Time, Asymptotics, and Recursion

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Admin

- Pairs assigned for lab 3 (randomly)
- Some office hours today if you’re still finishing up lab 2
- Important place in the course for you to check in if you’re feeling behind

Time and Space Analysis

How efficient is a given method?

- We saw how to do `contains` in a `Vector`. How many items did we have to look through in the worst case?
- Let’s say I’m looking through a literal dictionary. Is my `contains()` method very efficient? Do you have a faster way?
- What if I say I’m a really fast reader. Is your method still faster?
    - Probably
    - Unless the dictionary is really short. A fast reader may be able to read through a dictionary with 10 elements better than a more clever search method
- Idea here: analyze the efficiency of a *methodology*. Your speed—or your computer’s speed—shouldn’t be a factor.

What do we mean by efficiency?

- Perhaps: how long does a method take to run in seconds?

- How much space does it take? (How many bits do we need to store on our computer during the calculation)?

Algorithmic Efficiency

- We are looking for worst-case guarantees
- When you write a piece of code, the goal here is to say "I promise that my code will *always* run efficiently."
  - It's a much more widely applicable statement than "I tested my code out and it seems to run efficiently."
  - What if your tests didn't take into account a key scenario?

The Challenge of Analyzing Time

- Different computers run at different speeds
- Computers are complicated! Adding two numbers together (for example) can take drastically different times depending on context.
- Good news: often times these details don't change much
- Example: It doesn't matter (too much) how fast I read if I'm scanning thousands of extra dictionary pages.

Counting up time

- When we look at some Java code, how can we estimate how fast it is?
- Let's look at the operations the code requires
  - By operations, I mean built-in operations like `+`, `-`, `==`, `if`, `=`, array operations, etc.
  - If any methods are called, should count up their operations as well
- If we sum the time of all operations, we can figure out how long the code takes.
- Let's do a quick example

Counting up time example

```
int i = 0;     c1 time (integer variable assignment)
int count = 0; c2 time (integer variable assignment)
while(i < arr.length) { c3 time (accessing length and comparing)
   count += arr[i]; c4 time (array access, addition, and assignment)
   i++;  c5 time (variable assignment and addition)
}
```

In total, this code takes time *at most*:

$$c_1 + c_2 + (c_3 + c_4 + c_5) \cdot \texttt{arr.length}$$

Counting up time comparison

```
int count = arr[0] % 27; c6 time
```

In total, this code takes time at most $c6$.

If our array is at all large, this is going to be faster than the loop on *any* computer.

Let’s formalize this

Goal in analyzing efficiency:

- We don’t care that much about constants
- We care about scaling: what happens when the data in question is fairly large?
- Big-O notation: way of comparing two running times with this in mind

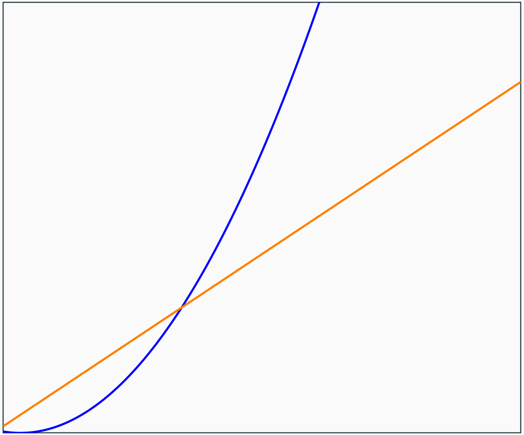
Big-O Notation

Definition 1

$f(n)$ is $O(g(n))$ if there exist constants $c > 1$ and n_0 such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$

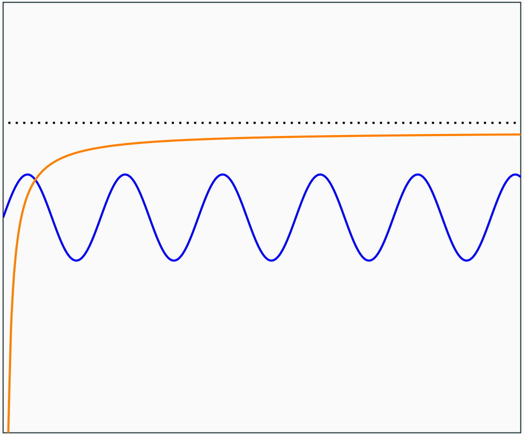
That is to say: If n is large enough ($n > n_0$), then ignoring constants (we compare to $c \cdot g(n)$), then $g(n)$ is larger.

