P, NP, NP-hard, and NP-complete

Admin

- Assignment 7 out right after class
- We've caught up a little with the schedule; thanks for being flexible in the last couple minutes with class
  - (Today we won't go over)
- Anything else?

Goal of Today

- Most of the class has been about how to efficiently solve problems
- Now we're going to shift to a higher-level question
  - What problems can a computer efficiently solve?

Goal of Today

- Number of inversions—can solve in $O(n^2)$ easily, but $O(n \log n)$ with a clever algorithm
- Weighted interval scheduling—can try all combinations in $O(2^n)$, but can solve in $O(n^2)$ using Dynamic programming
- Network flow seems very difficult to solve, but we saw how to solve it in $O(nmC)$ and even $O(n^2m)$

Goal of Today

- What problems can a computer solve in polynomial time?
- What problems can a computer (probably) **not** solve in polynomial time?

Technical Setup

- We will now focus on **decision problems** — problems with a yes or no answer
  - "Does this DAG have a topological order?"
  - Is this graph bipartite?
  - Do these two strings have Edit Distance at most 10?
  - Does this flow network have a max flow of at least 20?

Technical Setup

- Most problems have a decision analog
  - Find the flow of this network -> “does this network have flow at least k?”
  - Find the optimal schedule of these intervals -> “can we schedule at least k intervals?”

- These are (mostly) the same—-after all, can always binary search for the optimal value

Technical Setup

- Decision problem means that every solution is “yes” or “no”
- Can represent this using a set of possible inputs A:
  - $x \in A$ means that the solution to x is “yes”
  - $x \notin A$ means that the solution to x is “no”
- So can have (for example): A is the set of all flow networks which permit flow at least k
- Or can have: A is the set of all pairs of strings (a, b) where the edit distance between a and b is at most k

Class P

- P: The decision problems that can be solved by a computer in polynomial time
  - Edit distance is in P
  - Max flow is in P
  - Bipartite matching is in P
  - Knapsack?
    - The algorithm we saw is pseudo-polynomial! So we don’t know yet

Class NP

Class NP—Intuition

- NP is the class of problems that can be *verified* in polynomial time
- If I give you helpful information, you can easily get the answer

Class NP—Intuition

8								
		3	6					
	7			9		2		
	5				7			
				4	5	7		
			1				3	
		1					6	8
		8	5				1	
	9					4		

☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

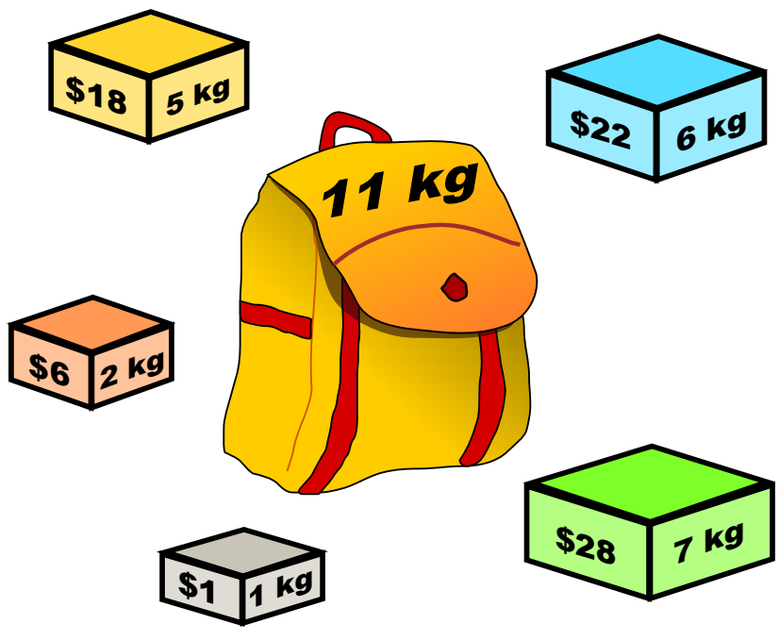
Sudoku is easy if I give you information (by giving you the solution). So sudoku is in NP

Class NP—Intuition

- Example (Knapsack capacity C = 11)
  - {3, 4} has value $40 (and weight 11)

i	v_i	w_i
1	$1	1 kg
2	$6	2 kg
3	$18	5 kg
4	$22	6 kg
5	$28	7 kg

knapsack instance (weight limit W = 11)



Knapsack is easy if I give you information (by giving you the solution). So knapsack is in NP

Class NP

Definition. Algorithm $V(s, c)$ is a verifier for problem X if for every string $s \in X$ iff there exists a certificate, a string c, such that $V(s, c) =$ yes iff $s \in X$.

Definition. NP = set of decision problems for which there exists a polynomial-time verifier

- $V(s, c)$ is a polynomial time algorithm
- Certificate c is of polynomial size:
    - $|c| \leq p(|s|)$ for some polynomial $p(\,.\,)$

Quick question

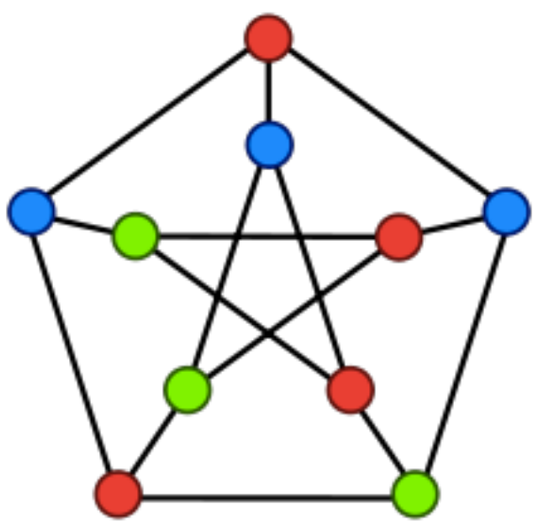
- Is $P \subseteq NP$?
- That is to say: if a problem is in P, does that mean that it is in NP?

