # of times they are compared (easier to do in this case)

An Aside about Randomized Analysis

- There are often multiple ways to determine a randomized algorithm’s cost
- We can split into phases, or count the cost directly. We can calculate each probability, or use linearity of expectation
- Intrinsically some “cleverness” involved in choosing the way that gets you a clean answer
- In this class I’m going to try to ask you problems where there’s a clear path to finding the solution (either it follows directly from the question, or I’ll ask about problems you’ve seen before)
- That said, here’s a very clever way to calculate Quicksort’s running time

Counting Total Comparisons

- Just for analysis, let B denote the sorted version of input array A, that is, $B[i]$ is the ith smallest element in A
- Define random variable X_{ij} as the number of times Quicksort compares $B[i]$ and $B[j]$
- Observation: $X_{ij} = 0$ or $X_{ij} = 1$, why?
  - $B[i]$, $B[j]$ only compared when one of them is the current pivot; pivots are excluded from future recursive calls
- Let $T = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$ be the total number of comparisons made by randomized Quicksort

Expected Running Time

- Goal: $E[T] = E\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n}\sum_{j=i+1}^{n} E[X_{ij}]$
- $E[X_{ij}] = \Pr[X_{ij} = 1]$
- When is $X_{ij} = 1$? That is, when are $B[i]$ and $B[j]$ compared?
- Consider a particular recursive call. Let rank of pivot p be r.
  - Case 1. One of them is the pivot: $r = i$ or $r = j$
  - Case 2. Pivot is between them: $r > i$ and $r < j$
  - Case 3. Both less than the pivot: $r > i, j$
  - Case 4. Both greater than the pivot: $r < i, j$

Comparisons for Each Case

- Case 1. $r = i$ or $r = j$
  - $B[i]$ and $B[j]$ are compared once and one of them is excluded from all future calls
- Case 2. $r > i$ and $r < j$
  - $B[i]$ and $B[j]$ are both compared to the pivot but not to each other, after which they are in different recursive calls: will never be compared again
- Case 3. $r > i, j$ and Case 4. $r < i, j$
  - $B[i]$ and $B[j]$ are not compared to each other, they are both in the same subarray and may be compared in the future
- **Takeaway:** $B[i]$, $B[j]$ are compared for the 1st time when one of them is chosen as pivot from $B[i], B[i+1], \ldots, B[j]$ & never again

Expected Running Time

- $\Pr[X_{ij} = 1] = \Pr($one of them is picked as pivot from $B[i], B[i+1], \ldots, B[j]$
- $\Pr[X_{ij} = 1] = \dfrac{2}{j - i + 1}$
- $E[T] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \dfrac{1}{j - i + 1}$

Expected Running Time

- $B[i]$ and $B[j]$ are compared iff one of them is the first pivot chosen from the range $B[i], B[i+1], \ldots, B[j]$
- Pr$[X_{ij} = 1] = \dfrac{2}{j - i + 1}$
- $E[T] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}] = 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \dfrac{1}{j - i + 1}$
- For fixed i, inner sum is $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots \dfrac{1}{n - i + 1} \leq \sum_{\ell=2}^{n} \dfrac{1}{\ell} = O(\log n)$
- Thus, expected number of comparisons is: $E[T] = O(n \log n + n) = O(n \log n)$

Quick Sort Summary

- Las Vegas algorithms like Quicksort and Selection are always correct but their running time guarantees hold in expectation
- We can actually prove that the number of comparisons made by Quicksort is $O(n \log n)$ **with high probability**
  - This means the the probability that the running time of quicksort is more than a constant factor away from its expectation is very small (polynomially small in n)
  - Called **concentration bounds**

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)
  - Hamiltonian cycle reduction images from Michael Sipser's Theory of Computation Book