Greedy: Scheduling and Minimum Spanning Trees

Admin

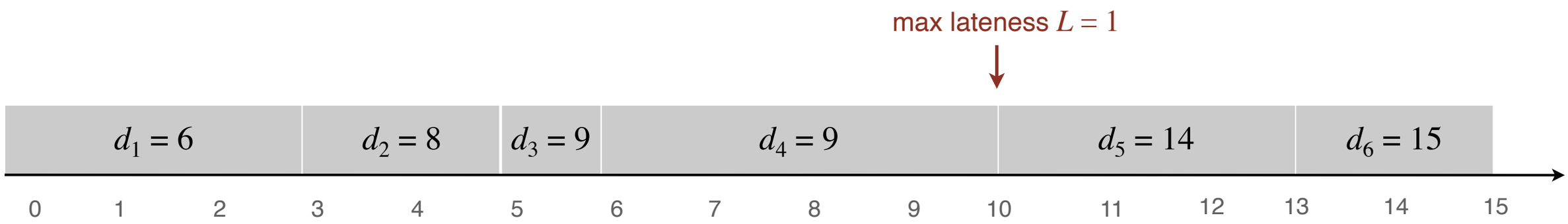
- Readings: we're a tad behind the schedule on the website—I'll revisit the schedule after mountain day

- Questions or comments?

Recall: Scheduling with Deadlines

Given interval length t_i and deadline d_i for $i \in \{1, \ldots, n\}$ jobs, schedule all tasks, that is, assign start and finish times $(t_i, d_i) \rightarrow (s_i, f_i)$, where $f_i = s_i + t_i$, so as to minimize the maximum lateness.

- Lateness of process i : $L_i = \max\{0, f_i - d_i\}$



Greedy: Earliest-Deadline First

EARLIEST-DEADLINE-FIRST $(n, t_1, t_2, \ldots, t_n, d_1, d_2, \ldots, d_n)$

SORT jobs by due times and renumber so that $d_1 \leq d_2 \leq \ldots \leq d_n$.

$t \leftarrow 0$.

FOR $j = 1$ TO n

  Assign job j to interval $[t, t + t_j]$.

  $s_j \leftarrow t$; $f_j \leftarrow t + t_j$.

  $t \leftarrow t + t_j$.

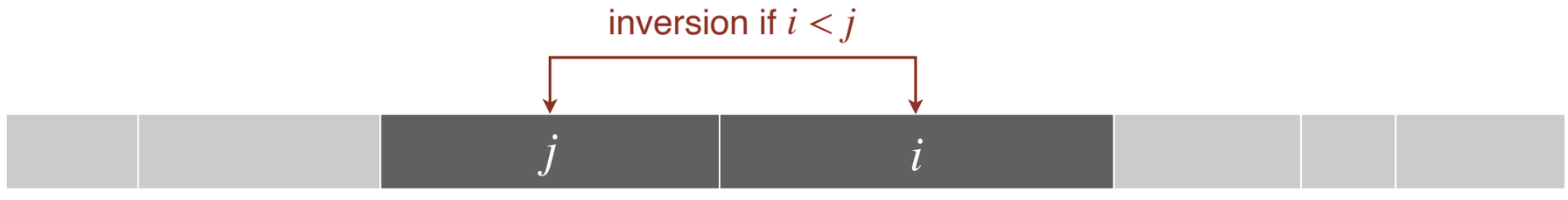
RETURN intervals $[s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n]$.

Minimizing Lateness: Greedy

Observations about our greedy algorithm

- It produces a schedule with no idle time
- It produces a schedule with no inversions
  - i, j is an inversion if job j is scheduled before i but i's deadline is earlier ($d_i < d_j$)

a schedule with an inversion



recall: we assume the jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$

Structure of the Solution

- **Notice:** All schedules with no inversions and no idle time have the same maximum lateness
  - Distinct deadlines, unique schedule
  - Non-distinct deadlines: Consider two jobs with deadline d; the maximum lateness does not depend on the order in which they are scheduled
- Say the two jobs have duration t_i, t_j and same deadline d
- If i is scheduled first at time s, the max lateness is: $\max\{0,(s + t_i + t_j) - d\}$
- If j is scheduled first at time s, the max lateness is the same: $\max\{0,(s + t_i + t_j) - d\}$

Where We Are Going

- **Notice:** All schedules with no inversions and no idle time have the same maximum lateness
  - Distinct deadlines, unique schedule
  - Non-distinct deadlines: Consider two jobs with deadline d; the maximum lateness does not depend on the order in which they are scheduled
- **Goal.** show there exists an optimal schedule with no inversions and no idle time
- Then, we have shown that the optimal schedule has the same maximum lateness as greedy!
- We will show this via **an exchange argument**
- Second proof technique to prove greedy is optimal

Minimizing Lateness: Greedy

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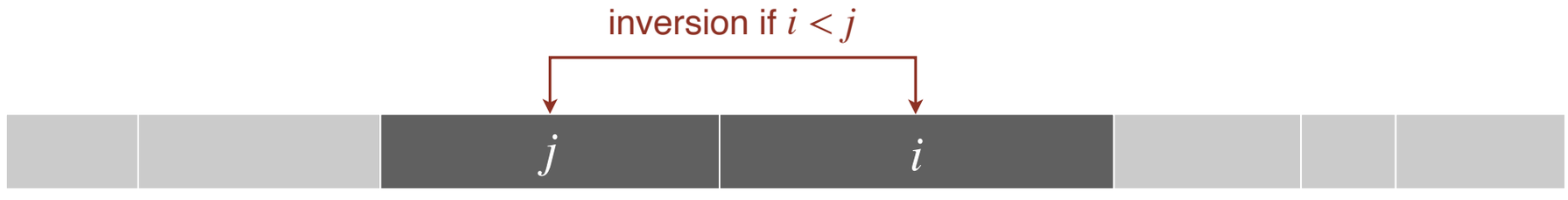
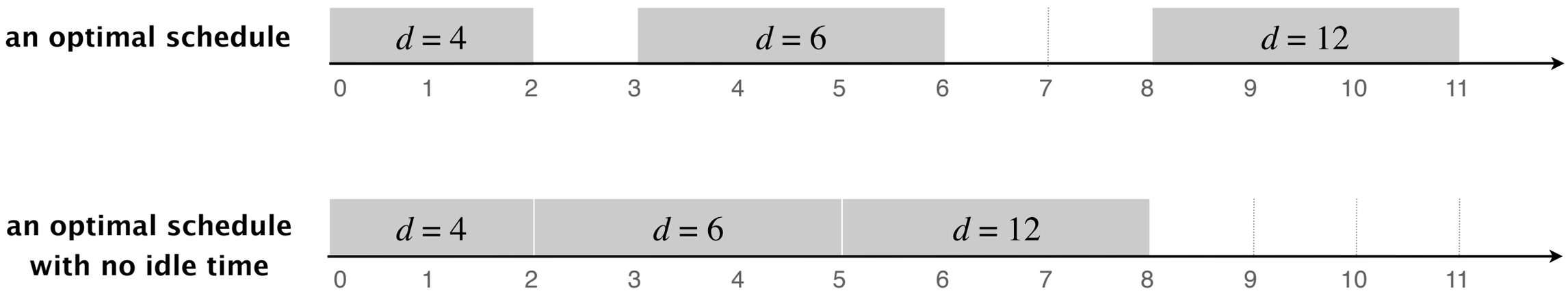


recall: we assume the jobs are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$

Structure of Optimal

Observation about optimal.

- There exists an optimal schedule with no idle time.
- (Can always schedule jobs earlier to prevent idleness!)



Structure of Optimal: Inversions

Observation. If an idle-free schedule has an inversion, then it has an adjacent inversion.

Recall. i, j is an inversion if job j is scheduled before i but i's deadline is earlier ($d_i < d_j$)

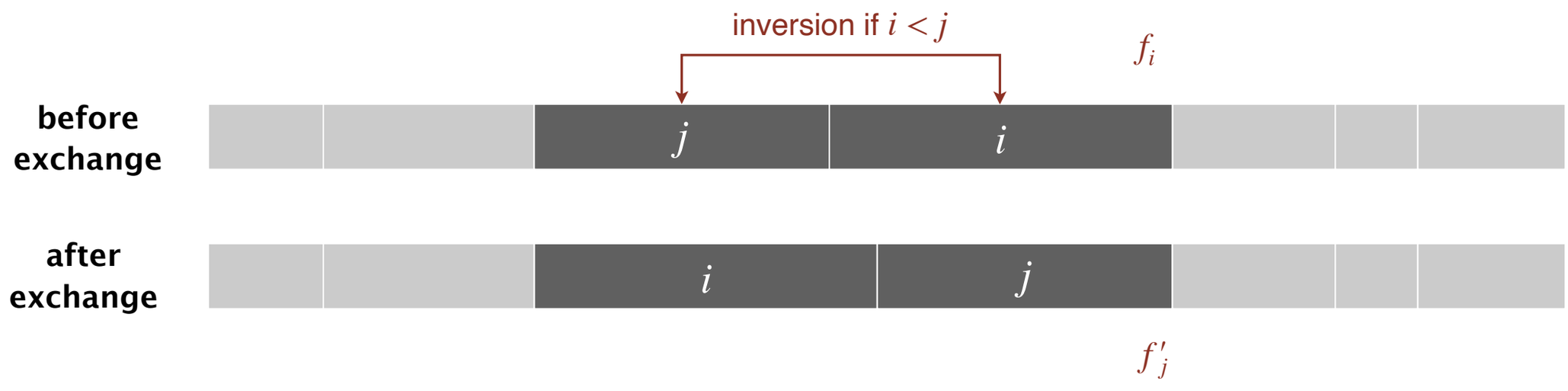
Proof. [Contradiction]

- Let i, j be any two non-adjacent inversions without another inversion between them
- Let k be element immediately to the right of j.
- Case 1. $[d_j > d_k]$ Then j, k is an adjacent and closer inversion ($\Rightarrow\Leftarrow$)
- Case 2. $[d_j < d_k]$ Since $d_i < d_j$, this means that i, k is a closer inversion ($\Rightarrow\Leftarrow$) ∎

Structure of Optimal: Inversions

Claim. Given a schedule with k inversions, we can modify it to schedule with $k - 1$ inversions (without increasing the maximum lateness).