- Yes! If a problem can be solved in polynomial time, it can be verified in polynomial time.
- (Can just set $c = ""$)

Graph-Coloring $\in$ NP

Graph-Coloring. Given a graph $G = (V, E)$, is it possible to color the vertices of G using only three colors, such that no edge has both end points colored with the same color.

- Graph-Coloring $\in$ NP
- **Certificate:** assignment of colors to vertices
- **Poly-time verifier:** check if at most 3 colors used, check for each edge if ends points same color or not

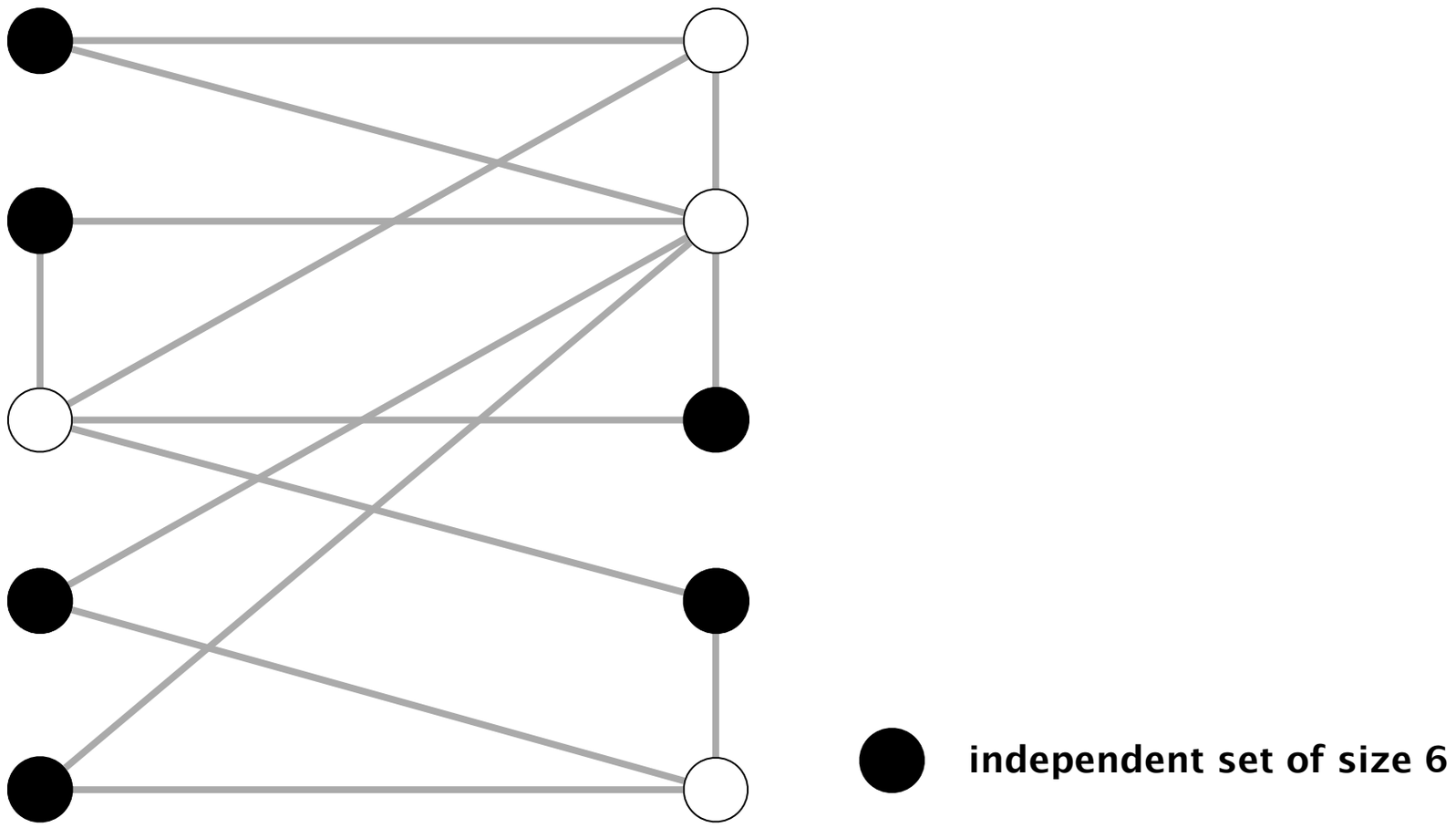


A 3-colorable graph

Independent Set

- Given a graph $G = (V, E)$, an independent set is a subset of vertices $S \subseteq V$ such that no two of them are adjacent, that is, for any $x, y \in S$, $(x, y) \notin E$

- **IND-SET Problem.**
  Given a graph $G = (V, E)$ and an integer k, does G have an independent set of size at least k?



IND-SET $\in$ NP

- Given a graph $G = (V, E)$, an independent set is a subset of vertices $S \subseteq V$ such that no two of them are adjacent, that is, for any $x, y \in S$, $(x, y) \notin E$

- **IND-SET Problem.** Given a graph $G = (V, E)$ and an integer k, does G have an independent set of size at least k?

- **IND-SET $\in$ NP.**
  - **Certificate:** a subset of vertices
  - **Poly-time verifier:** check if any two vertices are adjacent and check if size is at least k

Satisfiability

- The next problem is the classic example of a problem in NP
  - (and, as we'll soon see, probably not in P)
- Many different small variations on the same problem (we'll see a couple)
- Idea: given a logical equation, can we assign "true" and "false" to the variables to satisfy the equation?

