Dijkstra’s Shortest Path Algorithm

Instructors: Sam McCauley and Dan Barowy

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Admin

- Final: Sunday, May 22, 9:30 AM, in Physics 203
- Lab 8 back later today
    - If you think you want to use lab 9 for resubmission let me know. (Can't use lab 10—mentioned in syllabus)
- “Practice exam” (really just sample exam questions) posted on handouts page. Solutions today or tomorrow
- All practice quizzes also on handouts page!
- Any questions?

Heaps and Priority Queues

Heap vs priority queue

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- Heap is the specific implementation
- Like Map vs Hashtable. There are other ways to implement a Map; similarly, there are other ways to implement a priority queue

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- Extremely common in practice!
- HeapSort is one of the most common sorting methods, especially if you want $O(n \log n)$ guaranteed worst-case running time
- We saw min heaps. Can get a “max heap” by flipping the requirement: the root element must be largest in the heap. Then can get good `removeMax` performance

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- Where to start?

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- Where do we go from here? Can we find the shortest path to any other vertex?

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  This path has length $\sum_i e_i > e_1$. Then v_1 is closer to v than u is.

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- Idea: adding more edges only makes the path longer

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- Idea: must be a neighbor of one of the vertices we’ve already visited

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  - Add them to a priority queue; priority based on the *total* length of the path: length of the path to this vertex, plus the length of the outgoing edge

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- Now, let’s look at the actual code

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- Dijkstra’s demo!

Review!

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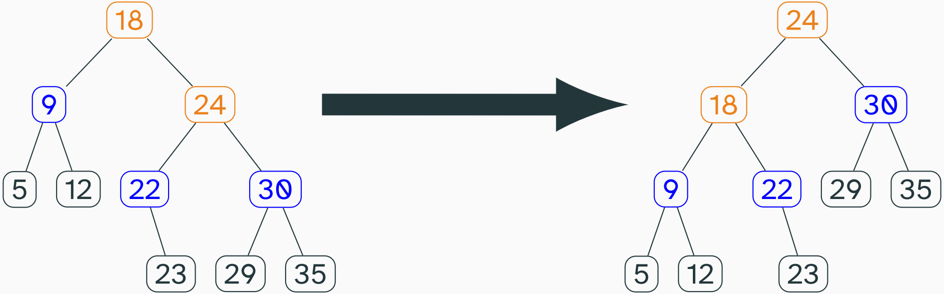
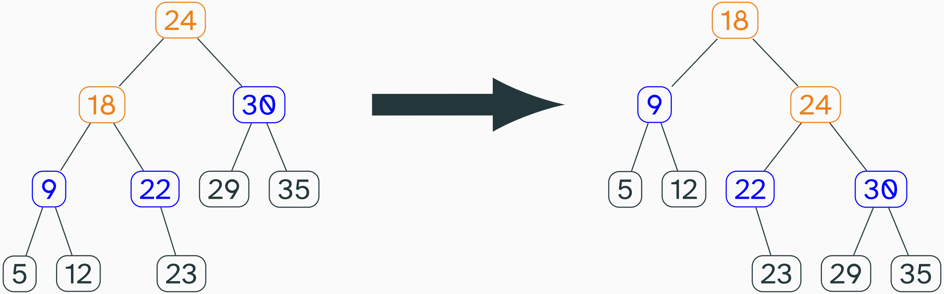
  - AVL trees use rotations to maintain balance.
  - It is possible to delete from a binary search tree, and a balanced binary search tree. (But you don't need to know how!)

Tree Rotation: Rotate Left



This rotation is on the orange nodes (18 and 24); for a left rotation one must be a right child of the other. We rearrange the *children* of these nodes (in blue).

Tree Rotation: Rotate Right



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Last Week’s Practice Quiz

Any Other Questions?

Wrapping Up CSCI 136

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  - Proofs
  - How can we be *sure* that our methodology is correct?

SCS Forms