Proof. (Key Idea) Exchanging two adjacent, inverted jobs i and j reduces the number of inversions by 1 and does not increase the maximum lateness.

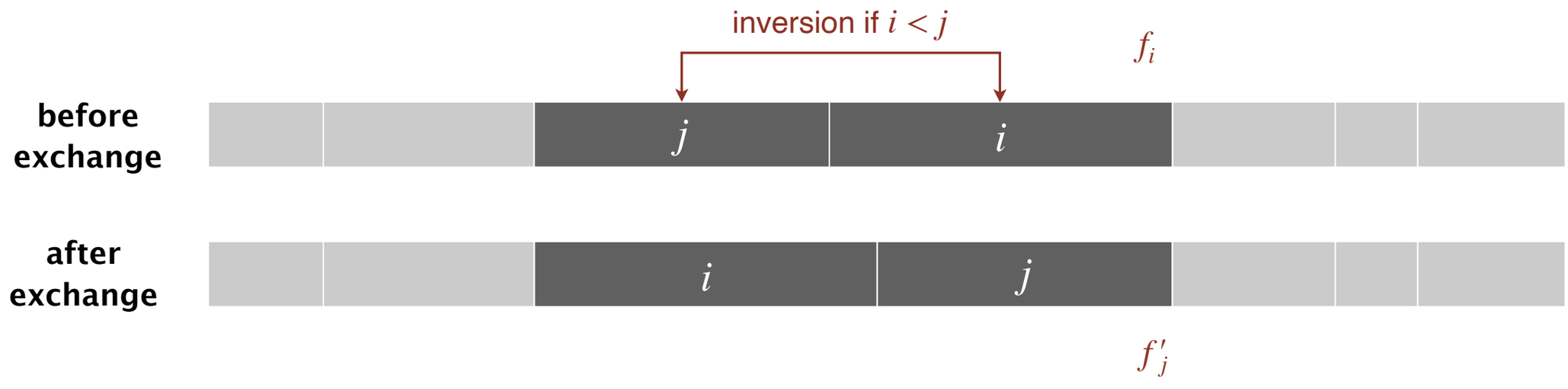


Structure of Optimal: Inversions

Claim. Given a schedule with k inversions, we can modify it to schedule with $k - 1$ inversions (without increasing the maximum lateness).

Proof. Let i, j be inverted jobs with $i < j$. Let ℓ be the lateness before swapping them and ℓ' after the swap.

- $\ell_k = \ell'_k \qquad \forall k \neq i, j$ (swap doesn't affect other jobs)
- $\ell'_i \leq \ell_i$ (lateness of i improves after swap)
- $\ell'_j = f'_j - d_j = f_i - d_j \leq f_i - d_i \leq \ell_i$ ∎



Optimality of Greedy

Summarizing the proof.

- All schedules with no inversions and no idle time have the same maximum lateness
- Greedy schedule has no inversions and no idle time
- Consider an optimal schedule $\mathcal{O}$, without loss of generality, we can assume that
  - $\mathcal{O}$ has no idle time
  - $\mathcal{O}$ has no inversions, why?
  - **[Iterate and exchange].** If there is an inversion, must be adjacent, exchanging them decreases # of inversions by 1 without increasing max lateness (repeat until no inversions)
- Greedy and $\mathcal{O}$ have same max lateness. ∎

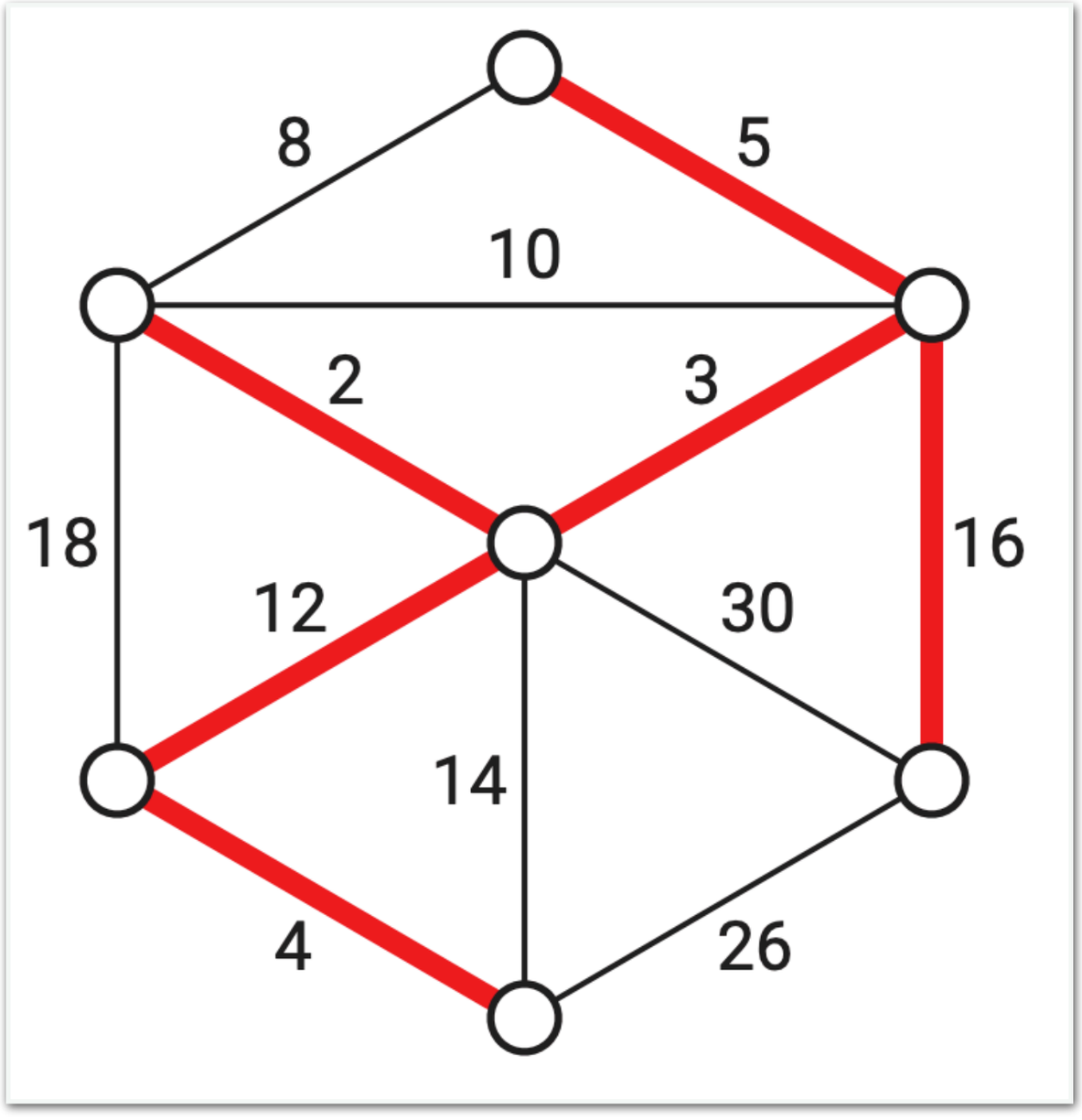
Exchange Argument

General Pattern.