SAT, 3SAT $\in$ NP

- SAT. Given a CNF formula ϕ, does it have a satisfying truth assignment?
- 3SAT. A SAT formula where each clause contains exactly 3 literals (corresponding to different variables)
- $\phi = (\overline{x_1} \vee x_2, \vee x_3) \wedge (x_1, \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- Satisfying instance: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$, where 1 : true, 0 : false
- SAT, 3-SAT $\in$ NP
    - Certificate: truth assignment to variables
    - Poly-time verifier: check if assignment evaluates to true

P versus NP

P vs NP

- We know that every problem in P is also in NP
- What about the reverse? That is to say:
  - If a problem can be efficiently *verified*, does that mean it can be efficiently solved in the first place?
  - Or, do there exist problems that can be verified quickly that are *impossible* to solve quickly?

Why do we care?

- If $P = NP$, some consequences:
  - Lots of important problems can be solved quickly!
  - Can build things better, faster, more efficiently
    - (Public key) cryptography does not exist
- If $P \neq NP$:
  - Some problems can't be solved quickly
  - Can stop trying to solve them

P vs NP

- The biggest open problem in computer science
- One of the biggest in math as well
- We are not even close to solving it
  - Know that $ACC_0 \subsetneq NEXP$ [Williams ’10]

NP-hard and NP-Complete Problems

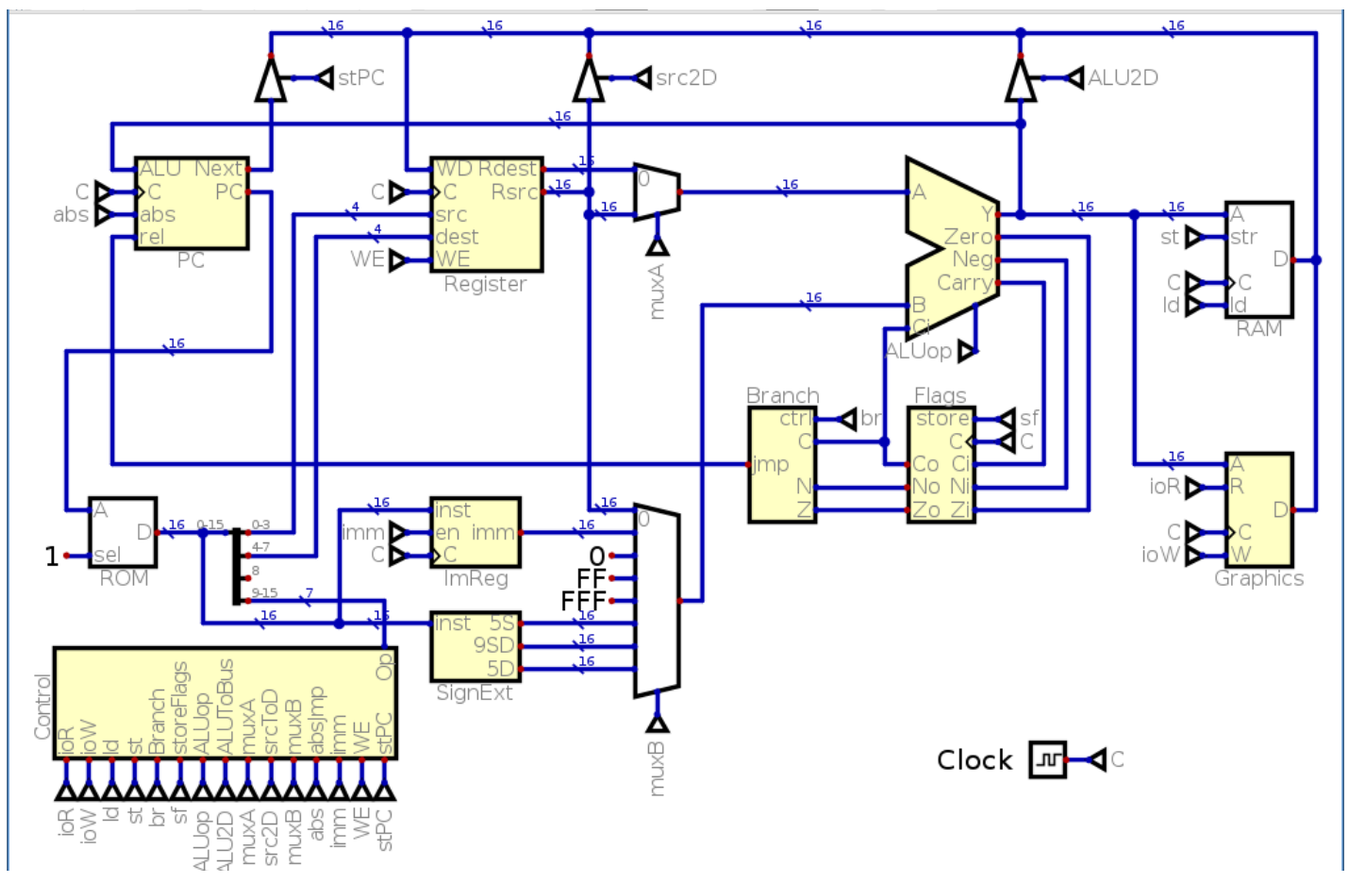
Cook-Levin Theorem

- If Satisfiability can be solved in polynomial time, then *any* problem in NP can be solved in polynomial time
- So: if Satisfiability can be solved in polynomial time, then P = NP
- How is this possible?

Cook-Levin Theorem

- Idea: any computer program can be represented by a circuit.

- Solve SAT in poly time -> can figure out the answer given by the circuit for NP problem in poly time



You'll see the proof in 361

NP-hard

- A problem X is **NP-hard** if:
  - If X can be solved in polynomial time, then any problem in NP can be solved in polynomial time
  - Therefore, if X can be solved in polynomial time, then P = NP

What does this mean?

