Introduction to Graphs

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May 2, 2022

Admin

- Last week’s quiz back Wednesday
- Any questions?

Graphs

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- Much of them were essentially implementations of a `Map` (or Dictionary)
- Trees were, in some cases, able to store relationships between pieces of data
  - Family tree: parent/child relationships
  - Lexicon trie: relationship between nodes in the trie represented stored words
- Graphs: a new data structure to store relationships between data. (Graphs are not particularly useful for Dictionary-like operations.)

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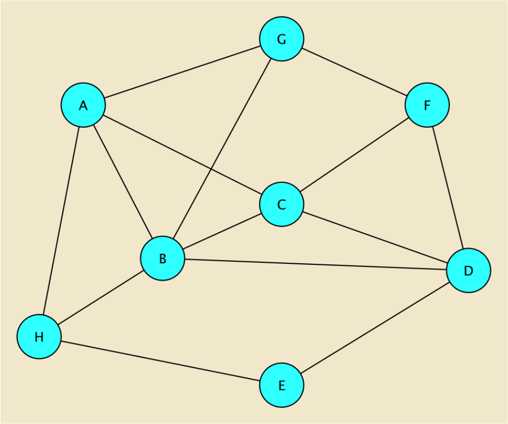
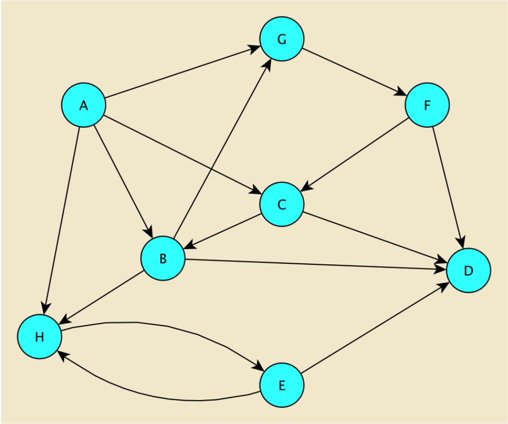
- Much like a tree! But no restrictions on how edges may connect nodes
- An edge may be *directed* or *undirected*
  - Directed edges represent a relationship from (say) node *A* to node *B*. Undirected represent a relationship between node *A* and node *B* (no direction on the relationship)

Drawing Graphs



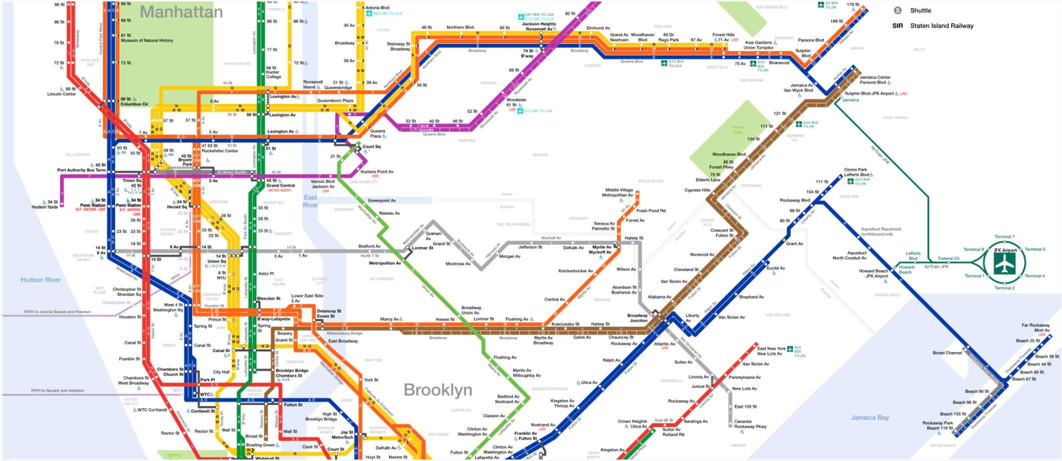
An undirected graph



A directed graph

We usually draw graphs much like we drew trees. In directed graphs, we show the direction of an edge with an arrow.