- You start with an arbitrary optimal solution
- Prove that WLOG it must have certain nice properties
- Assume there is an optimal solution that is different from the greedy solution
- Find the “first” difference between the two solutions
- Argue that we can exchange the optimal choice for the greedy choice without making the solution worse (although the exchange may make it better)
- Show that you can iteratively perform the exchange step until you get the greedy solution

Greedy Graph Algorithms: Minimum Spanning Trees

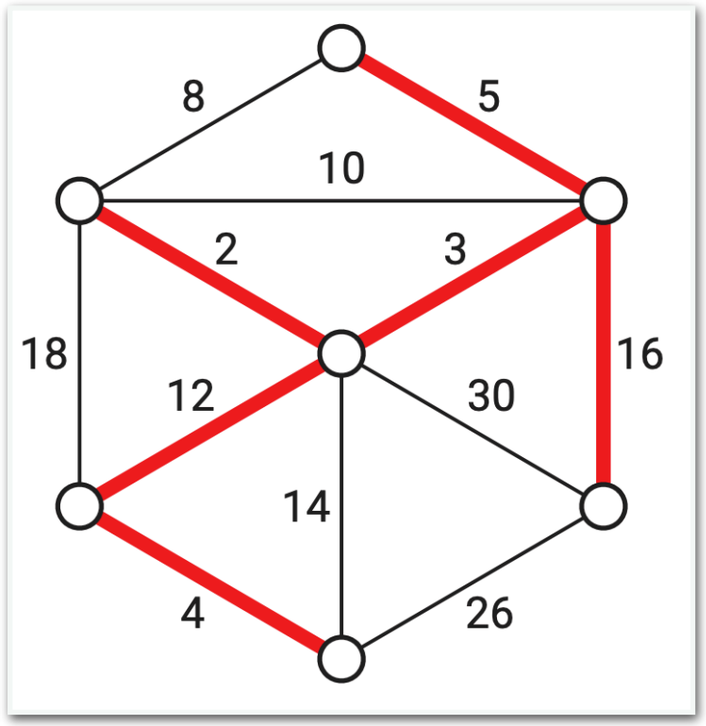
Minimum Cost Spanning Trees



Minimum Spanning Trees

Problem. Given a connected, undirected graph $G = (V, E)$ with edge costs w_e, output **a minimum spanning tree**, i.e., set of edges $T \subseteq E$ such that

- (a spanning tree of G): T connects all vertices
- (has minimum weight): for any other spanning tree T' of G, we have $\sum_{e \in T} w_e \leq \sum_{e \in T'} w_e$

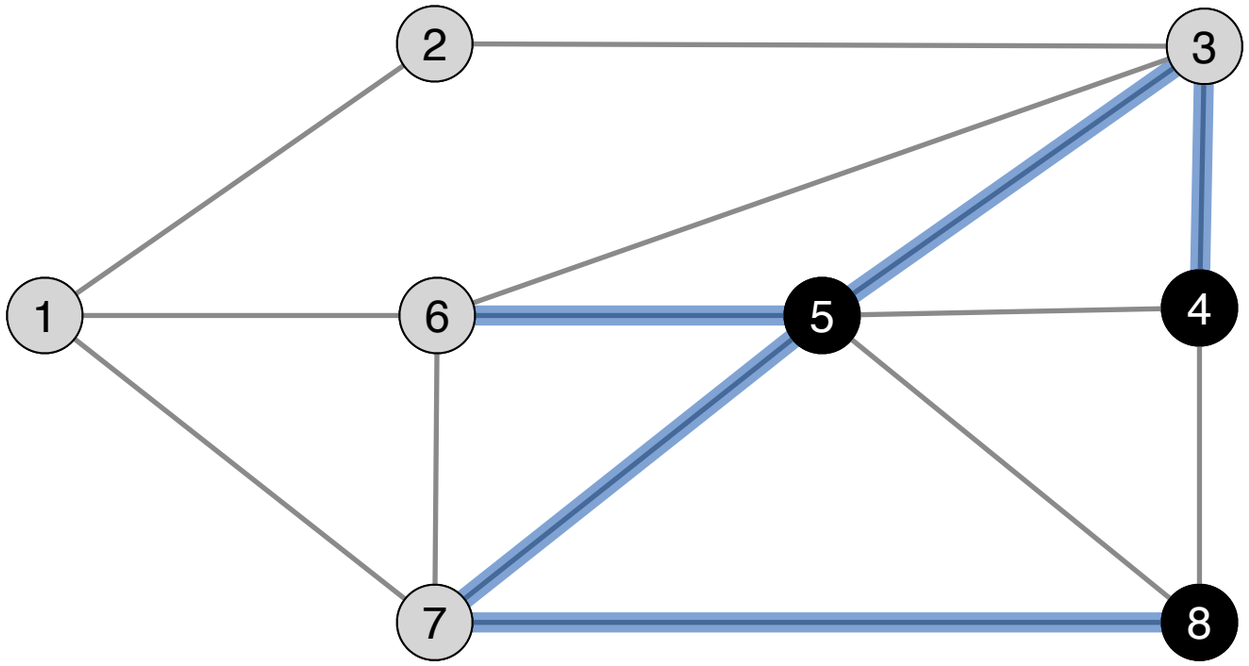


Minimum Spanning Trees

- Many applications!
  - Classic application:
    - Underground cable (Power, Telecom, etc)
  - Efficient broadcasting on a computer network (Note: different from shortest paths)
  - Approximate solutions to harder problems, such at TSP
  - Real-time face verification

Spanning Trees and Cuts

A **cut** is a partition of the vertices into two **nonempty** subsets S and $V - S$. A **cut edge** of a cut S is an edge with one end point in S and another in $V - S$.



cut S = { 4, 5, 8 }

Cut edges = { (3, 4), (3, 5), (5, 6), (5, 7), (8, 7) }

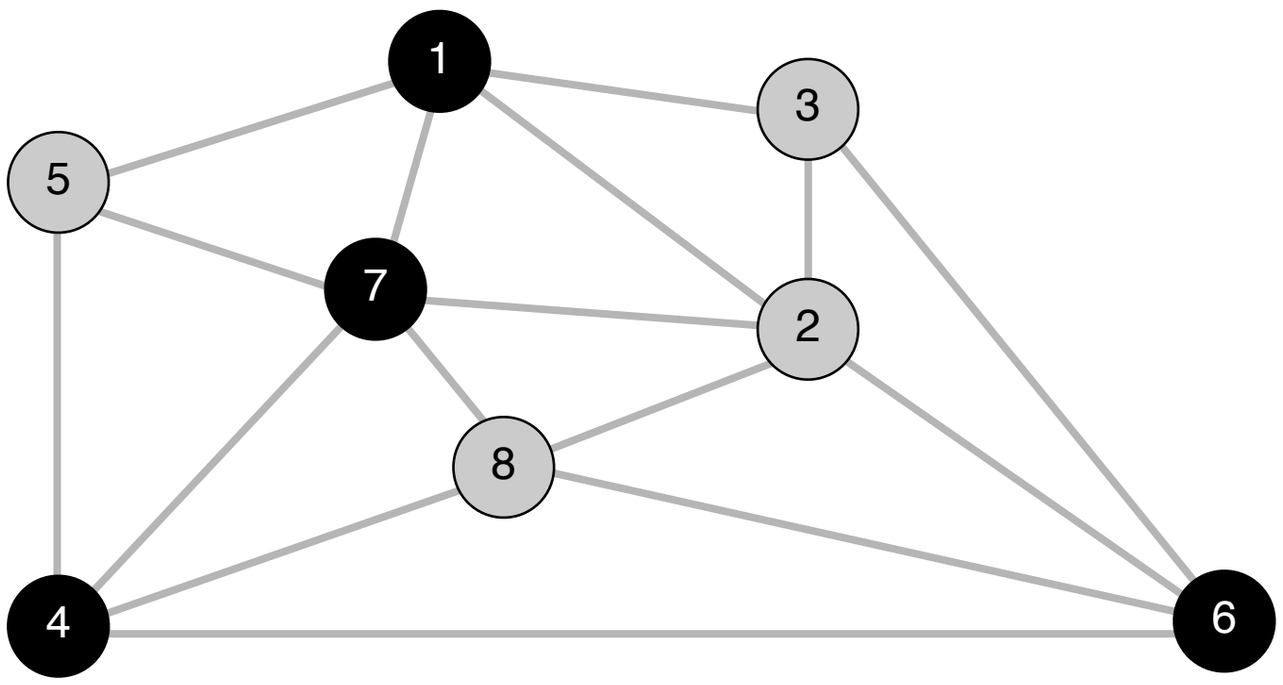
Spanning Trees and Cuts

Question. Consider the cut $S = \{1,4,6,7\}$. Which of the following edges are cut edges with respect to this cut?

A. (1, 7)

B. (5, 7)

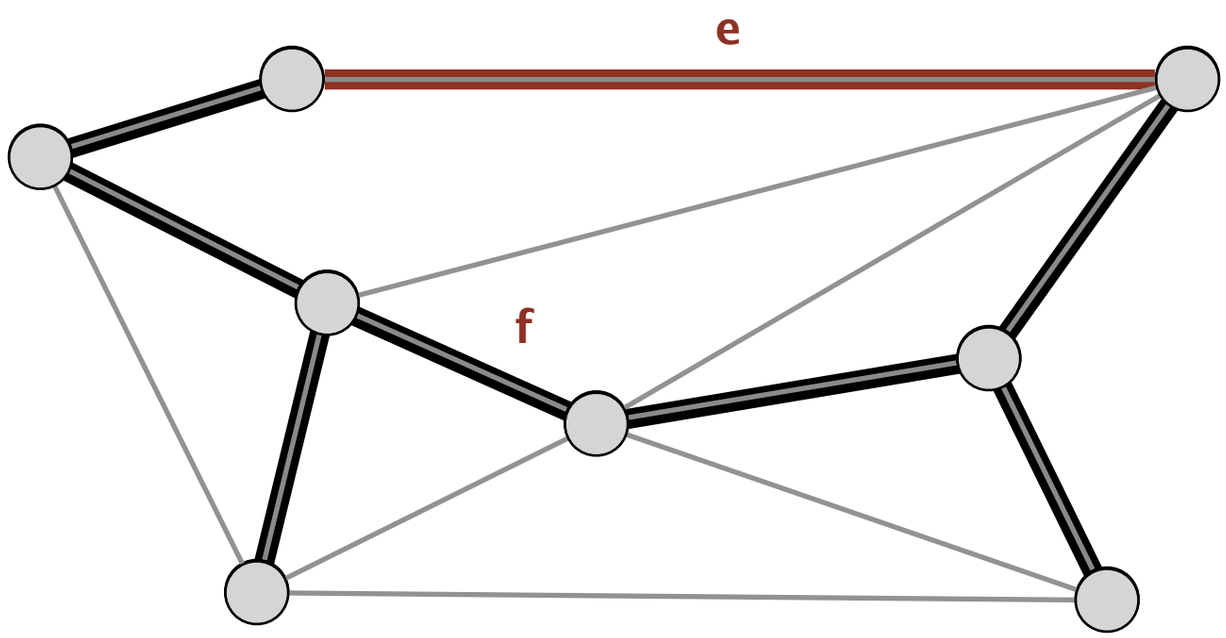
C. (2, 3)



Fundamental Cycle

Let T be a spanning tree of G.