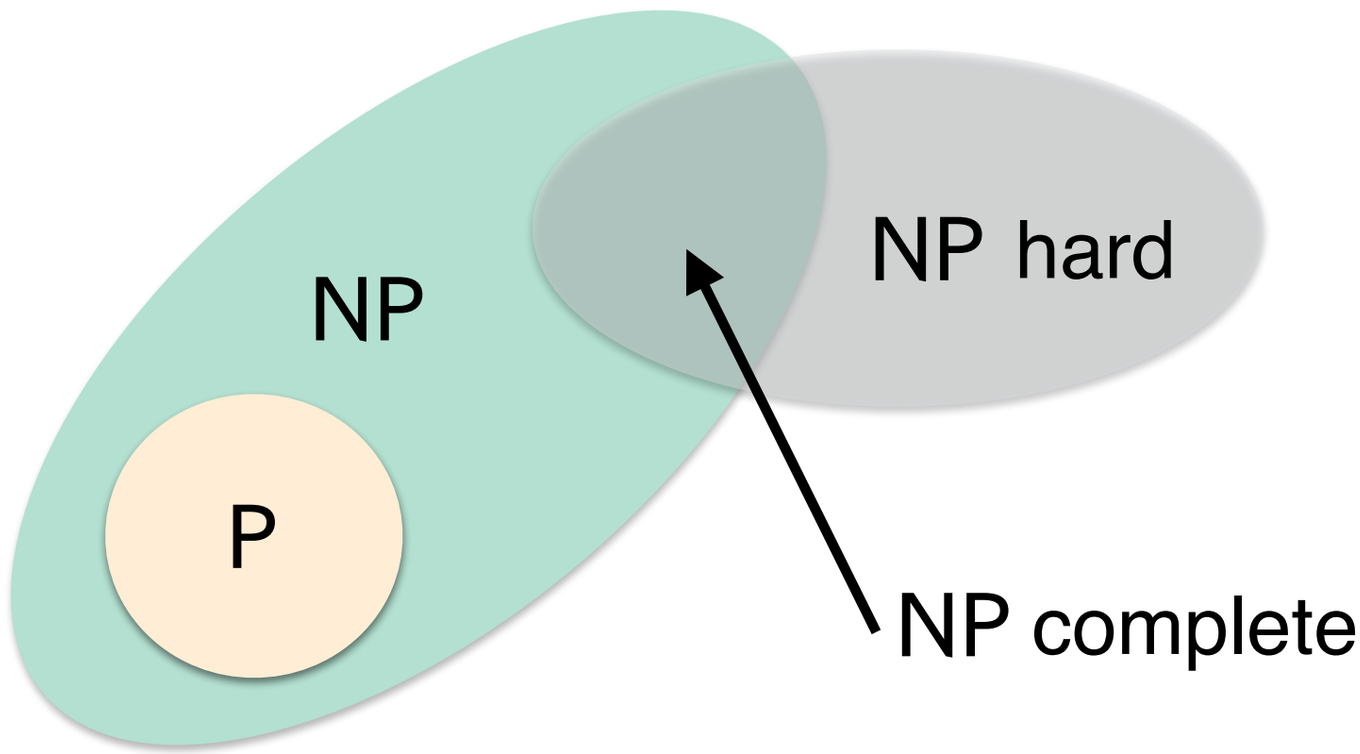
- We think that, probably, $P \neq NP$
- So if a problem is NP-hard, then you probably cannot obtain a polynomial-time algorithm for it

Classifying Problems as Hard

- We are frustratingly unable to prove a lot of problems are **impossible** to solve efficiently
- Instead, we say problem X is likely very hard to solve by saying, *if a polynomial-time algorithm was found for* X*, then something we all believe is impossible will happen*
- We say X is NP-hard $\Leftrightarrow$ if $X \in \mathsf{P}$, then $\mathsf{P} = \mathsf{NP}$
- (Erickson) Calling a problem NP hard is like saying, *"If I own a dog, then it can speak fluent English"*
  - You probably don't know whether or not I own a dog, but you are definitely sure I don't own *a talking dog*
  - Corollary: No one should believe that I own a dog
- If a problem is NP hard, no one should believe it can be solved in polynomial time