height



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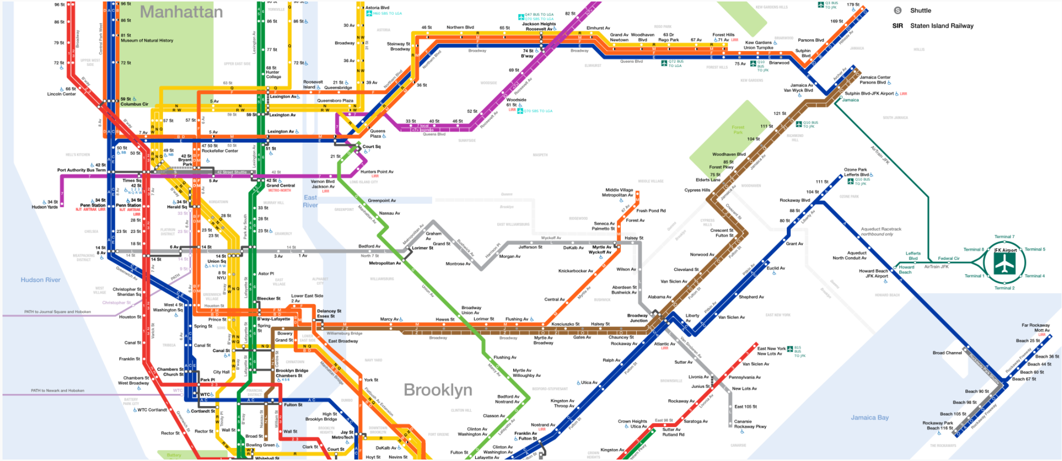
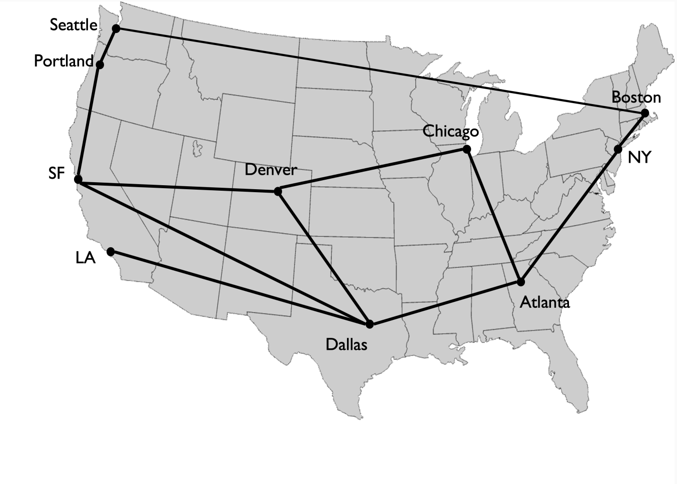


What are the nodes here? Edges?

Nodes: subway stops. An edge between two stops if there is a train between them.

(Simplified) US Train Map



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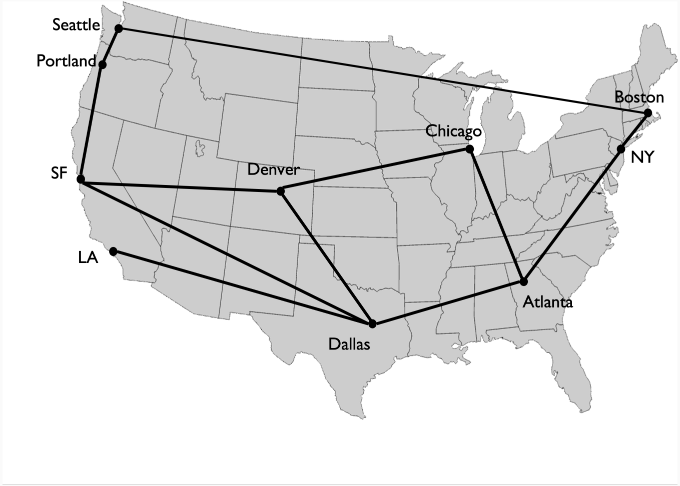

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Nodes: cities. An edge between two cities if there is a train between them. Note that it's not important how we draw the edges.