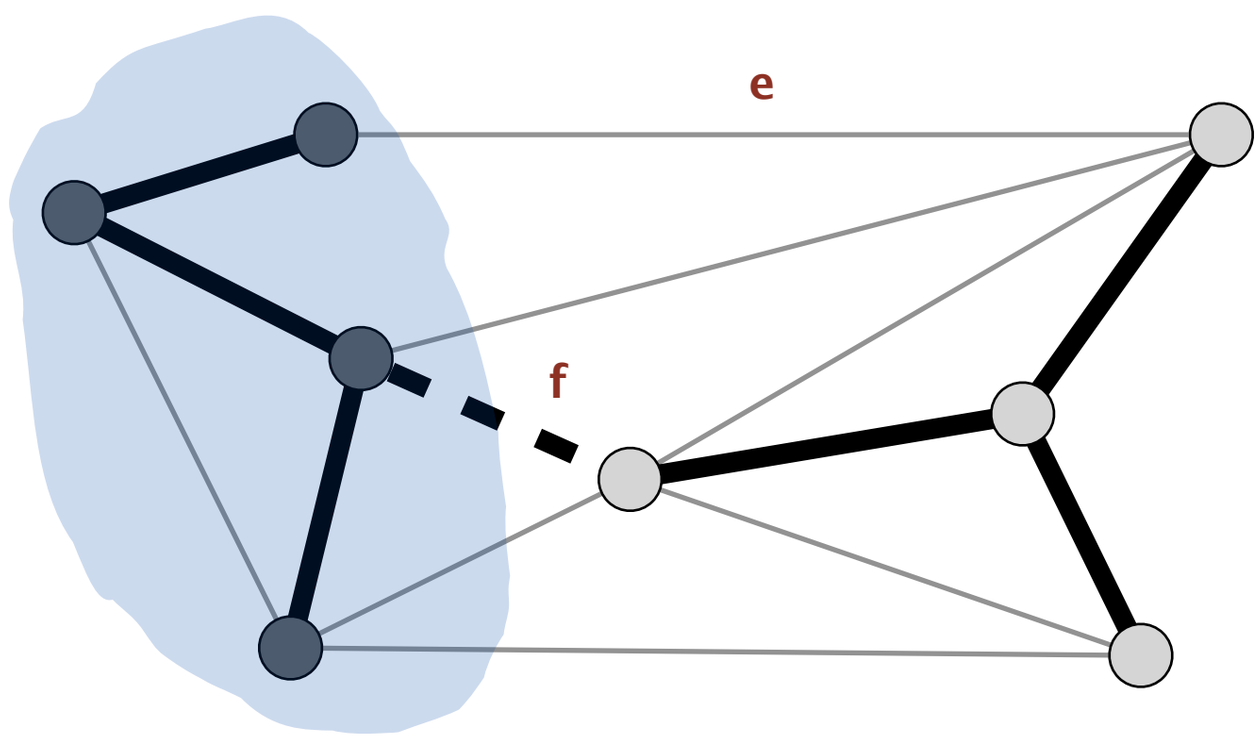
- For any edge $e \notin T$, $T \cup \{e\}$ creates a unique cycle C
- For any edge $f \in C$: $T \cup \{e\} - \{f\}$ is a spanning tree



Fundamental Cut

Let T be a spanning tree of G.

- For any edge $f \in T$, $T - \{f\}$ breaks the graph into two connected components, let D be the set of cut edges with end points in each component
- For any edge $e \in D : T - \{f\} \cup \{e\}$ is a spanning tree



Spanning Trees and Cuts

Lemma (Cut Property). For any cut $S \subset V$, if $e = (u, v)$ is the strictly smallest edge connecting any vertex in S to a vertex in $V - S$, then every minimum spanning tree must include e.

Proof. (By contradiction via an exchange argument)

Suppose T is a spanning tree that does not contain $e = (u, v)$.

Main Idea: We will construct another spanning tree $T' = T \cup e - e'$ with weight less than T ($\Rightarrow\Leftarrow$)

How to find such an edge e'?

Exchange argument

Spanning Trees and Cuts

Proof (Cut Property).

Suppose T is a spanning tree that does not contain $e = (u, v)$.

- Adding e to T results in a unique cycle C (why?)
- C must "enter" and "leave" cut S, that is, $\exists e' = (u', v') \in C$ s.t. $u' \in S, v' \in V - S$
- $w(e') > w(e)$ (why?)
- $T' = T \cup e - e'$ is a spanning tree (why?)
- $w(T') < w(T)$

$(\Rightarrow\Leftarrow)$ ∎

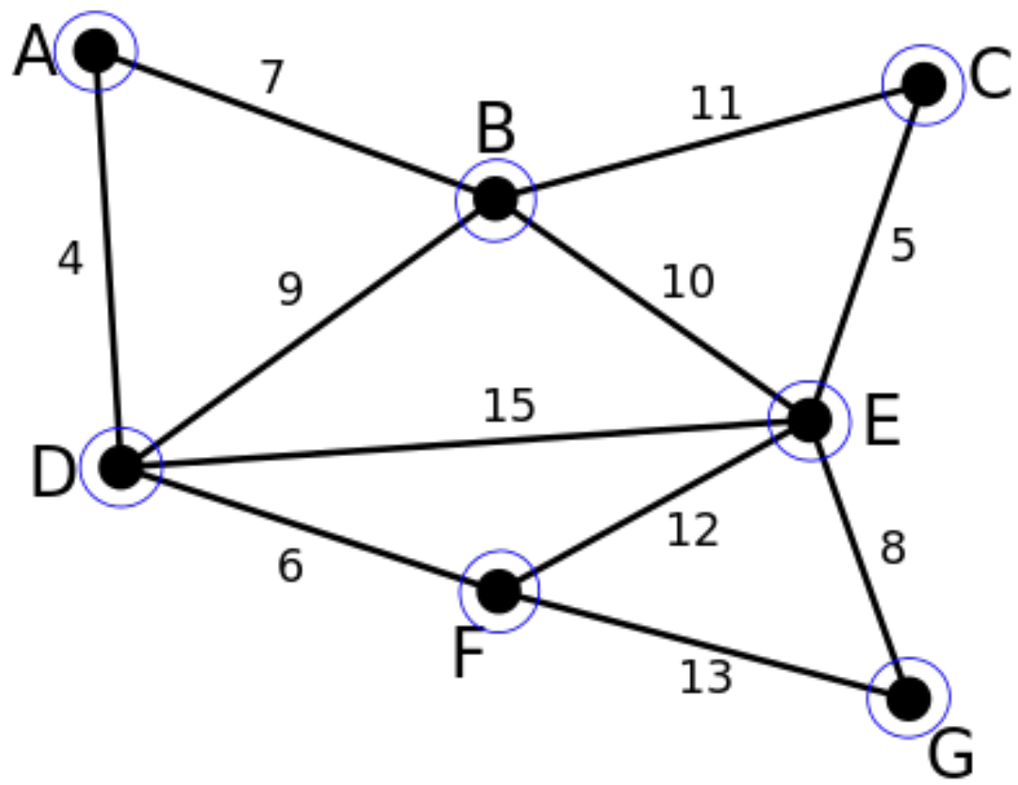
Spanning Trees and Cuts

- What if there's no unique minimum?

Lemma (Cut Property). For any cut $S \subset V$, if $e_1, e_2, \ldots, e_k$ are the smallest edges connecting any vertex in S to a vertex in $V - S$, then every minimum spanning tree must include some e_i.

Jarník's ("Prims Algorithm")

- Initialize $S = \{u\}$ for any vertex $u \in V$ and $T = \emptyset$
- While $|T| \leq n - 1$:
    - Find the min-cost edge $e = (u, v)$ with one end $u \in S$ and $v \in V - S$
    - $T \leftarrow T \cup \{e\}$
    - $S \leftarrow S \cup \{v\}$



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    - $T \leftarrow T \cup \{e\}$
    - $S \leftarrow S \cup \{v\}$
- **Implementation crux.** Find and add min-cost edge for the cut $(S, V - S)$ and add it to the tree in each iteration, update cut edges
- How can we prove that this finds the MST?
    - Cut property! (On board.)

Jarník's ("Prims Algorithm")

- Initialize $S = \{u\}$ for any vertex $u \in V$ and $T = \emptyset$
- While $|T| \leq n - 1$:
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    - $S \leftarrow S \cup \{v\}$
- **Implementation crux.** Find and add min-cost edge for the cut $(S, V - S)$ and add it to the tree in each iteration, update cut edges
- Running time?
    - Naive implementation may take $O(nm)$
    - Need to maintain set of edges adjacent to nodes in T and extract min-cost cut edge from it each time
    - Which data structure from CS 136 can we use?