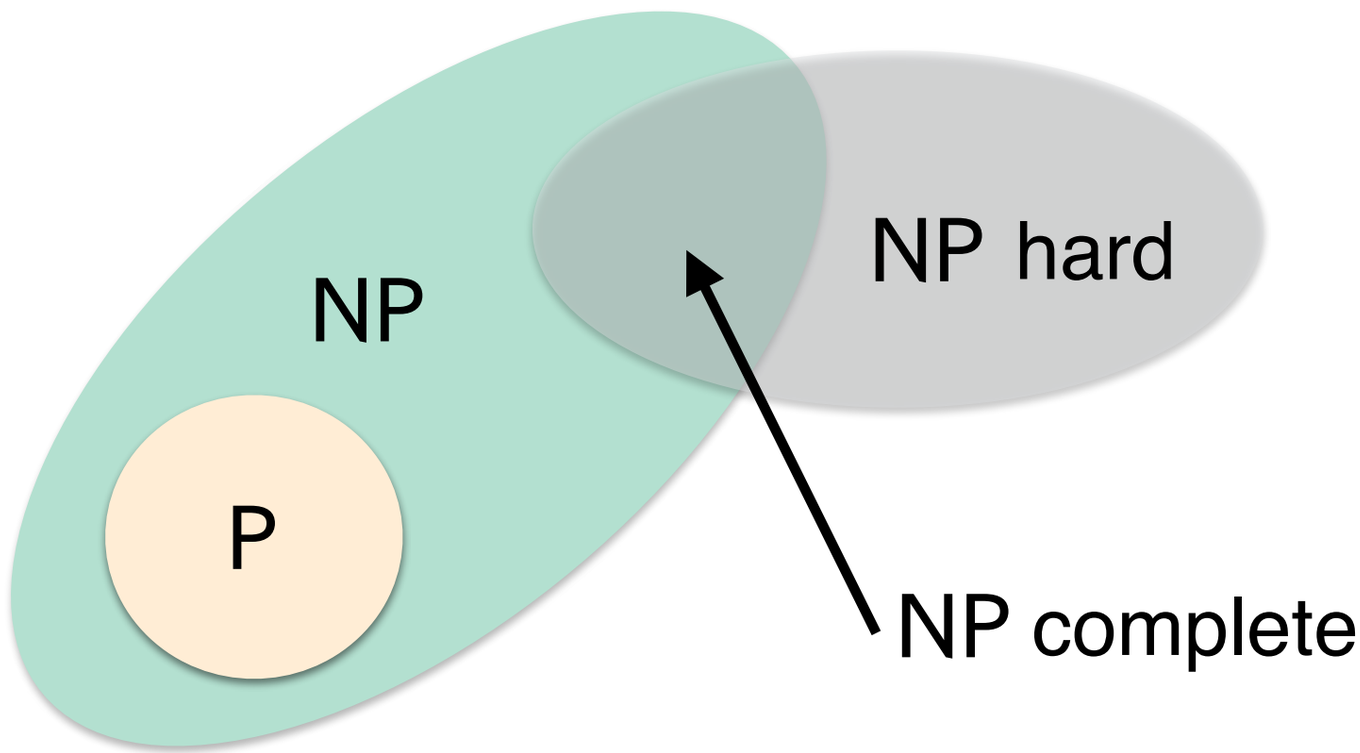
NP Completeness

- **Definition.** A problem X is NP complete if X is NP hard and $X \in$ **NP**
- SAT is **NP** complete
    - SAT $\in$ **NP**: given an assignment to input gates (certificate), can verify whether output is one or zero in poly-time
    - SAT is **NP** hard (Cook-Levin Theorem)



Summary

- X is **NP**-hard **NP**-hard $\Leftrightarrow$ if $X \in$ **P**, then **P** = **NP**
- A problem X is NP complete if X is NP hard and $X \in$ **NP**
- Alternate definition of NP hard:
  - X is NP hard if all languages in NP reduce it to in polynomial time
- Thus, NP-complete problems are the hardest problems in NP



Relative Hardness

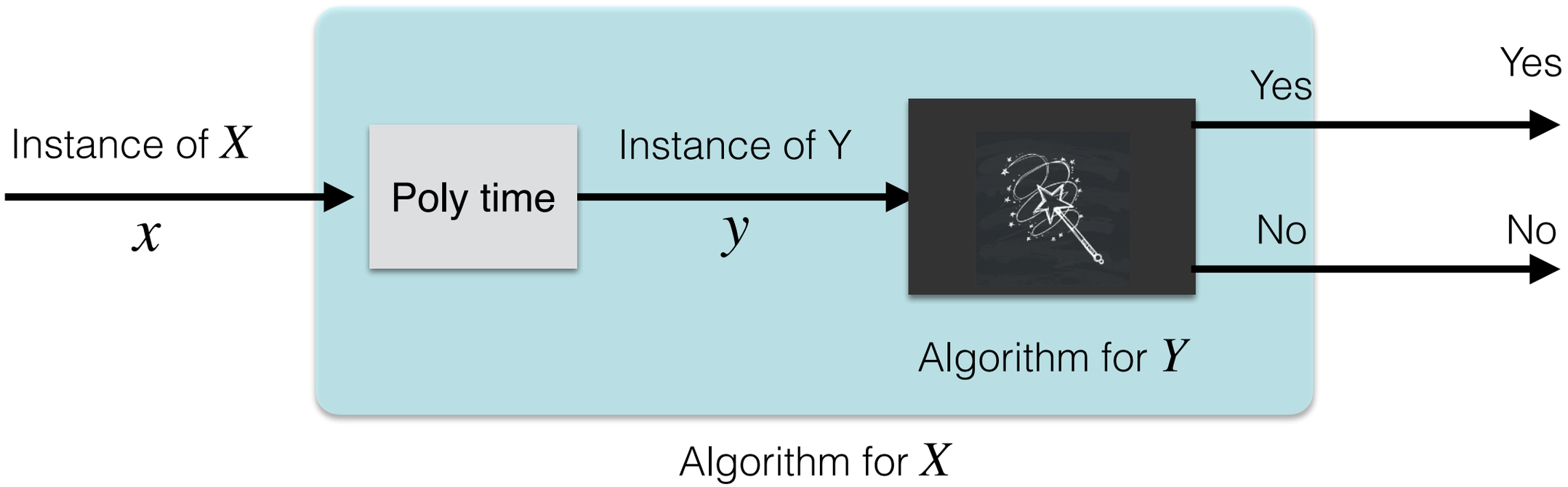
- Suppose we know problem X is NP hard, how can we use that to show problem Y is also hard to solve?
- How do we compare the relative hardness of problems
- Recurring idea in this class: **reductions!**
- Informally, we say a problem X reduces to a problem Y, if can use an algorithm for Y to solve X
    - Bipartite matching reduces to max flow
    - Edge-disjoint paths reduces to max flow

Intuitively, if problem X reduces to problem Y, then solving X is no harder than solving Y

[Karp] Reductions

Definition. Decision problem X polynomial-time (Karp) reduces to decision problem Y if given any instance x of X, we can construct an instance y of Y in polynomial time s.t $x \in X$ if and only if $y \in Y$.