Plotting Big-O



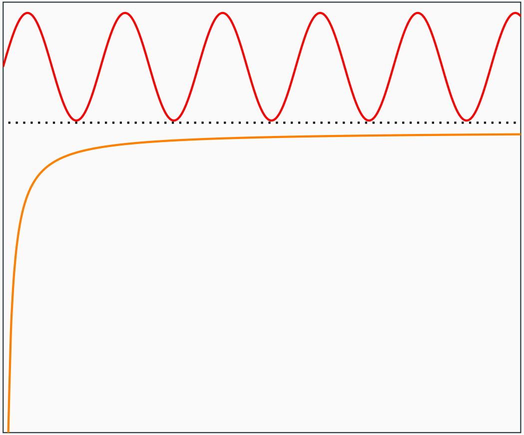
Let $g(n)$ be the blue function, and $f(n)$ be the orange function. If $g(n)$ and $f(n)$ continue increasing in the same way, then $f(n) = O(g(n))$.

Plotting Big-O



Let $g(n)$ be the blue function, and $f(n)$ be the orange function; assume that $f(n)$ is bounded above by the dotted line. Since $f(n) < c \cdot g(n)$, we still have $f(n) = O(g(n))$.

Plotting Big-*O*



Continued from last slide: once we multiply g by a constant, we obtain the plot shown in red; this is larger than f.

Proving Big-O

Reminder: $f(n)$ is $O(g(n))$ if there exist constants $c > 1$ and n_0 such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$

- Let's say we have two functions $f(n)$ and $g(n)$, and we want to show that $f(n)$ is $O(g(n))$.
- We need to come up with a $c > 1$ and an n_0 such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$

Counting up time example

```
int i = 0;   c1 time
int count = 0; c2 time
while(i < arr.length) { c3 time
   count += arr[i]; c4 time
   i++;  c5 time
}
```

Let's define $n =$ arr.length. That is: we are analyzing the running time in terms of the array length

In total, this code takes time at most:

$$f(n) = c_1 + c_2 + (c_3 + c_4 + c_5)n$$

Let's prove that this is $O(n)$.

Counting up time example

$$f(n) = c_1 + c_2 + (c_3 + c_4 + c_5)n$$

Let's prove that $f(n) = O(n)$.

Let's set $n_0 = 1$ and $c = c_1 + c_2 + c_3 + c_4 + c_5$. Then we want to show that for all $n > 1$, $f(n) \leq c \cdot g(n)$:

$$\begin{aligned} f(n) &= c_1 + c_2 + (c_3 + c_4 + c_5)n \\ &\leq c_1 n + c_2 n + (c_3 + c_4 + c_5)n \\ &= (c_1 + c_2 + c_3 + c_4 + c_5)n \\ &= c \cdot n \\ &= c \cdot g(n) \end{aligned}$$

So for $n > n_0$, $f(n) \leq c \cdot g(n)$; therefore, $f(n) = O(n)$

Counting up time example

```
int i = 0;   c1 time
int count = 0; c2 time
while(i < arr.length) { c3 time
   count += arr[i]; c4 time
   i++;  c5 time
}
```

This code takes $O(n)$ time.

Simplifying

- None of the c_i really mattered in the above analysis
- What we want to do: count the *number of operations* in a code segment
- Don't need to be too careful about it: `count += arr[i]` counting as 1 operation or 3 operations isn't going to change our final result
- Let's consider the above code again

Counting up time example

```
int i = 0;   2 operations
int count = 0;
while(i < arr.length) { 1 operation
   count += arr[i]; 4 operations
   i++;
}
```

This code takes $2 + 5n$ operations. Since each operation takes constant time, this is $O(n)$ time.

Counting up time comparison

```
int count = arr[0] % 27; 1 operation
```

If n is the length of the array, how long does this code take?

It takes a constant number of operations. $O(1)$ time.

(Note: $O(1)$ is means *bounded above by a constant*—this function does not get larger as n increases. How does this relate to the definition of big-O?)

Wrapping Up Asymptotics

- We want to count the amount of time taken by a method
- Our analysis should apply regardless of how fast the computer is
- Idea: Look at how many operations are used. Use big-O notation
    - Ignore constants
    - Only care about sufficiently large inputs

Recursion

Recursion

- We've seen methods call other methods in Java
- Methods can also call themselves. This is called recursion
- Works just like any other method call! Execution continues from the beginning of the method; goes back to previous point after it returns
- Recursion allows for *simpler* and *clearer* code in some cases
    - But not in others
    - Anything that can be solved with recursion can be solved without recursion
    - It's just one more tool in your toolbox

Classic example: factorial

- The factorial function, written $n!$, is useful in combinatorics
  - (It counts the number of ways to order n objects.)
- $n!$ is the product of the first n numbers:

$$n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$$

- So $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$.
- Can also define $n! = n \cdot (n-1)!$

Two implementations of factorial

```
public static int factorial(int n) {
      int ret= 1;
      for(int current = n; current >= 1; current--) {
         ret *= current;
      }
      return ret;
}
```

```
public static int factorial(int n) {
      if(n == 1) {
         return 1;
      }
      return n * factorial(n-1);
}
```

Factorial discussion

- Which of these methods is better?
  - A matter of taste
  - What are some advantages of the recursive method? What are some disadvantages?
- If this method calls itself, why doesn't it loop forever?