CS136 Review: Priority Queue

Managing such a set typically involves the following operations on S

- **Insert.** Insert a new element into S
- **Delete.** Delete an element from S
- **ExtractMin.** Retrieve highest priority element in S

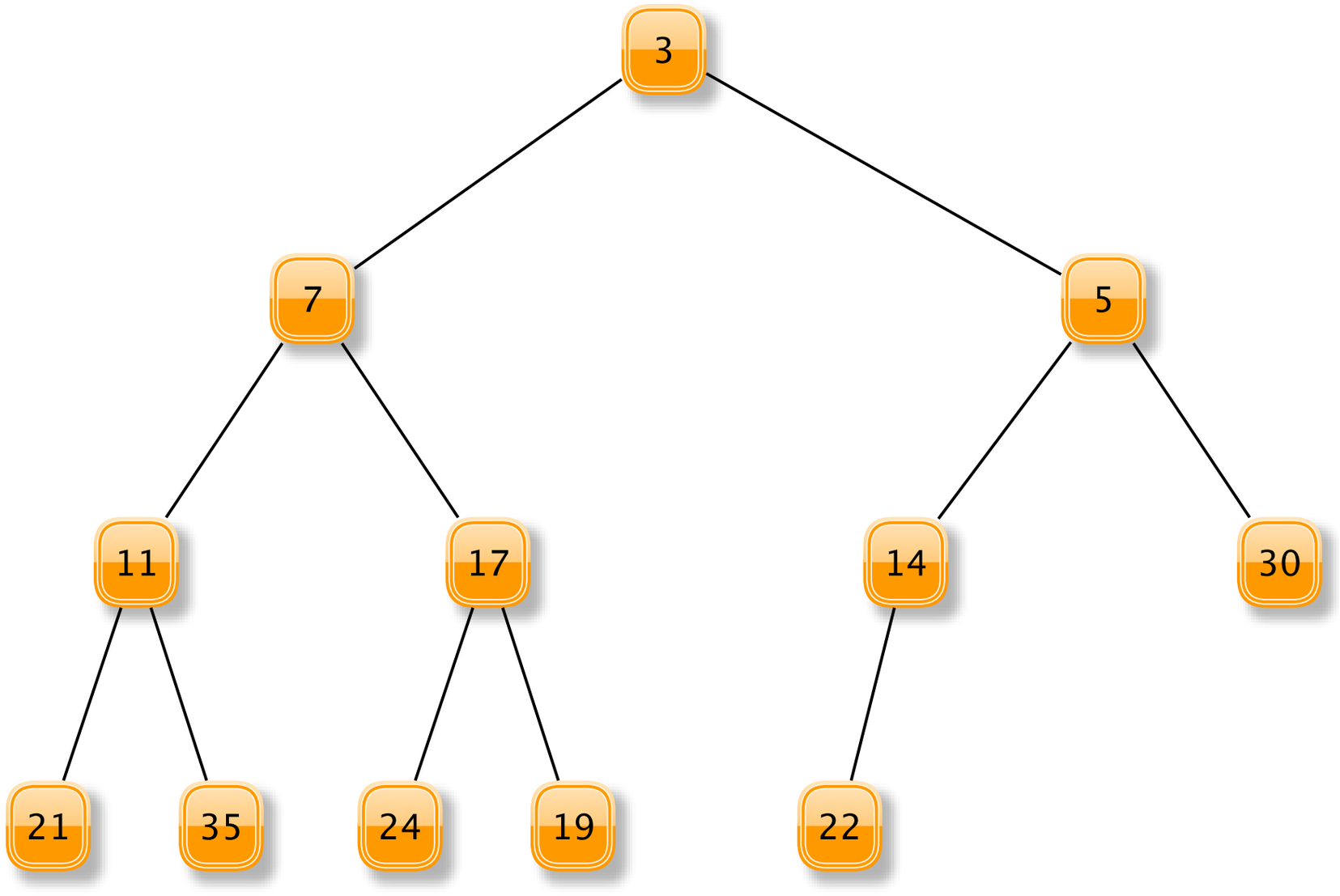
Priorities are encoded as a ‘key’ value

Typically: higher priority <—> lower key value

Heap as Priority Queue. Combines tree structure with array access

- Insert and delete: $O(\log n)$ time (‘tree’ traversal & moves)
- **Extract min.** Delete item with minimum key value: $O(\log n)$

Heap Example



H

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
X	3	7	5	11	17	14	30	21	35	24	19	22	-	-	-

“Prims” Implementation

- Use **Binary heaps**
  - Create a priority queue initially holding all edges incident to u.
  - At each step, dequeue edges from the priority queue until we find an edge (x, y) where $x \in S$ and$y \notin S$.
  - Add (x, y) to T.
  - Add to the queue all edges incident to y whose endpoints aren't in S.
  - Each edge is enqueued and dequeued at most once
  - Total runtime: $O(m \log m)$
    - In any graph, $m = O(n^2)$
    - So $O(m \log m) = O(m \log n)$

“Prims” Implementation

- Implementation using **Binary heaps**
  - Total runtime: $O(m \log n)$

- If a **Fibonacci heap** is used instead of binary heap:
  - Runs in $O(m + n \log n)$ **“amortized time”**
  - Support amortized $O(1)$-time inserts, $O(\log n)$ time extract min

Definition. If k operations take total time $O(t \cdot k)$, then the amortized time per operation is $O(t)$.

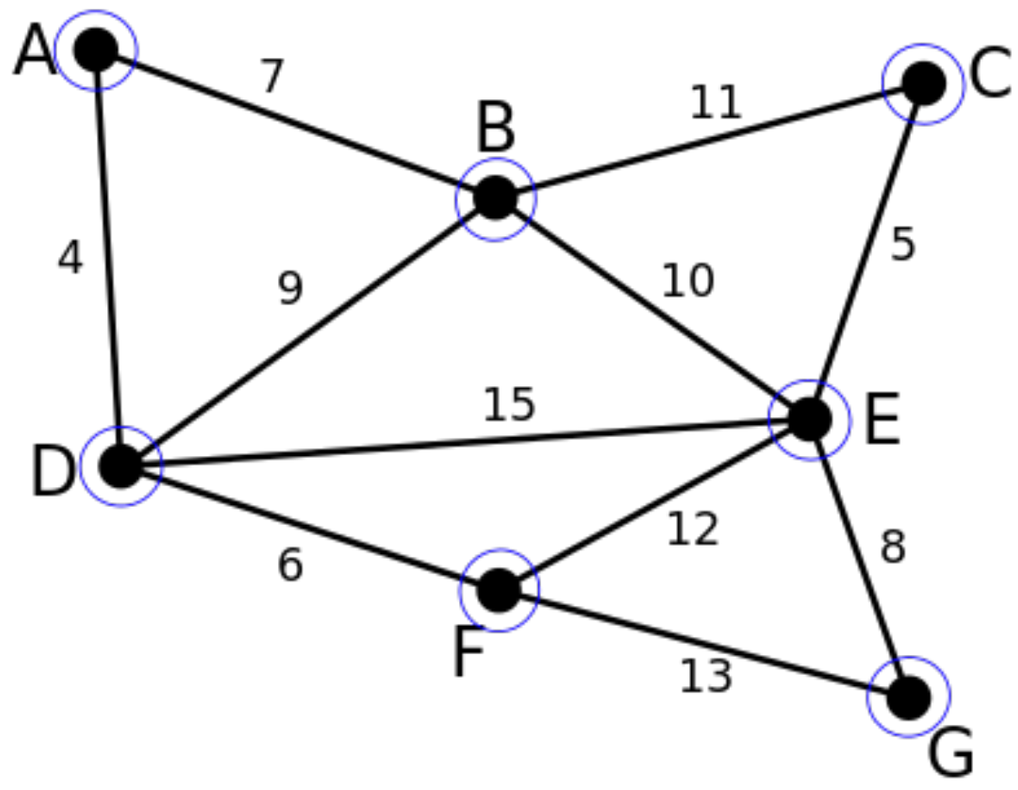
Kruskal's Algorithm

- Another MST algorithm
- Why do you think we're looking at a second one?

Kruskal's Algorithm

Idea: Add the cheapest remaining edge that does not create a cycle.

- Initialize $T = \emptyset$, $H \leftarrow E$
- While $|T| < n - 1$:
    - Remove cheapest edge e from H
    - If adding e to T does not create a cycle
    - $T \leftarrow T \cup \{e\}$



Kruskal's Algorithm

- Does it give us the correct MST?
  - Proof?
  - How quickly can we find the minimum remaining edge?
  - How quickly can we determine if an edge creates a cycle?

Kruskal's Implementation

- Sort edges by weight or extra edges stored in a min-heap (weight as priorities): $O(m \log m)$
    - Turns out this is the dominant cost
- Determine whether $T \cup \{e\}$ contains a cycle
    - Maintain a partition of V: components of T
    - Let $[u]$ denote component of u
    - Adding edge $e = (v, w)$ creates a cycle if and only if $[v] = [w]$
- Add an edge to T: update components

Union-Find Data Structure

Manages a **dynamic partition** of a set S

- Provides the following methods:
    - `MakeUnionFind()`: Initialize
    - `Find(x)`: Return name of set containing x
    - `Union(X, Y)`: Replace sets `X`, `Y` with $X \cup Y$

Kruskal's Algorithm can then use

- `Find` for cycle checking
- `Union` to update after adding an edge to T
- Sorting is still dominant step but imagine if edges were presented in sorted order!

