Sorting: Selection Sort and Insertion Sort

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Admin

- Any questions?

Sorting

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Maximum so far:

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Maximum so far: 10 at pos 0

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Maximum so far: 21 at pos 1

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Maximum so far: 40 at pos 3

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- (This is essentially a recursive algorithm)

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Selection Sort

10	11	-3	-4	17	13	21	40

Maximum so far: 10 at pos 0

Selection Sort

10	11	-3	-4	17	13	21	40

Maximum so far: 11 at pos 1

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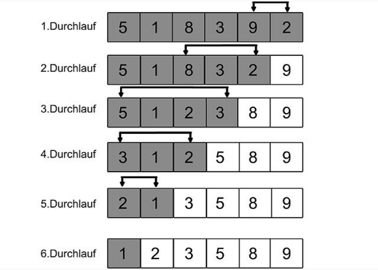
- Let's assume we have a `swap(int[], int, int)` method that swaps two indices of an array

Selection Sort Code

```
public static void selectionSort(int data[], int n) {
   int numUnsorted = n;
   int index; // general index
   int max; // index of largest value
   while (numUnsorted > 0) {
      // determine maximum value in array
      max = 0;
      for (index = 1; index < numUnsorted; index++) {
         if (data[max] < data[index]) max = index;
      }
      swap(data,max,numUnsorted-1);
      numUnsorted--;
   }
}
```

How can we prove that this works?

- Why does it work?



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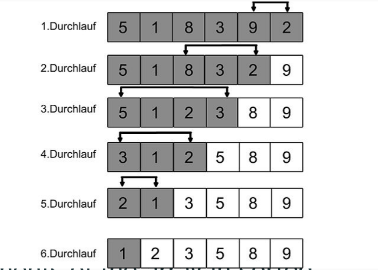

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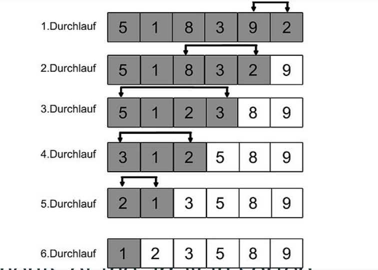

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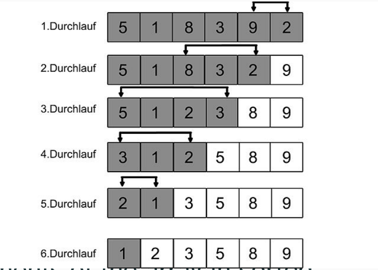

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- Prove using induction. (Kind of like recursive algorithms.)

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- Base case: $k = 0$. Already satisfied
- Inductive hypothesis: for some k, after the loop iterates k times, the last k slots of the array contain the k largest elements of the array in sorted order.
- Inductive step: by the inductive hypothesis, after the kth iteration of the outer loop, the last k slots of the array contain the k largest array items in sorted order. We scan through the array and find the largest element excluding the last k slots; this is the $k + 1$st largest item. The swap moves it into the $k + 1$st slot from the end of the array.

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- Summing: $\sum_{i=1}^{n} O(n - i + 1) = \sum_{j=1}^{n} O(j) = O(n^2)$

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- This time we’ll start with why it works, and derive the algorithm

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  - Needs to insert the $k + 1$st item among the first k items in sorted order.

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- Want to take the new item and move it into sorted position
- Idea: need to move it down until the previous element is smaller
- Inner loop: store element we are trying to insert. Shift elements down while it is smaller.

Insertion Sort Code

```java
public static void insertionSort(int data[], int n) {
   int numSorted = 1; // number of values in place
   int index; // general index
   while (numSorted < n) {
      int temp = data[numSorted]; // first unsorted value
      for (index = numSorted; index > 0; index--) {
         if (temp < data[index-1]) {
            data[index] = data[index-1];
         } else {
            break;
         }
      }
      data[index] = temp; // reinsert value
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         } else {
            break;
         }
      }
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   }
}
```

Can we get rid of the `break` command in this code?

Insertion Sort Code # 2

```java
public static void insertionSort(int data[], int n) {
   int numSorted = 1; // number of values in place
   while (numSorted < n) {
      int temp = data[numSorted]; // first unsorted value
      int index = numSorted;
      while(index > 0 && temp < data[index - 1]) {
         data[index] = data[index-1];
         index--;
      }
      data[index] = temp; // reinsert value
      numSorted++;
   }
}
```

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  - Insertion sort may stop early! Can lead to better performance in practice (and is never worse)
- To be clear: both are still $O(n^2)$ in terms of worst-case performance. Insertion sort just has better constants, and better best-case performance

Sorting Objects

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- Reminder: we interact with objects using methods
- What methods do we need in order to sort objects?
  - Need to be able to determine if one item is less than another
- Two ways that this may work. Both are good depending on use case.
  - First: *only* sort objects of a type with a `compareTo()` method, allowing two objects of that type to be compared
  - Second: create a *new method* that allows us to compare the objects

Sorting with `compareTo`

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- Let’s try sorting `Students` with a `compareTo` method

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- With an interface!

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- Let's tell Java that our `Student` class implements this interface

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  - `public static void <E> insertionSort(Vector<E> vec)`

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- Really: want to make one method generic. Can we do this in Java?
- Yes! Looks something like this:
    - `public static void <E> insertionSort(Vector<E> vec)`
- Problem: can't use *any* `E`. Needs to be comparable with other objects of type `E`

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- What do we want for our `insertionSort` method?
  - Want `<E extends Comparable<E>>`
  - That is to say: we want a type `E` that implements `Comparable<E>`. That is to say: need that objects of type `E` have a `compareTo` method that takes objects of type `E` as argument

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  - Can only sort objects one way. (What if we want to sort `Students` by grade? Would need to rewrite the `Student` class!
- There are upsides as well; we'll come back to this after we talk about `Comparators`