Notation. $X \leq_p Y$



Reductions Quiz

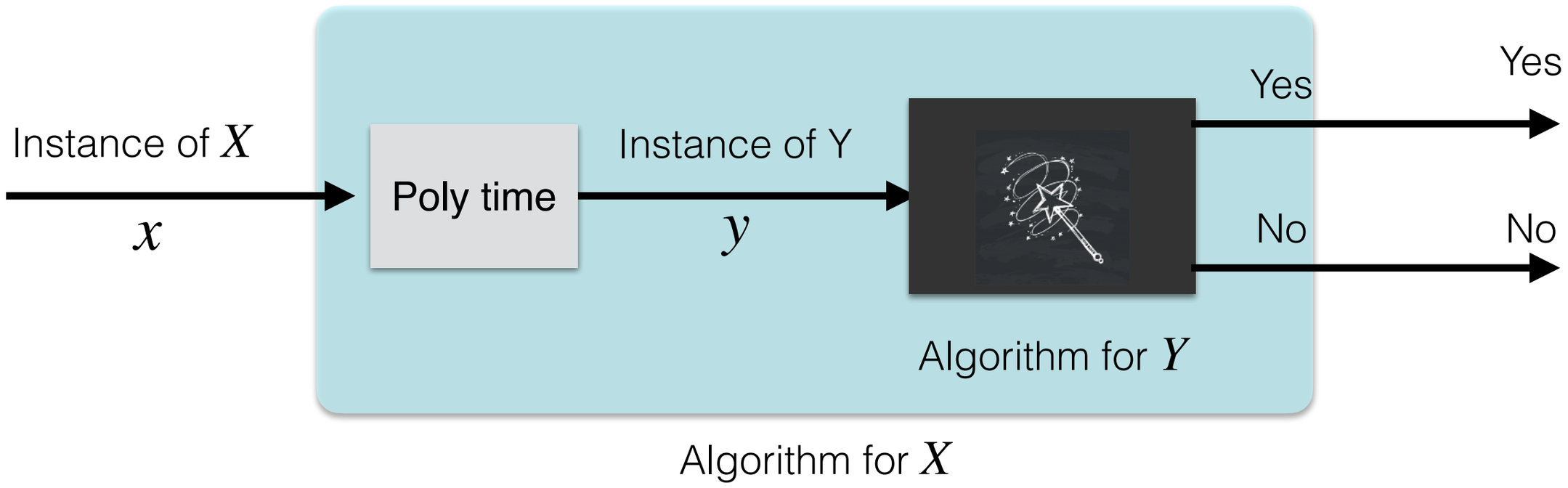
Say $X \leq_p Y$. Which of the following can we infer?

A. If X can be solved in polynomial time, then so can Y.

B. X can be solved in poly time iff Y can be solved in poly time.

C. If X cannot be solved in polynomial time, then neither can Y.

D. If Y cannot be solved in polynomial time, then neither can X.



Digging Deeper

- **Graph 2-Color** reduces to **Graph 3-color**
    - Just replace the third color with either of the two
- **Graph 2-Color** can be solved in polynomial time
    - How?
        - We can decide if a graph is bipartite in $O(n + m)$ time using traversal
- **Graph 3-color** (we'll show) is NP hard

Intuitively, if problem X reduces to problem Y, then solving X is no harder than solving Y

Use of Reductions: $X \leq_p Y$

Design algorithms:

- If Y can be solved in polynomial time, we know X can also be solved in polynomial time

Establish intractability:

- If we know that X is known to be impossible/hard to solve in polynomial-time, then we can conclude the same about problem Y

Establish Equivalence:

- If $X \leq_p Y$ and $Y \leq_p X$ then X can be solved in poly-time iff Y can be solved in poly time and we use the notation $X \equiv_p Y$

NP hard: Definition

- **New definition of NP hard using reductions.**
  - A problem Y is NP hard, if for any problem $X \in \mathsf{NP}$, $X \leq_p Y$
- Recall we said Y is NP hard $\iff Y \in \mathsf{P}$, then $\mathsf{P} = \mathsf{NP}$.
- Lets show that both definitions are equivalent
  - $(\Rightarrow)$ every problem in NP reduces to Y, and if $Y \in \mathsf{P}$, then $\mathsf{P} = \mathsf{NP}$
  - $(\Leftarrow)$ if $Y \in \mathsf{P}$, then $\mathsf{P} = \mathsf{NP}$: every problem in $\mathsf{NP}(= \mathsf{P})$ reduces to Y

Proving NP Hardness

- To prove problem Y is **NP**-hard
  - Difficult to prove every problem in **NP** reduces to Y
  - Instead, we use a known-NP-hard problem Z
  - We know every problem X in **NP**, $X \leq_p Z$
  - Notice that $\leq_p$ is transitive
  - Thus, enough to prove $Z \leq_p Y$

To prove that a problem Y is NP hard, reduce a known NP hard problem Z to Y

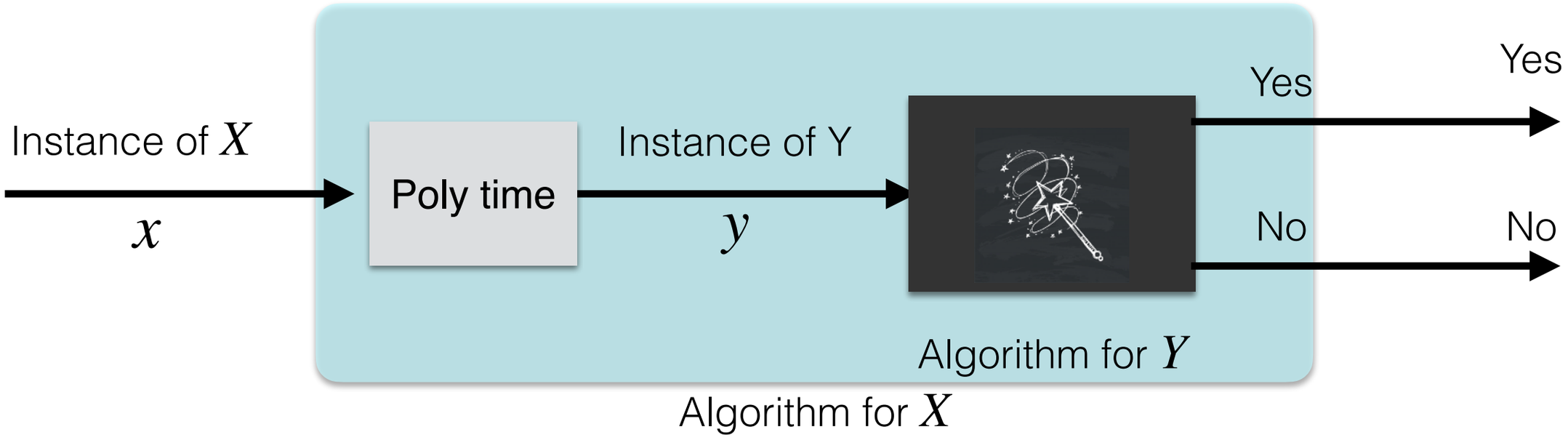
Known NP Hard Problems?