Union-Find: First Attempt

Let $S = \{1, 2, \ldots, n\}$ be the set.

Idea: Each element stores the label of its partition

- `MakeUnionFind()`: Set $L[x] = x$ for each $x \in S$: $O(n)$
- `Find(x)`: Return $L[x]$: $O(1)$
- `Union(X,Y)`:
  - For each $z \in X \cup Y$, set $L[z]$ to the label of the **larger** set
    - $O(n)$
  - Doing this changes fewer names; how many times can an element change labels?
    - Each time an element changes labels, size of its set increases by at least 2: $O(\log n)$ times

Aside: Amortized Complexity

- Way to account for the average cost of a sequence of operations, where some operations may be expensive but happen infrequently
- Here we do at most n unions and total time spent on unions is $O(n \log n)$
    - What is the amortized complexity of a union operation?
    - $O(\log n)$

Definition. If k operations take total time $O(t \cdot k)$, then the amortized time per operation is $O(t)$.

Union-Find: First Attempt

Let $S = \{1, 2, \ldots, n\}$ be the set.

Idea: Each element stores the label of its set

- `MakeUnionFind()`: Set $L[x] = x$ for each $x \in S$: $O(n)$
- `Find(x)`: Return $L[x]$: $O(1)$
- `Union(X,Y)`:
  - For each $z \in X \cup Y$, set $L[z]$ to the label of the **larger** set
    - $O(n)$
  - Can happen at most $O(\log n)$ times
    - A sequence of n union operations take $O(n \log n)$ time
    - **Amortized cost of union** $O(\log n)$

Kruskal's with Union-Find

Note: If sorting is the bottleneck, the first attempt at union-find is good enough for Kruskals:

- Each time label of a vertex changes, its component size grows by at least 2: happens $O(\log_2 n)$ times
- Total time spent updating vertex labels: $O(n \log n)$
- Running time: $O(m \log m + n \log n) = O(m \log n)$; space $O(n + m)$

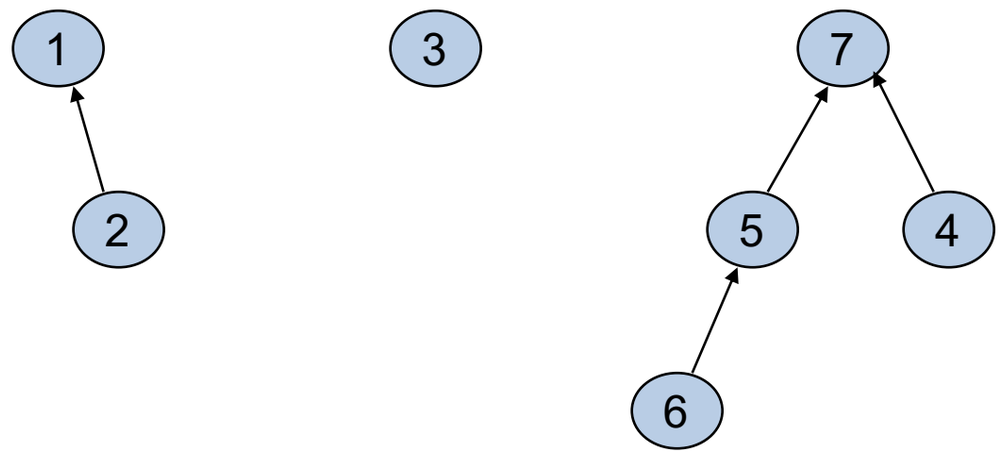
Not done with union-find. But as for union-find itself, we can still do a lot better…

Union-Find with Trees

- Trees let us find many elements given one root (i.e. representative); use one tree for each subset
- **Up tree.** If we reverse the pointers (child to parent), we can find a single root from many elements
- Union now will just require pointing one root to another

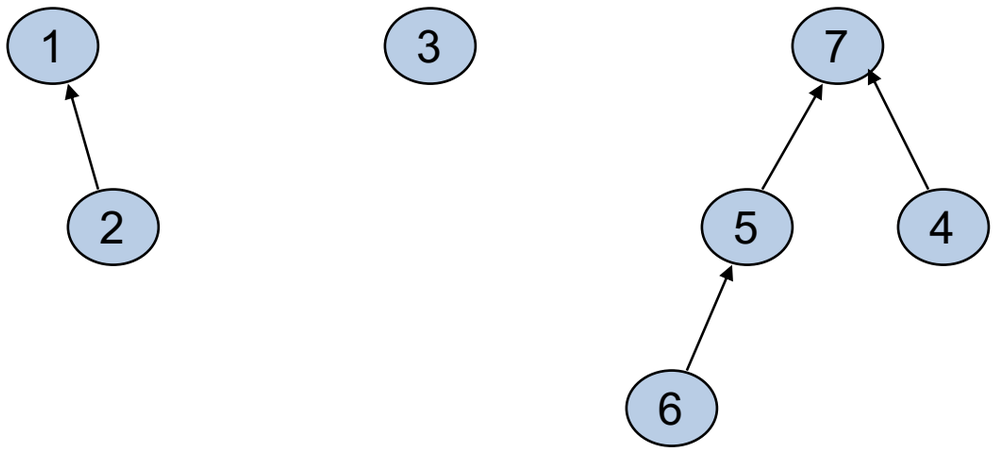
Initial state:

Intermediate state:

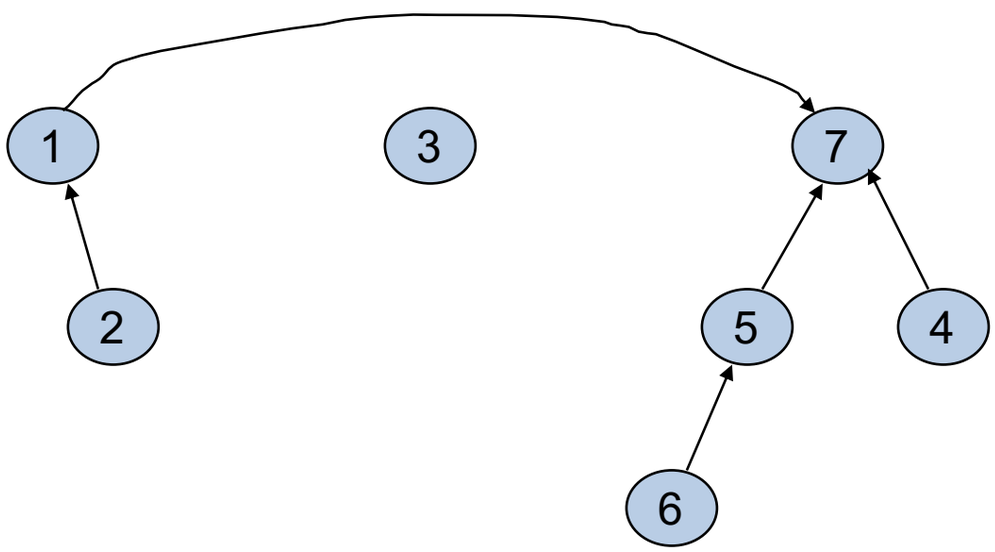


Union-Find with Trees

Intermediate state:

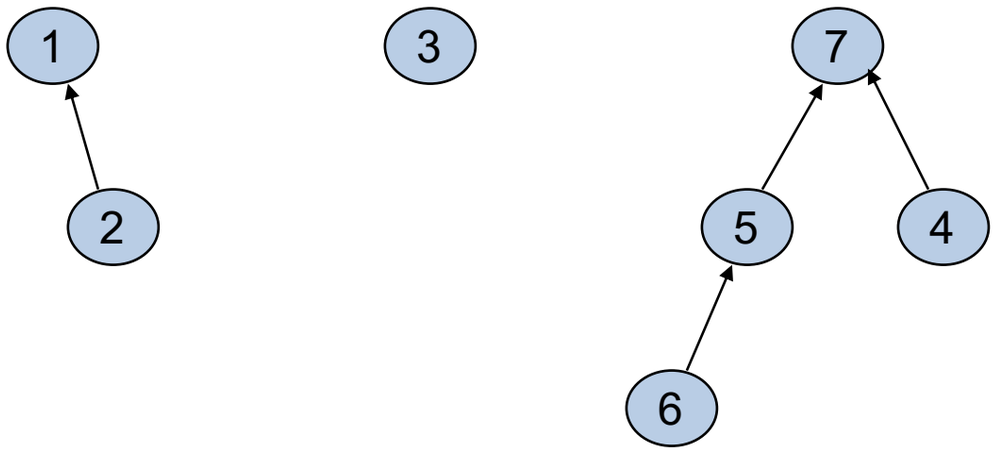


Union(1, 7)