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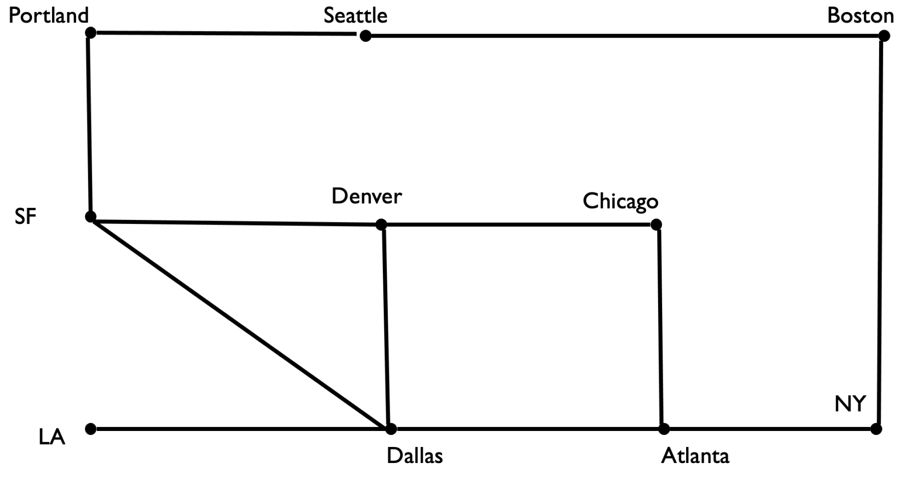

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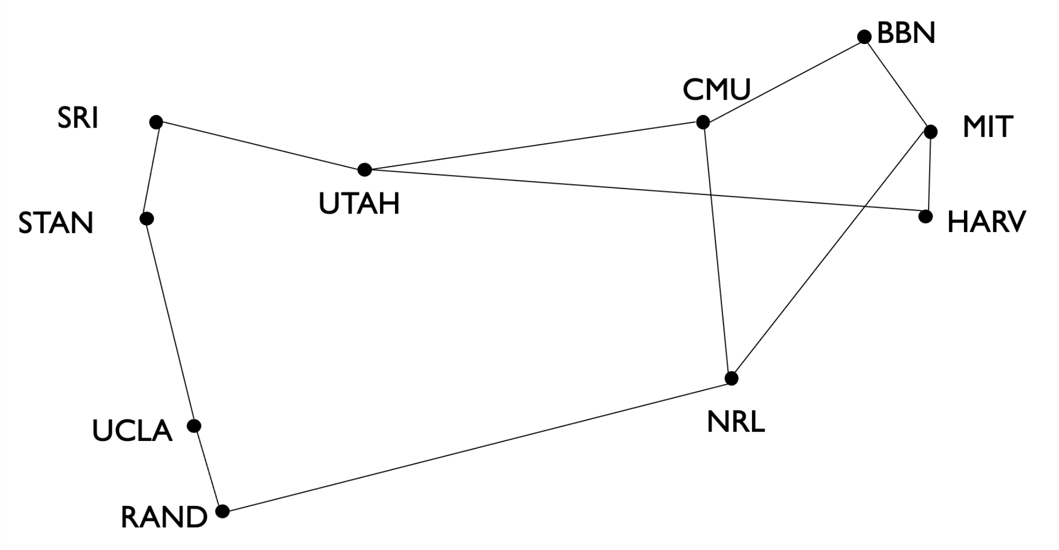