- For now: SAT (Cook-Levin Theorem)
- We will prove a whole repertoire of NP hard and NP complete problems by using reductions
- Before reducing SAT to other problems to prove them NP hard, let us practice some easier reductions first

To prove that a problem Y is NP hard, reduce a known NP hard problem Z to Y

Reductions: General Pattern

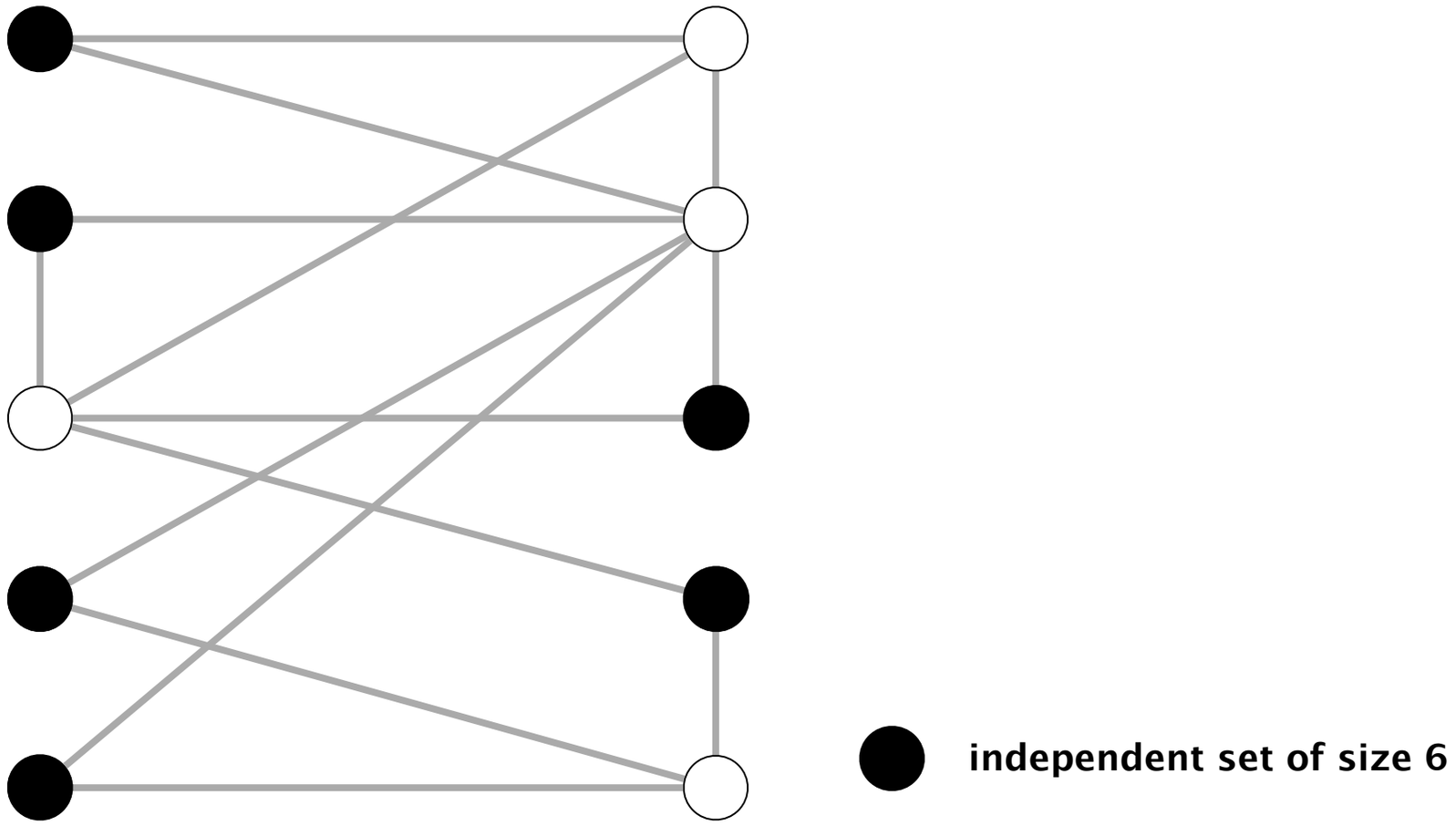
- Describe a polynomial-time algorithm to transform an arbitrary instance x of Problem X into a special instance y of Problem Y
- Prove that if x is a "yes" instance of X, then y is a "yes" instance of Y
- Prove that if y is a "yes" instance of Y, then x is a "yes" instance of X
- Notice that correctness of reductions are not symmetric:
  - the "if" proof needs to handle arbitrary instances of X
  - the "only if" needs to handle the special instance of Y