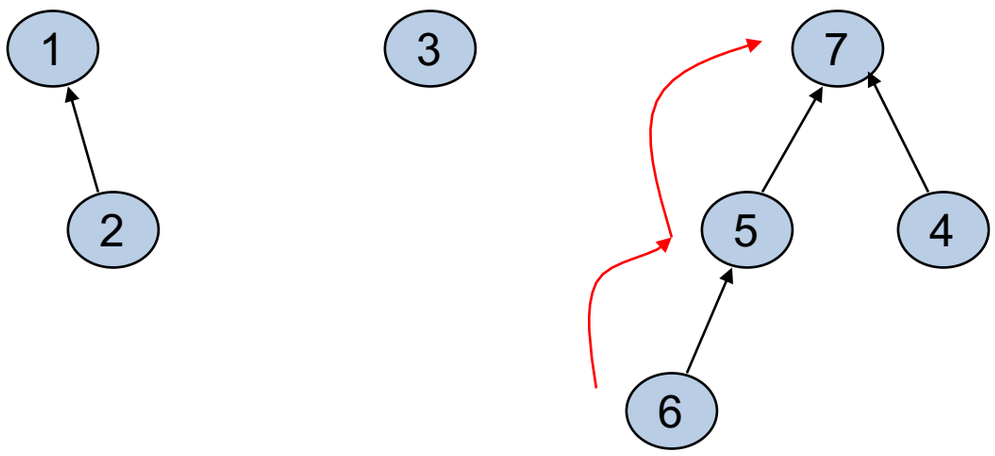


Union-Find with Trees

Intermediate state:

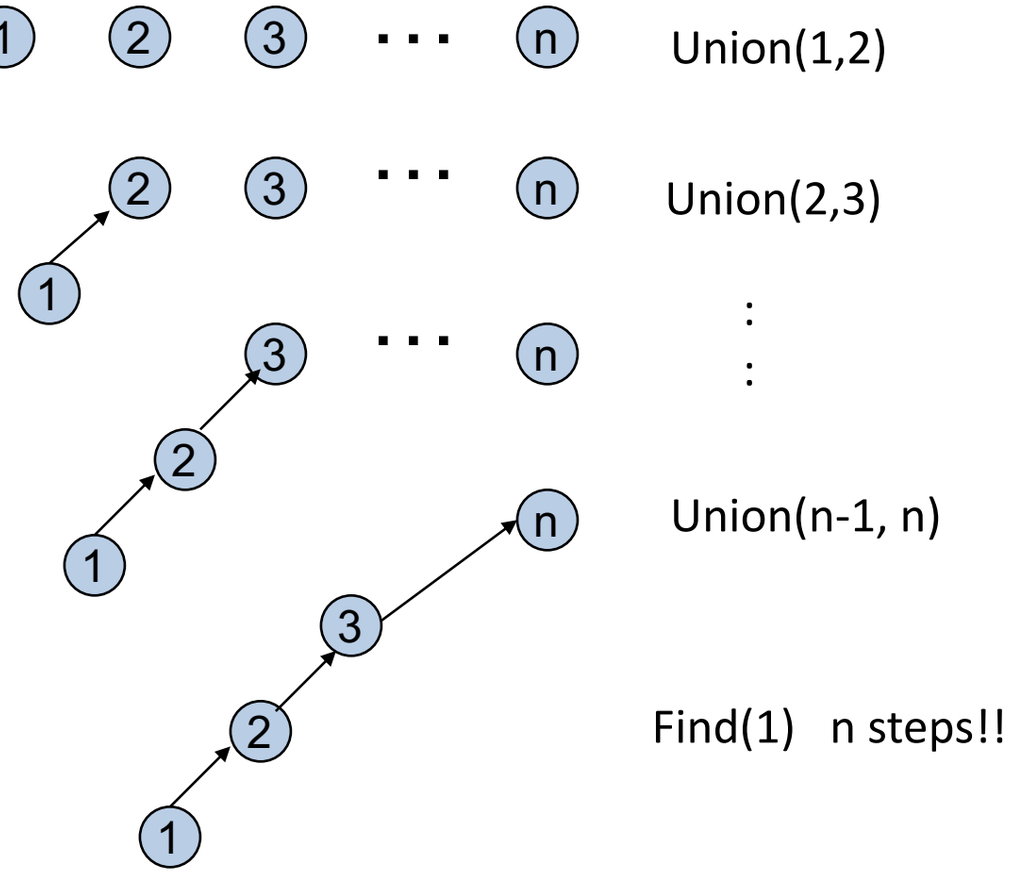


Find(6)



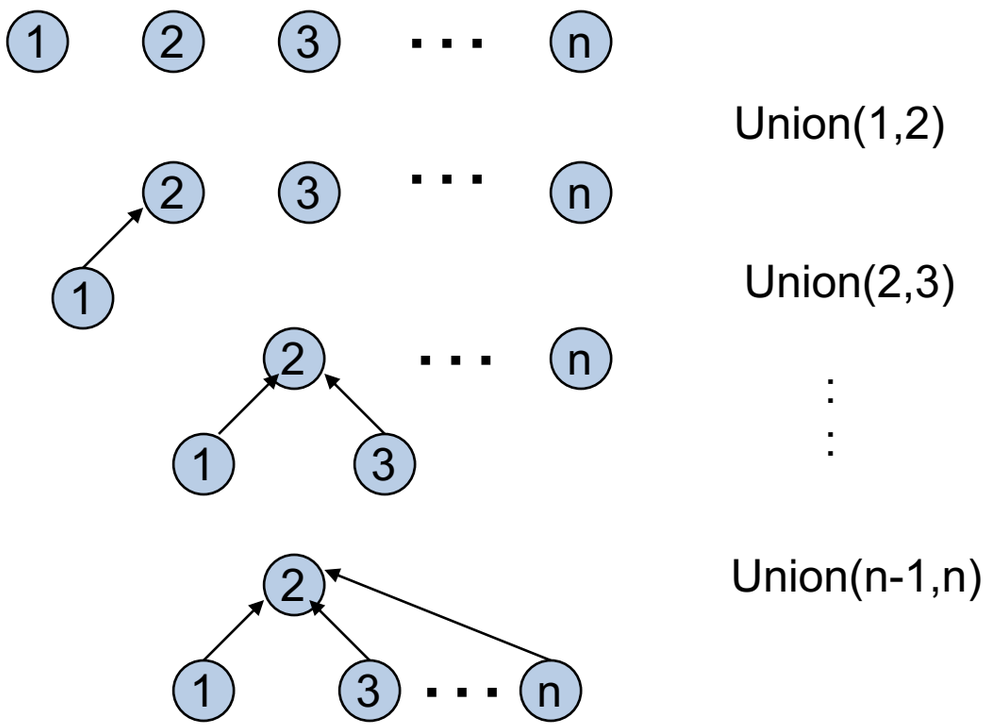
Union-Find with Trees

- Bad case!
- Find(1) takes $O(n)$ time



Union-Find with Trees

- Improvement (Union): point smaller to bigger tree



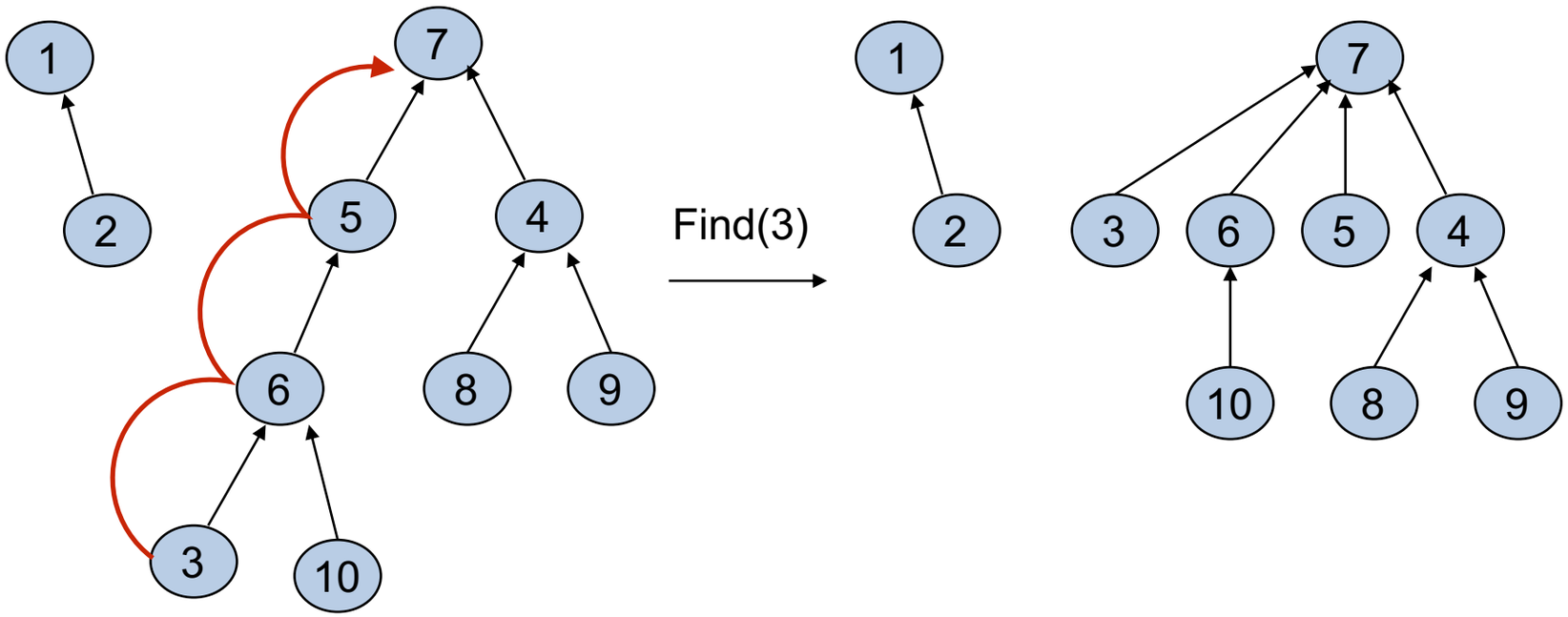
- Overall: Find $O(\log n)$ time (why?)
- Union: $O(1)$

Union-Find with Trees

- First attempt **(arrays)**
  - Union: amortized $O(\log n)$, worst-case $O(n)$
  - Find: $O(1)$
- Second attempt **(up trees)**
  - Union: $O(1)$
  - Find: $O(\log n)$
- **Question.** Do we always have to pay $\Omega(\log n)$ cost for one of union or find?
  - Surprisingly no, we can do better!
  - Possible to get cost of both essentially constant!