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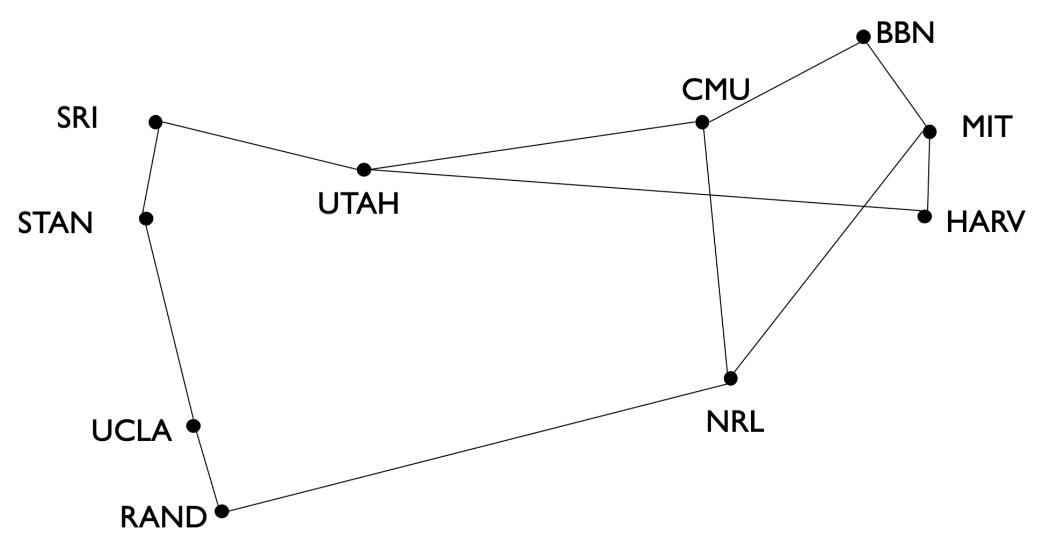
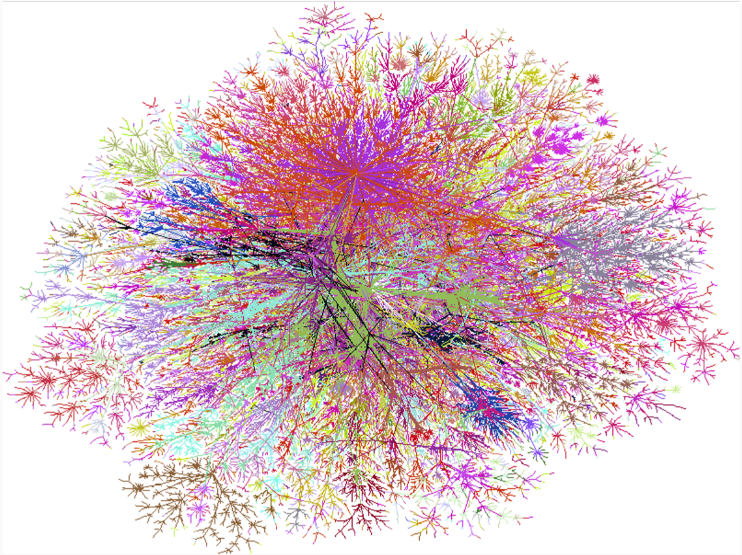


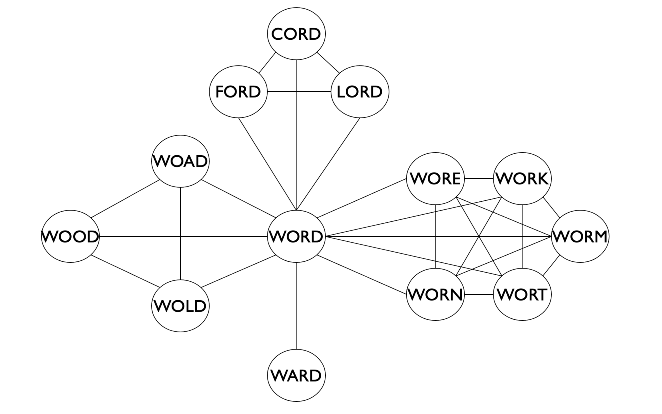
What are the nodes here? Edges?

Nodes: network access points. Edges represent a connection.

Internet Circa 1998



Word Game

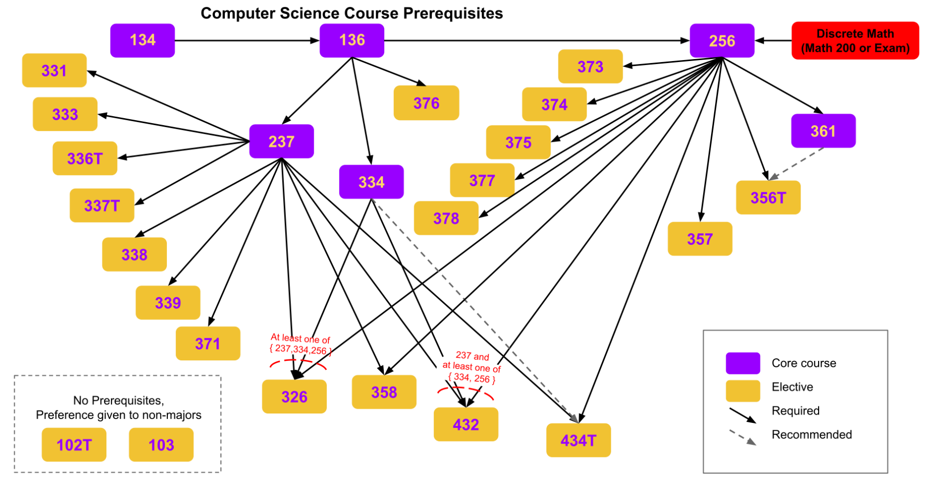


Goal of the game: given two words, transform one into the other by changing one letter at a time, always maintaining a legal word.