$$\text{VERTEX-COVER} \equiv_p \text{IND-SET}$$

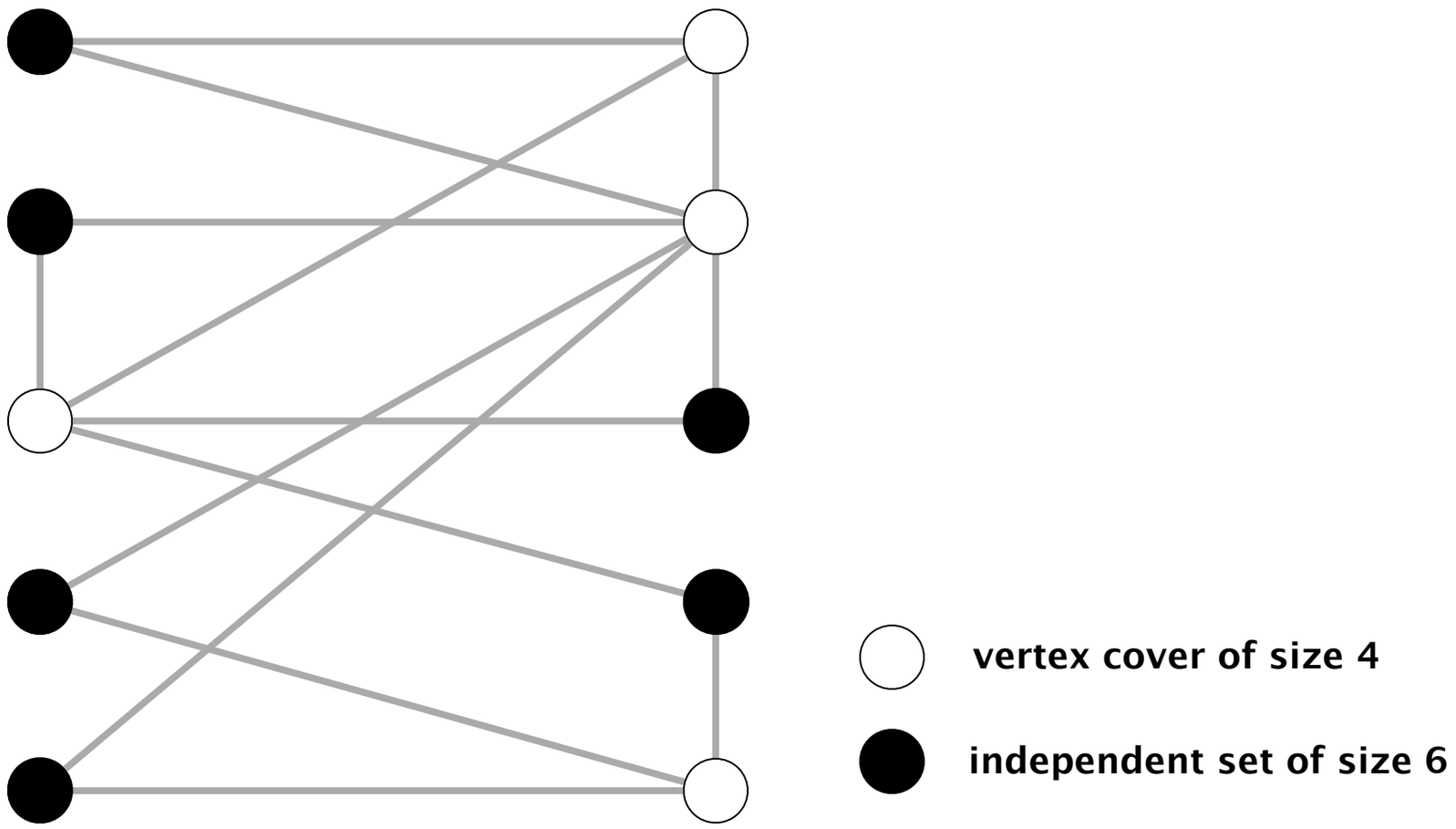
IND-SET

- Given a graph $G = (V, E)$, an independent set is a subset of vertices $S \subseteq V$ such that no two of them are adjacent, that is, for any $x, y \in S$, $(x, y) \notin E$
- **IND-SET Problem.** Given a graph $G = (V, E)$ and an integer k, does G have an independent set of size at least k?



Vertex-Cover

- Given a graph $G = (V, E)$, a vertex cover is a subset of vertices $T \subseteq V$ such that for every edge $e = (u, v) \in E$, either $u \in T$ or $v \in T$.
- **VERTEX-COVER Problem.** Given a graph $G = (V, E)$ and an integer k, does G have a vertex cover of size at most k?



Our First Reduction

- **VERTEX-COVER** $\leq_p$ **IND-SET**
    - Suppose we know how to solve independent set, can we use it to solve vertex cover?

- **Claim.** S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

- **Proof.** ($\Rightarrow$) Consider an edge $e = (u, v) \in E$
    - S is independent: u, v both cannot be in S
    - At least one of $u, v \in V - S$
    - $V - S$ covers e ∎

Our First Reduction

- **VERTEX-COVER** $\leq_p$ **IND-SET**
    - Suppose we know how to solve independent set, can we use it to solve vertex cover?

- **Claim.** S is an independent set of size k iff $V - S$ is a vertex cover of size $n - k$.

- **Proof.** ($\Leftarrow$) Consider an edge $e = (u, v) \in E$
    - $V - S$ is a vertex cover: at least one of u, v or both must be in $V - S$
    - Both u, v cannot be in S
    - Thus, S is an independent set. ∎

Vertex Cover $\equiv_p$ IND Set

- **VERTEX-COVER** $\leq_p$ **IND-SET**
    - Suppose we know how to solve independent set, can we use it to solve vertex cover?
- **Reduction.** Let $G' = G,\ k' = n - k$.
    - ($\Rightarrow$) If G has a vertex cover of size at most k then G' has an independent set of size at least k'
    - ($\Leftarrow$) If G' has an independent set of size at least k' then G has a vertex cover of size at most k
- **IND-SET** $\leq_p$ **VERTEX-COVER**
    - Same reduction works: $G' = G$, $k' = n - k$
- **VERTEX-COVER** $\equiv_p$ **IND-SET**

Acknowledgments

- Some of the material in these slides are taken from
  - Kleinberg Tardos Slides by Kevin Wayne (https://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/04GreedyAlgorithmsI.pdf)
  - Jeff Erickson's Algorithms Book (http://jeffe.cs.illinois.edu/teaching/algorithms/book/Algorithms-JeffE.pdf)