Recipe for Recursion

- Need a *base case*: on a sufficiently small input, can easily return the correct solution without a recursive call

- If we're not in the base case, can split into *smaller* instances of the same problem

Searching a (Physical) Dictionary

- We agreed that my dictionary lookup method wasn't very effective
- Can we describe a lookup methodology that works faster?
- (This is a method for humans, not code: let's just talk about how it works on the board.)

Binary Search

- Recursive algorithm for searching in a sorted list
- Can be implemented without recursion! That is to say: you can implement a perfectly good binary search method with a loop instead of recursion
- We’ll be seeing a lot more of binary search soon.

Helper methods

- May be helpful to use extra information when recursing
    - We didn’t just search in the dictionary, we kept track of which *portion* of the dictionary we were recursing on

- Can create *helper methods* that have more parameters

- Let’s say we want to search a dictionary. We can “help” using a method that searches a portion of a dictionary.

Another Recurion Example: Scheduling

A scheduling problem: Creating Office Hours

- Let's say there are 6 enrolled students enrolled in a course. I want to schedule office hours so that every single student has a chance to attend office hours
- I create a doodle poll with 10 options for my office hours
- Each student states which of the office hours they can attend
- What is the minimum number of office hours I can hold so that every student can make at least one hour?

Creating Office Hours

The possible time slots are $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- Student 1 can make slots $\{1, 6, 8\}$
- Student 2 can make slots $\{2, 5, 8\}$
- Student 3 can make slots $\{3, 4, 9, 10\}$
- Student 4 can make slots $\{6, 7, 8, 9\}$
- Student 5 can make slots $\{2, 3, 4\}$
- Student 6 can make slots $\{1, 3, 4, 5, 9\}$

Creating Office Hours

The possible time slots are $\{1, \mathbf{2}, 3, 4, 5, \mathbf{6}, 7, 8, \mathbf{9}, 10\}$.

- Student 1 can make slots $\{1, \mathbf{6}, 8\}$
- Student 2 can make slots $\{\mathbf{2}, 5, 8\}$
- Student 3 can make slots $\{3, 4, \mathbf{9}, 10\}$
- Student 4 can make slots $\{\mathbf{6}, 7, 8, 9\}$
- Student 5 can make slots $\{\mathbf{2}, 3, 4\}$
- Student 6 can make slots $\{1, 3, 4, 5, \mathbf{9}\}$

This is solvable with 3 slots. (I think that's optimal?)

Solving the Office Hours Problem Recursively

- Where to start?
- Can someone come up with a base case?
    - When there’s only one time slot, our only choice is to take it or not take it
    - Second base case option: if there is one student, if they have an hour that matches with a time slot, then 1 slot is optimal. Otherwise, can’t solve.
    - Another option: zero students or zero time slots

Office Hours Scheduling: Breaking into a Smaller Subproblem

- How can we make this subproblem smaller?
- Let’s look at the first possible time slot
- There are two options: either this time slot is in the solution, or it isn’t
  - Let’s assume we take the first time slot. Then we can remove that time slot from our list, and remove all students who can attend that time slot. That gives us a new instance of office hours scheduling!
  - Let’s assume we *don’t* take the first time slot. Then we can remove that time slot from our list. That gives us a new instance of office hours scheduling!

Office Hours Scheduling Solution

- If there is only remaining slot, just determine if it meets all students' needs. Return 1 if so; -1 otherwise.
- Otherwise:
  - Recursively find the office hours scheduling solution with the first slot removed, and with all students whose availability matches that slot removed. Store this optimal solution in `solWithSlot`
  - Recursively find the office hours scheduling solution with the first slot removed. Store this optimal solution in `solWithOutSlot`
- If both `solWithSlot` and `solWithOutSlot` are not -1, return the minimum of $1+$ `solWithSlot` and `solWithOutSlot`
- If just one is -1, return the other
- If both are -1, return -1.

Discussion

- Why does this method work? What do we need to guarantee for a recursion to terminate?
    - Need to make progress towards the base case!
    - Each recursive call reduces the number of slots by 1
- Is this method fast? Is that OK?
    - No, this is not fast at all.
    - In algorithms you will learn that this problem is computationally intensive—there's no known solution that's efficient and always correct