CORD $\rightarrow$ WORD $\rightarrow$ WORM

Two words are connected if they differ by one letter.

CS Prerequisite Graph



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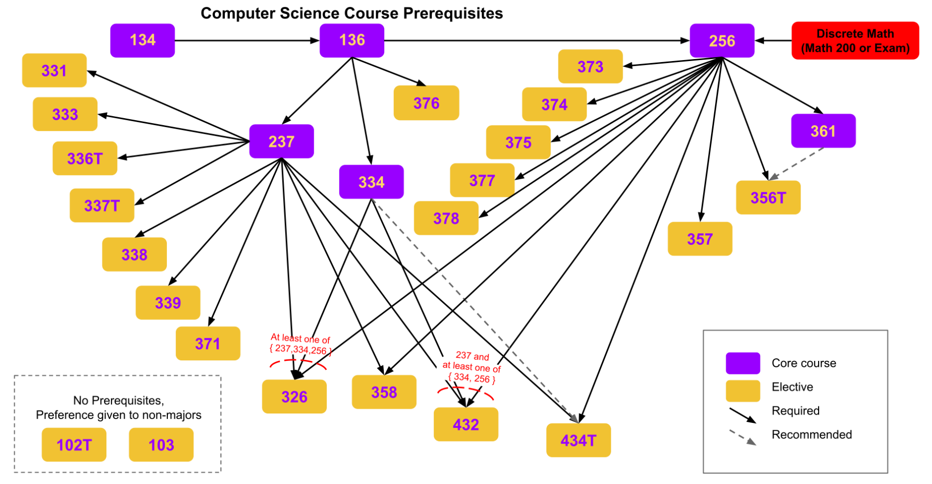

What are the nodes? Edges?

Is this graph a tree?

Relationships to Trees

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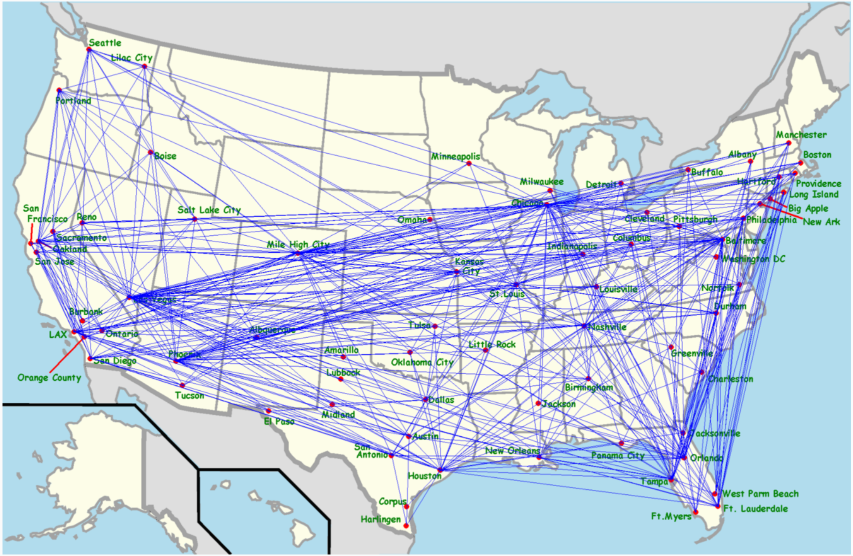
- Every tree is a graph!
- But not every graph is a tree.

Flight Routes



Basic Definitions

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- We say that u and v are *adjacent* (they are connected by an edge)
- The *neighbors* of v are all nodes u that are adjacent to v

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  - Note that this needs to follow the direction of the edge in a directed graph; for an undirected graph the can go in either direction
- No edge can appear more than once: $e_i \neq e_j$ if $i \neq j$
- In a *simple path*, no vertex appears more than once

Path Continued

- A *cycle* is a simple path that begins and ends at the same vertex

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- A *cycle* is a simple path that begins and ends at the same vertex
- The *length* of a path or cycle is the number of edges in the sequence