Union-Find: Path Compression

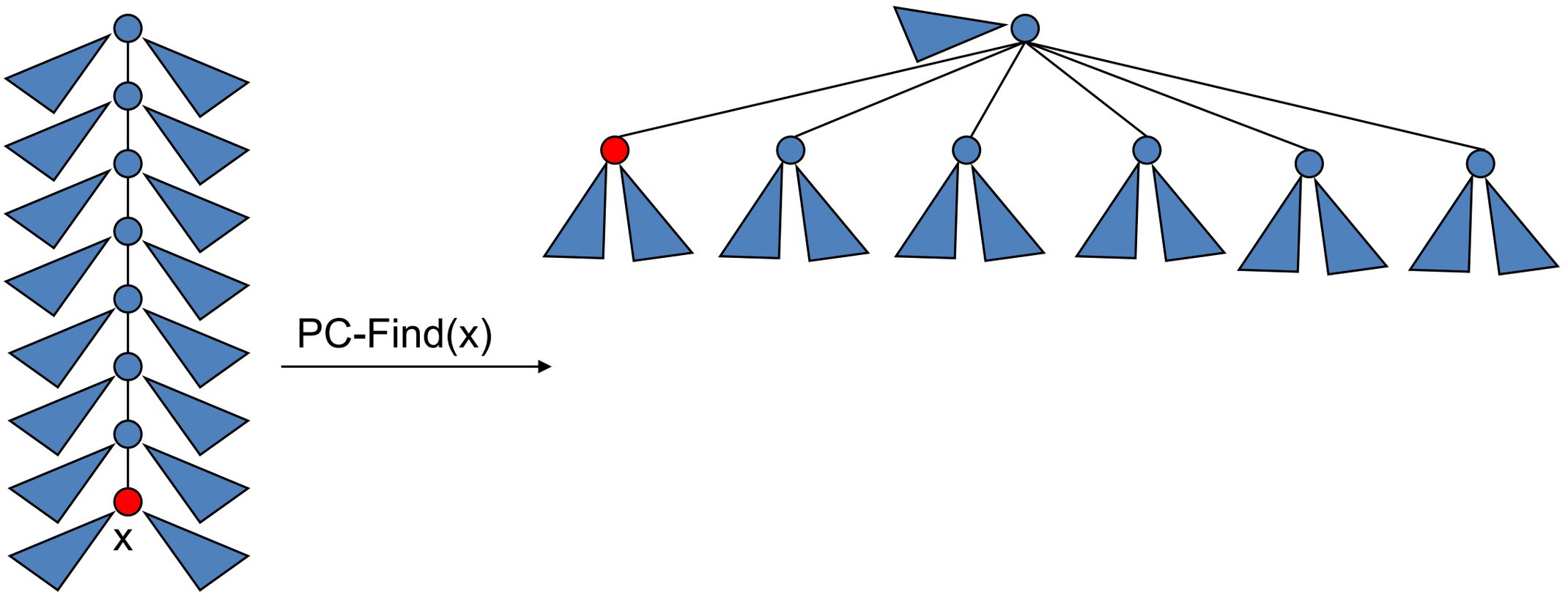
- **Heuristic.** On a Find operation point all the nodes on the search path directly to the root



- This does not change the worst case time complexity, which is still $O(\log n)$ and $O(1)$ for Union

Union-Find: Path Compression

- Self adjustment improves amortized complexity!



- Amortized complexity of find becomes very close to constant!

Surprising Result: Hopcroft Ulman'73

- Amortized complexity of union find with path compression improves significantly!
- Time complexity for n union and find operations on n elements is $O(n \log^* n)$
- $\log^* n$ is the number of times you need to apply the log function before you get to a number <= 1
- **Very small! Less than 5 for all reasonable values**

$$\log^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log n) & \text{if } n > 1 \end{cases}$$

n	1	2	$4 = 2^2$	$16 = 2^4$	$65,536 = 2^{16}$	$2^{65,536}$
$\log^*(n)$	0	1	2	3	4	5

Surprising Result: Tarjan ‘75

- Improved bound on amortized complexity of union-find with path compression
- Time complexity for n union and find operations on n elements is $O(n\alpha(n))$, where
    - $\alpha(n)$ is extremely slow-growing, **inverse-Ackermann function**
        - Essentially a constant
- Grows much **muuchch morrree** slowly than $\log^*$
- $\alpha(n) \leq 4$ for all values in practice
- **Result.** Union and Find become (essentially) amortized constant time in practice (just short of $O(1)$ in theory) !

Ackermann & Inverse Ackermann

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m,n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

$$\alpha(m,n) = \min\{i \geq 1 : A(i, \lfloor m/n \rfloor) \geq \log_2 n\}$$

“I am not smart enough to understand this” — Seidel

Inverse Ackermann

- **Inverse Ackerman:** The function $\alpha(n)$ grows much more slowly than $\log^{*c} n$ for **any fixed c**

- With $\log^*$, you count how many times does applying $\log$ over and over gets the result to become small
- With the inverse Ackermann, essentially you count how many times does applying $\log^*$ (not log!) over and over gets the result to become small

- $\alpha(n) = \min\{k \mid \log^{\overbrace{****\cdots*}^{k}}(n) \leq 2\}$

- $\alpha(n) = 4$ for $n = 2^{2^{2^{2^{16}}}}$

Many Applications of Union-Find

- Good for applications in need of clustering
    - cities connected by roads
    - cities belonging to the same country
    - connected components of a graph
- Maintaining equivalence classes
- Maze creation!

MST Algorithms History

- **Borůvka's Algorithm** (1926)
    - The Borvka / Choquet / Florek-ukaziewicz-Perkal-Steinhaus-Zubrzycki / Prim / Sollin / Brosh algorithm
    - Oldest, most-ignored MST algorithm, but actually very good
- **Jarník's Algorithm** ("Prims Algorithm", 1929)
    - Published by Jarník, independently discovered by Kruskal in 1956, by Prims in 1957
- **Kruskal's Algorithm** (1956)
    - Kruskal designed this because he found Borůvka's algorithm "unnecessarily complicated"

Can we do better?

Best known algorithm by Chazelle (1999)

A Minimum Spanning Tree Algorithm with Inverse-Ackermann Type Complexity*

BERNARD CHAZELLE†

NECI Research Tech Report 99-099 (July 1999)
Journal of the ACM, 47(6), 2000, pp. 1028–1047.

Abstract

A deterministic algorithm for computing a minimum spanning tree of a connected graph is presented. Its running time is $O(m\alpha(m,n))$, where α is the classical functional inverse of Ackermann's function and n (resp. m) is the number of vertices (resp. edges). The algorithm is comparison-based: it uses pointers, not arrays, and it makes no numeric assumptions on the edge costs.

1 Introduction

The history of the minimum spanning tree (MST) problem is long and rich, going as far back as Borůvka's work in 1926 [1, 9, 13]. In fact, MST is perhaps the oldest open problem in computer science. According to Nešetřil [13], "this is a cornerstone problem of combinatorial optimization and in a sense its cradle." Textbook algorithms run in $O(m \log n)$ time, where n

Can we do better?

Using randomness, can get $O(n + m)$ time!

A Randomized Linear-Time Algorithm to Find Minimum Spanning Trees

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Abstract. We present a randomized linear-time algorithm to find a minimum spanning tree in a connected graph with edge weights. The algorithm uses random sampling in combination with a recently discovered linear-time algorithm for verifying a minimum spanning tree. Our computational model is a unit-cost random-access machine with the restriction that the only operations allowed on edge weights are binary comparisons.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*computations on discrete structures*; G.2.2 [**Discrete**

Optimal MST Algorithm?

Has been discovered but don't know its running time!

An Optimal Minimum Spanning Tree Algorithm

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Abstract. We establish that the algorithmic complexity of the minimum spanning tree problem is equal to its decision-tree complexity. Specifically, we present a deterministic algorithm to find a minimum spanning tree of a graph with n vertices and m edges that runs in time $O(\mathcal{T}^*(m, n))$ where $\mathcal{T}^*$ is the minimum number of edge-weight comparisons needed to determine the solution. The algorithm is quite simple and can be implemented on a pointer machine.

Although our time bound is optimal, the exact function describing it is not known at present. The current best bounds known for $\mathcal{T}^*$ are $\mathcal{T}^*(m, n) = \Omega(m)$ and $\mathcal{T}^*(m, n) = O(m \cdot \alpha(m, n))$, where α is a certain natural inverse of Ackermann's function.

Even under the assumption that $\mathcal{T}^*$ is superlinear, we show that if the input graph is selected from $G_{n,m}$, our algorithm runs in linear time with high probability, regardless of n, m, or the permutation of edge weights. The analysis uses a new martingale for $G_{n,m}$ similar to the edge-exposure martingale for G