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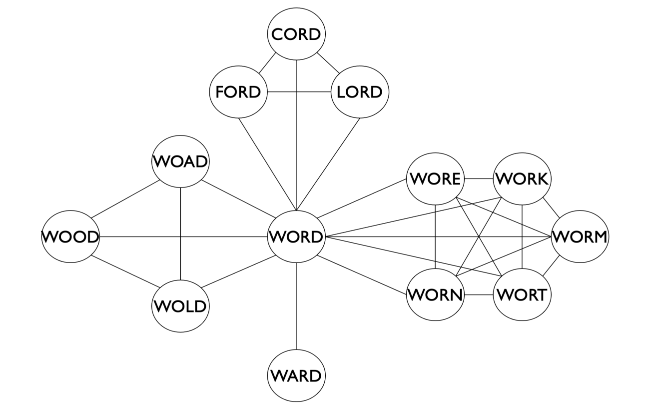
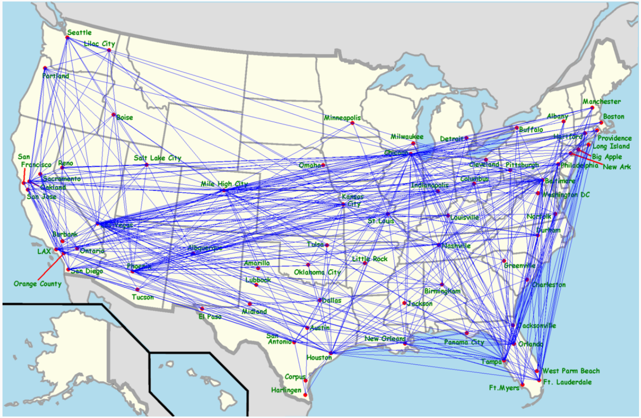


Goal of the game: given two words, transform one into the other by changing one letter at a time, always maintaining a legal word.

What does a *path* mean in this graph? What is the meaning of the length of the path?

Flight Routes



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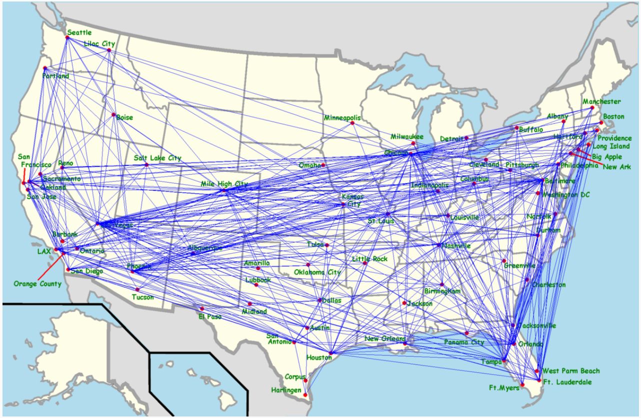

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Takeaway: graphs really can represent a very broad variety of real-world problems.

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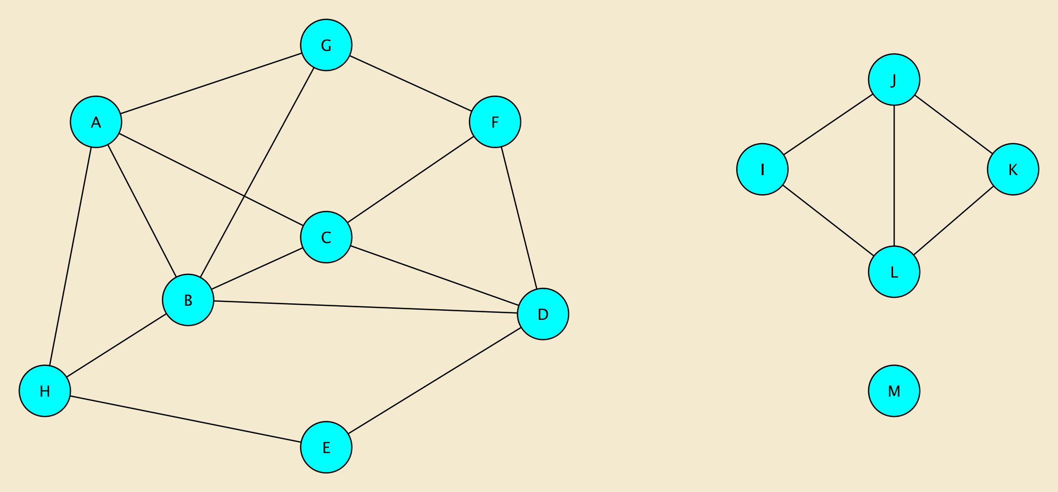
- A graph is *connected* if for every path of vertices u and v, v is reachable from u.
- What does it mean if one vertex is reachable from another in the graph of flights? What does it mean if the flight graph is connected?

Connected Component

- All vertices reachable from v, along with all edges of G connecting two of them, constitute the *connected component* of v.

Reachability Example



Determining Reachability

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- How can we implement this?

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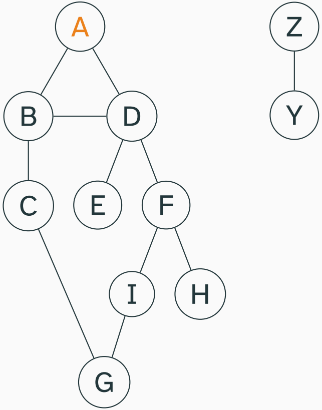
- What does this look like?

Breadth-First Search Example 1



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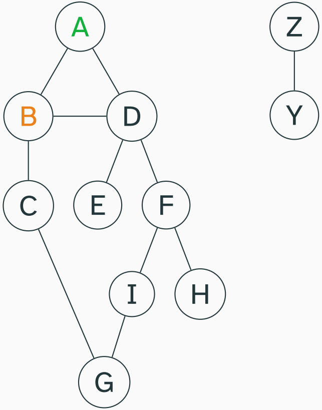

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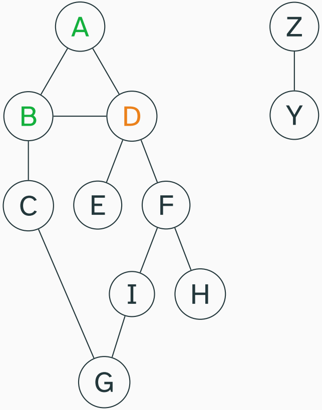

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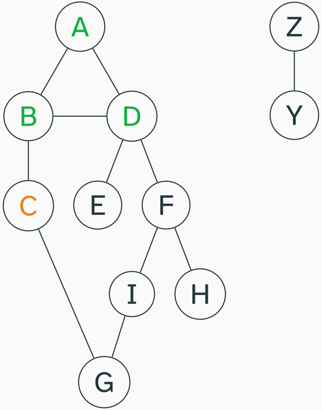

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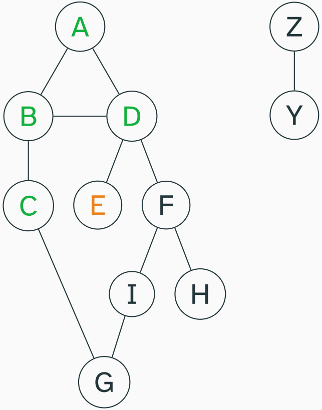

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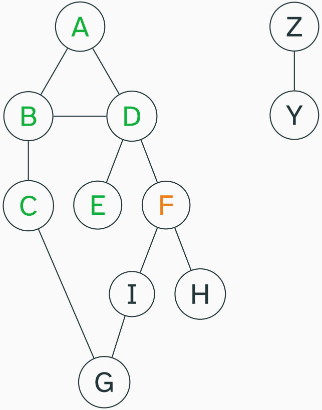

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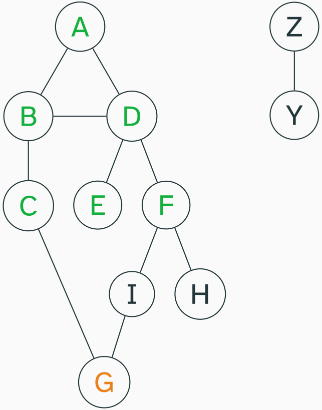

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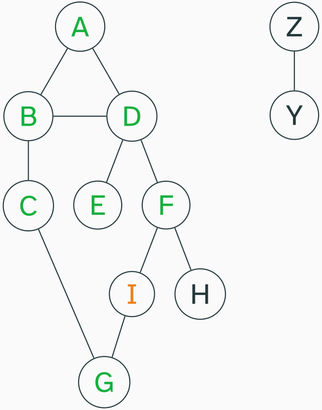

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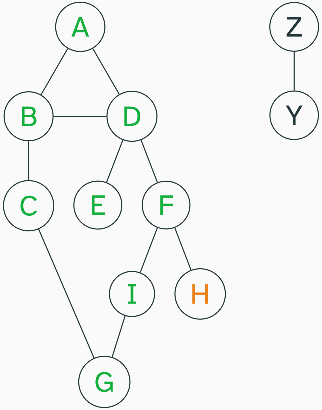

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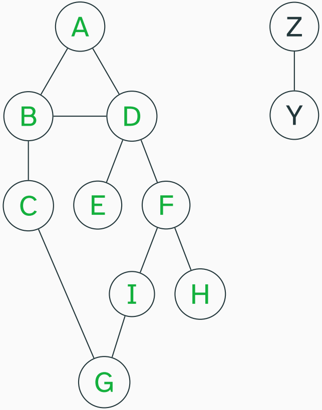

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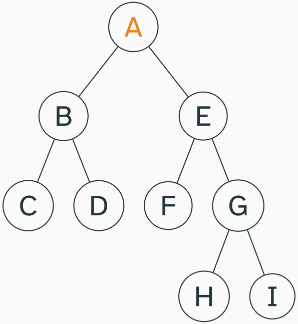

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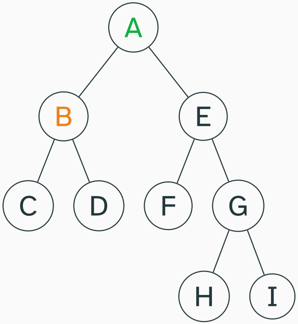

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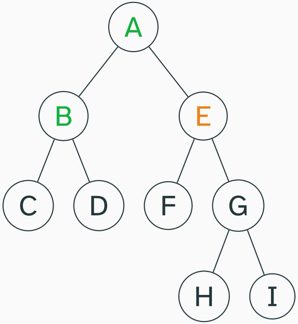

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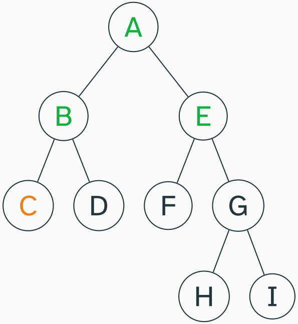

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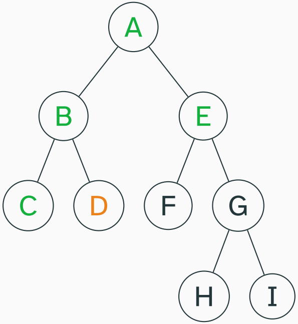

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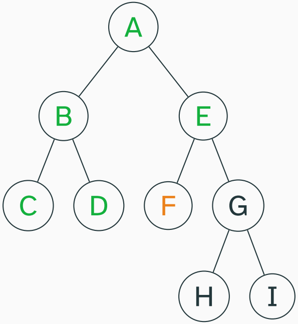

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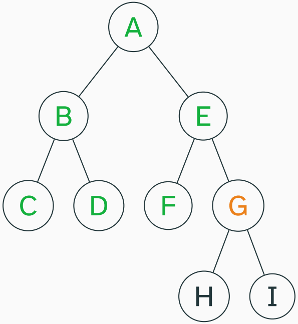

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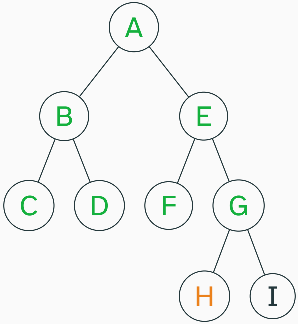

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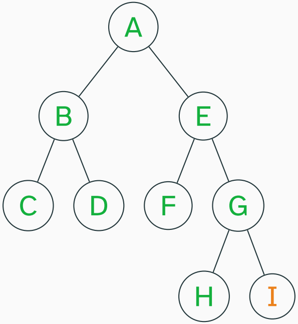

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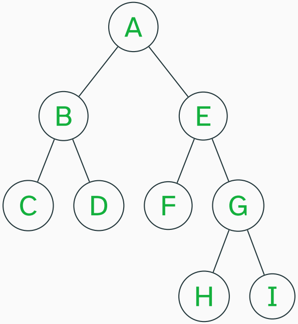

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Plan

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- Use *pseudocode*: a description of an algorithm in code-like notation (without worrying about language-specific details)

Breadth-First Search

```
// pre: all vertices are marked as unvisited
BFS(G, v) // Do a breadth-first search of G starting at v
   count ← 0
   Create empty queue Q
   enqueue v
   mark v as visited
   count++
   while Q isn't empty:
      current ← Q.dequeue()
      for each unvisited neighbor u of current:
         add u to Q
         mark u as visited
         count++
   return count;
```

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- How can we store the edges?